Numerical Treatment of General Third Order Ordinary Differential Equations Using Taylor Series as Predictor

Abstract

This work considers the direct solution of general third order ordinary differential equation. The method is derived by collocating and interpolating the approximate solution in power series. A single hybrid three-step method is developed. Taylor series is used to generate the independent solution at selected grid and off grid points. The order, zero stability and convergence of the method were established. The developed method is then applied to solve some initial value problems of third order ODEs. The numerical results of the method confirm the superiority of the new method over the existing method.

Keywords: collocation, interpolation, hybrid, Error constant, Zero stability, linear multistep method

1.Introduction

This paper examines the solution of third order ordinary differential equations of the form

$$y'''(x) = f(x, y, y', y''), y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2 \quad (1)$$

In the past, equation (1) is solved by method of reducing it to its equivalent system of first order ordinary differential equations and thereafter appropriate numerical method for first order would be applied to solve the systems. However, it is shown in [1, 2], that reduction of higher order ordinary differential equations to a system of first order has serious problems which include consumption of human effort, computational burden and non-economization of computer time.

This is also discussed by [3], [4], [5] and [6]. In order to cater for the setbacks encountered in reduction method and also bring about improvement on numerical method [7, 8, 9], [10] and [11] developed block methods for solving higher order ordinary differential equations directly in which the accuracy is better than when it is reduced to system of first order ordinary differential equations.

Linear multistep methods for solving equation (1) directly have been proposed by some researchers such as [12] developed a block method for the solution of third order ordinary differential equation whereby the accuracy of the method is not efficient enough in terms of error. A P-stable linear multistep method for direct solution of (1) was developed by [13] which was implemented in predictor-corrector mode.

Also, various authors such as [14], [15] developed the hybrid method. This method, while retaining certain charateristics of the continuous linear multistep method, share with Runge-Kutta methods the property of utilizing data at other points, other than the step point $x_{n+j}, j = 0,1 \cdots n-1$. This method is useful in reducing the step number of a method and still remains zero stable.

But, in [16], he stated that block method has a setback of not being able to exhaust all the possible interpolation points because the interpolation points cannot exceed the order of the differential equation. With this drawback, [17], proposed Taylor series approximation method to improve on the setback usually faced with Predictor-Corrector and Block methods.

In this research, a one point hybrid linear multistep method based on collocation and interpolation at selected grid and off-grid point is developed for the direct solution of third order initial value problems of ordinary differential equation using hybrid method $x_{n+j,}\ j = 0,1\cdots n-1$ with Taylor's series being used to analyze and implement the method.

2. **Methodology**

We consider power series of the form

$$y(x) = \sum_{j=0}^{s+r-1} a_j x^j \qquad (2)$$

as an approximate solution to equation (1), where s is the number of collocation points for $0 \le s \le 5$, r is the number of interpolation points for $0 \le r \le 3$, and a_j are the unknown parameters to be determined. The first, second and third derivatives of (2) give

$$y^{'}(x) = \sum_{j=1}^{s+r-1} j a_j x^{j-1} \qquad (3)$$

$$y^{''}(x) = \sum_{j=2}^{s+r-1} j(j-1) a_j x^{j-2} \qquad (4)$$

$$y'''(x) = \sum_{j=0}^{s+r-1} j(j-2)(j-1) a_j x^{j-3} = f(x, y, y^{'}, y^{''}) \qquad (5)$$

Interpolating equation (2) at the points $x = x_{n+i}, i = 1, \frac{3}{2}, 2$ and collocating equation (5) at the points $x = x_{n+i}, i = 0, 1, \frac{3}{2}, 2, 3$ produces the following equations

$$\sum_{j=0}^{s+r-1} a_j x^j = y_{n+i} \qquad (6)$$

$$\sum_{j=0}^{s+r-1} j(j-2)(j-1) a_j x^{j-3} = f_{n+i} \qquad (7)$$

Solving equations (6) and (7) by using Gaussian elimination method to determine the values of the unknown parameters a_j which are then substituted into equation (2). This gives a continuous implicit hybrid scheme of the form

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^3 \left(\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_v(x) f_{n+v} \right) \qquad (8)$$

Using the transformation $t = \frac{x - x_{n+2}}{h}, \frac{dt}{dx} = \frac{1}{h}$ (9)

