DISCRETE PHASE SPACE, STRING-LIKE PHASE CELLS, AND RELATIVISTIC QUANTUM MECHANICS

Abstract

The discrete phase space representation of quantum mechanics involving a characteristic length is investigated. The continuous (1+1)-dimensional phase space is first discussed for the sake of simplicity. It is discretized into denumerable infinite number of concentric circles. These circles, endowed with "unit area", are degenerate phase cells resembling *closed strings*.

Next, Schrödinger wave equation for one particle in the three dimensional space under the influence of a static potential is studied in the discrete phase space representation of wave mechanics. The Schrödinger equation in the arena of discrete phase space is *a partial difference equation*. The energy eigenvalue problem for a three dimensional oscillator is exactly solved.

Next, *relativistic wave equations* in the scenario of three dimensional discrete phase space and continuous time are explored. Specially, *the partial finite difference-differential equation* for a scalar field is investigated for the sake of simplicity. The exact relativistic invariance of the partial finite difference-differential version of the Klein-Gordon equation is rigorously proved. Moreover, it is shown that all nine important Green's functions of the partial finite difference-differential wave equation for the scalar field *are non-singular*.

In the appendix, a possible physical interpretation for the discrete orbits in the phase space as degenerate, string-like phase cells is provided in a mathematically rigorous way.

§ 1. Introduction

In 1960, the quantum field theory of interacting fields was proposed[1] in the arena of *a discrete space-time* involving a characteristic length. The corresponding Green's functions of the *partial difference-equations* representing wave fields in discrete space-time were all non-singular. Moreover, *divergence difficulties of the usual S-matrix theory were eliminated.* However, all the invariance and covariance of the continuous Poincaré group were lost !

In 1994, a new representation of quantum mechanics (or wave mechanics) in the setting of the discrete phase space (involving a characteristic length) was formulated.[2,3] The corresponding classical wave equations were expressed as partial difference equations. Every Green's function of these partial difference equations was *non-singular.* Furthermore, every partial difference wave equation turned out to be invariant or covariant under the continuous Poincaré group !

In 2010, quantum mechanics was explored under *the mixed representation* involving the background of three dimensional discrete phase space and one dimensional continuous time.[4–6] The resulting wave equations were expressed as *partial finite difference-differential equations.* (It is interesting to note that Hamilton used[7] a partial finite difference-differential equation for the light propagation through ether-lattice !)

It was rigorously proved that every partial finite difference-differential equation (corresponding to the usual relativistic partial differential wave equation in continuous space-time) remains exactly invariant or covariant under *the continuous Poincaré group.* Moreover, every Green's function turned out to be *non-singular.* Finally, quantum electrodynamics was investigated in the background of discrete phase space and continuous time.[6] The corresponding S-matrix elements in every order turned out to be *divergence-free.*

In the present *paper*, physical interpretation of discrete concentric circles as degenerate phase cells is enunciated. However, a phase cell respecting the uncertainty principle of quantum mechanics must be of an area $|\Delta p \cdot \Delta q| \geq \hbar$. Then, the puzzling situation arises of a circular orbit in a phase plane possessing an area ! Fortunately, in pure mathematics, there are examples of continuous *Peano curves* covering completely a unit area already exit.[8] In the appendix, a particular example of Peano curves which covers an annular phase cell of unit area is explained. In fact a sequence of such annular phase cells is constructed such that in the limiting case, the sequence of annular cells collapse into one circular orbit in the $(1+1)$-dimensional continuous phase space. Such an orbit resembles a *closed string*[9] which with passage of time creates a two dimensional *world sheet*[9] in the three dimensional space of a phase plane and continuous time.

Next, in the $(3+3)$-dimensional continuous phase space, three dimensional discrete orbits $S^1 \times S^1 \times S^1$ are considered. These are the closed brane-like degenerate phase cells applicable to the real physical phenomena. The arena of wave equations considered is the three discrete variables together with one continuous time variable. The scalar wave equation comprises of *one* partial finite difference-differential equation.[4,5] *The relativistic invariance* of such an equation is rigorously proved. Furthermore, corresponding Green's functions are investigated. All of the *nine* important Green's functions of the partial finite difference-differential equation are shown to be *non-singular.*

§ 2. Notations and preliminary definitions

There is a characteristic length $l > 0$ implicit in this article. We choose physical units such that $c = \hbar = l = 1$. All physical quantities are expressed as dimensionless numbers. Greek indices take values from $\{1, 2, 3, 4\}$, whereas roman indices take (special) values from $\{1, 2, 3\}$. Einstein's summation convention is followed in both cases. We denote the flat space-time metric of signature $+2$ by $\eta_{\mu\nu}$ and the diagonal matrix $[\eta_{\mu\nu}] := \mathrm{diag}[1, 1, 1, -1]$. We denote the set of all non-negative integers by $\mathbb{N} := \{0, 1, 2, 3\}$. An element $n \equiv \left(n^1, n^2, n^3, n^4\right) \in \mathbb{N}^4$ and an element $\left(\boldsymbol{n}, x^4\right) \equiv \left(n^1, n^2, n^3\cancel{, n^4}; t\right) \in \mathbb{N}^3 \times \mathbb{R}$.

Let a function f be defined by

$$f: \ \mathbb{N}^3 \times \mathbb{R} \longrightarrow \mathbb{R} \quad \big(\text{or, } f: \ \mathbb{N}^3 \times \mathbb{R} \longrightarrow \mathbb{C}\big) \,. \tag{1}$$

The right partial difference-differential equation and the left partial difference operations are defined by[4,10]

$$\Delta_j f\,(\boldsymbol{n}; t) := f\left(\ldots, n^j + 1, \ldots; t\right) - f\left(\ldots, n^j, \ldots; t\right), \tag{2a}$$

$$\Delta'_j f\,(\boldsymbol{n}; t) := f\left(\ldots, n^j, \ldots; t\right) - f\left(\ldots, n^j - 1, \ldots; t\right), \tag{2b}$$

We define $f\,(\boldsymbol{n}; t) \equiv 0$ for the cases [illegible]y of [illegible] < 0 .

Note that

$$