Where k = 3, v =3/2 α_j and β_j are given as

$$\alpha_1(t) = 2t^2 + t$$

$$\alpha_{3/2}(t) = -(4t^2 + 4t)$$

$$\alpha_2(t) = (1 + 3t + 2t^2)$$

$$\beta_0(t) = \frac{h^3}{241920}\left[47t + 175t^2 - 560t^4 - 448t^5 + 112t^6 + 128t^7\right]$$

$$\beta_1(t) = \frac{h^3}{241920}\left[-61t - 7t^2 + 112t^4 + 672t^5 - 336t^6 - 128t^7\right] \quad (10)$$

$$\beta_{3/2}(t) = \frac{h^3}{945}\left[62t + 154t^2 - 35t^4 - 28t^5 + 28t^6 + 8t^7\right]$$

$$\beta_2(t) = \frac{h^3}{26880}\left[541t + 2653t^2 + 4480t^3 + 2800t^4 - 560t^6 - 128t^7\right]$$

$$\beta_3(t) = \frac{h^3}{241920}\left[-79t - 287t^2 + 1120t^4 + 1568t^5 + 112t^6 + 128t^7\right]$$

Evaluating (10) at the end point $x = x_{n+3}$ i.e. $t = 1$ gives our hybrid method as

$$y_{n+3} - 6y_{n+2} + 8y_{n+3/2} - 3y_{n+1} = \frac{1}{5760}\left[77f_{n+3} + 2097f_{n+2} + 512f_{n+3/2} + 207f_{n+1} - 13f_n\right] \quad (11)$$

With order p = 6 and error constant $c_{p+2} = -0.000439453125$.

Finding the first derivative of (10) gives:

$$\alpha_1^{'}(t) = 4t + 1$$
$$\alpha_{3/2}^{'}(t) = -8t - 4$$
$$\alpha_2^{'}(t) = 4t + 3$$
$$\beta_0^{'}(t) = \frac{h^3}{241920}\left[47 + 350t - 2240t^3 - 2440t^4 + 672t^5 + 896t^6\right]$$
$$\beta_1^{'}(t) = \frac{h^3}{26880}\left[-61 - 14t + 4490t^3 + 3360t^4 - 2016t^6\right]$$
$$\beta_{3/2}^{'}(t) = \frac{h^3}{945}\left[62 + 308t - 140t^3 - 140t^4 + 168t^5 + 56t^6\right]$$
$$\beta_2^{'}(t) = \frac{h^3}{26880}\left[541 + 5306t + 13440t^2 + 11200t^3 - 3360t^5 - 896t^6\right]$$
$$\beta_3^{'}(t) = \frac{h^3}{241920}\left[-79 - 574t + 4480t^3 + 7840t^4 + 672t^5 + 896t^6\right]$$

(12)

While the second derivative of (10) gives:

$$\alpha_1^{''}(t) = 4$$
$$\alpha_{3/2}^{''}(t) = -8$$
$$\alpha_2^{''}(t) = 4$$
$$\beta_0^{''}(t) = \frac{h^3}{241920}\left[350 - 6720t^2 - 9760t^3 + 3360t^4 + 5376t^5\right]$$
$$\beta_1^{''}(t) = \frac{h^3}{26880}\left[-14 + 13470t^2 + 13440t^3 - 10080t^4 - 5376t^5\right]$$
$$\beta_{3/2}^{''}(t) = \frac{h^3}{945}\left[308 - 420t^2 - 560t^3 + 840t^4 + 336t^5\right]$$
$$\beta_2^{''}(t) = \frac{h^3}{26880}\left[5306 + 26880t + 33600t^2 - 16800t^4 - 5376t^5\right]$$
$$\beta_3^{''}(t) = \frac{h^3}{241920}\left[-574 + 13440t^2 + 31360t^3 3360t^4 + 5376t^5\right]$$

(13)

Evaluating (12) and (13) at the end points i.e at $t = 1$ gives the following equations (14) and (15):

$$241920hy'_{n+3} - 1209600y_{n+1} + 2903040y_{n+3/2} - 1693440y_{n+2} = h^2\begin{bmatrix}-2515f_n + 38385f_{n+1} - 27136f_{n+3/2} \\ +236079f_{n+2} + 17267f_{n+3}\end{bmatrix} \quad (14)$$

$$5760h^2 y''_{n+3} - 23040 y_{n+1} + 46080 y_{n+3/2} - 23040 y_{n+2} = h\begin{bmatrix} -157 f_n + 2319 f_{n+1} - 46080 f_{n+3/2} \\ +11649 f_{n+2} + 1741 f_{n+3} \end{bmatrix} \quad (15)$$

3. Basic properties of the method

3.1 Order and Error Constant of the Method

In this paper, we adopt the method proposed by [4], with the linear operator:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j f_{n+j}$$

We associate the linear operator L with the scheme and define as

$$L\{y(x), h\} = \sum_{j=0}^{k} \left[\alpha_j y(x+jh) - h^3 \beta_j y''(x+jh) \right]$$

where α_0 and β_0 are both non-zero and $y(x)$ is an arbitrary function which is continuous and differentiable on the interval [a, b]. If we assume that $y(x)$ has as many higher derivatives as we require, then on Taylor's series expansion about the point x, we obtain

$$L\left[y(x,h)\right] = c_0 y(x) + c_1 h y'(x) + \cdots + cq h^q y^q(x) + \cdots$$

Accordingly we say that the method has order P if,

$$c_0 = c_1 = \cdots = c_p = c_{p+1} = 0, c_{p+2} \neq 0$$

Then, c_{p+2} is the error constant and $C_{p+2} h^{p+2} y^{p+2}(x_n)$ is the principal local truncation error at the point x_n .

In this paper, since $c_0 = c_1 = c_2 = c_3 = \cdots = c_6$ and $c_8 = c_{p+2} \neq 0$ which implies that the scheme is of order 6 and the error constant $c_{p+2} = -0.000439453125$.

3.2 Zero Stability of the Method

Definition (see [4])**:** A linear multistep method is said to be zero-stable, if no root of the first characteristics polynomials $\rho(r)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than two.

Our method is zero stable since no root of the first characteristics polynomial $\rho(r)$ has modulus greater than one that is $|r| \leq 1$. This implies that the method is zero stable if,

$\sum_{j=0}^{k} \alpha_j = 0$, where α_j are the coefficients of $\sum_{j=0}^{k} \alpha_j y_{n+j}$

$$\sum_{j=0}^{k} \alpha_j = \alpha_3 - 6\alpha_2 + 8\alpha_{3/2} - 3\alpha_1 = 1 - 6 + 8 - 3 = 0$$

3.3 Theorem 1: Convergence (see [4])

The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. From the theorem above, the method is convergent

4. Implementation

4.1 Taylor Series Expansion of the Method

Since Taylor's series expansion is used to approximate y variables in this research, to generate y values for the approximate solution of the scheme, the third derivative is expanded term by term up to the order of the scheme developed by Taylor's series expansion.

$$y_{n+i} = y(xn+ih) = y_{(xn)} + ihy'_{(xn)} + \frac{(ih)^2}{2!}y''_{(xn)} + \frac{(ih)^3}{3!}f_n + \frac{(ih)^4}{4!}f'_n + \frac{(ih)^5}{5!}f''_n + \frac{(ih)^6}{6!}f'''_n + ...$$

$$y'_{n+i} = y'(xn+ih) = y'_{(xn)} + ihy''_{(xn)} + \frac{(ih)^2}{2!}f_n + \frac{(ih)^3}{3!}f'_n + \frac{(ih)^4}{4!}f''_n + \frac{(ih)^5}{5!}f'' + \frac{(ih)^6}{6!}f_n^{iv} + ...$$

$$y''_{n+i} = y''\left(xn+ih\right) = y''_{(xn)} + ihf_n + \frac{(ih)^2}{2!}f'_n + \frac{(ih)^3}{3!}f''_n + \frac{(ih)^4}{4!}f'''_n + \frac{(ih)^5}{5!}f_n^{iv} + \frac{(ih)^6}{6!}f_n^{v} + ...$$

$$f_{n+i} = y'''(xn+ih) = f_n + ihf_n' + \frac{(ih)^2}{2!}f''_n + \frac{(ih)^3}{3!}f'''_n + \frac{(ih)^4}{4!}f_n^{iv} + \frac{(ih)^5}{5!}f_n^{v} + \frac{(ih)^6}{6!}f_n^{iv} + ...$$

$$Y(x) = \sum_{q=0}^{2}\frac{(jh)}{q!} + h^3\sum_{\lambda=0}^{p}\frac{\partial^\lambda}{\partial y^\lambda} f(x,y,y',y'')$$

Where

$$\frac{\partial f}{\partial y}(x,y,y',y'') = \left(\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + f\frac{\partial f}{\partial y''}\right)f_j = Df_j$$

$$f(x_j, y_j, y'_j, y''_j) = f_j.$$

$$D = \left(\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + f\frac{\partial f}{\partial y''}\right), D^2 = D(D)$$

Where p is the order of the method

4.2 Numerical Examples

Problem 1:

We consider the non-linear IVP which was solved by [13] for the step-size h=0.1

$$y''' = -y, \; y(0) = 1, \; y'(0) = -1, y''(0) = 1, h = 0.1$$

Exact solution: $y(x) = e^{-x}$

In this example, our method of order p = 6 is compared with the method in [13]. In terms of accuracy, our result performs better than those given in [13]. The details of the numerical result at some selected points are shown in table 1 below.

Table 1: Result of problem 1

X	Exact solution	Computed solution	Error in Taylor series K=3, h=0.1 Order, p = 6	Error in [13] K=3,Order, p = 5, h=0.1 (P-C)
0.30	0.740818220681717770	0.740818232946138580	1.226442E-08	3.991507930E-07
0.40	0.670320046035639330	0.670320061249500610	1.521386E-08	1.036855911E-06
0.50	0.606530659712633420	0.606530680594479790	2.088185E-08	2.128500409E-06
0.60	0.548811636094026390	0.548811665913141460	2.981912E-08	3.789530170E-06
0.70	0.496585303791409470	0.496585346306387790	4.251498E-08	6.130076711E-06
0.80	0.449328964117221560	0.449329023516181540	5.939896E-08	9.253856792E-06
0.90	0.406569659740599110	0.406569740582070790	8.084147E-08	1.325713611E-05
1.00	0.367879441171442330	0.367879548325104060	1.071537E-07	1.822776743E-05
1.10	0.332871083698079500	0.332871222284604030	1.385865E-07	2.424431283E-05
1.20	0.301194211912202080	0.301194387241549900	1.753293E-07	3.137525869E-05

Problem 2:

$$y''' = y'\left(2xy'' + y'\right),\ y(0) = 1,\ y'(0) = \frac{1}{2}, y''(0) = 0, h = 0.01$$

Exact solution: $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Our result was compared with [18] which is of order 6. Using thesame step size (h = 0.01), it is observed that our result performs better. The details of the numerical result at some selected points are in table 2 below:

Table 2: Result of problem 2

X	Exact solution	Computed solution	Error in Taylor Series K=3, h=0.01 Order, p = 6	Error in [18] K=3,Order , p = 6, h = 0.01 (block method)
0.21	1.015001125151899300	1.105388447837780900	7.178702E-13	8.037948 E – 11
0.31	1.156259497799360100	1.156259497796915600	2.444489E-12	6.043090 E – 10
0.41	1.207946365635211800	1.207946365629159600	6.052270E-12	2.581908 E – 09
0.51	1.260753316593162600	1.260753316580459900	1.270273E-11	8.158301E – 09
0.61	1.315023237096001100	1.315023237071823600	2.417755E-11	2.141286 E – 08
0.71	1.371153208259014500	1.371153208215620600	4.339396E-11	4.969641 E – 08
0.81	1.429615588111108300	1.429615588035786400	7.532197E-11	1.620387 E – 07

5. Discussion of Result

In table 1, the results of our three-step hybrid Taylor series method are more accurate than that of [13] which was executed by predictor-corrector method.

In table 2, the results of the three-step hybrid Taylor series method also perform better than [18] block method implemented scheme. Though we used the same parameters with that of [18] that is, order, P=6, K=3 and h=0.01 our method is still more accurate.

6. Conclusion

This research describes the development, analysis and implementation of three-step hybrid method for solving general third order initial value problems of ordinary differential equations directly using Taylor Series approximation method.

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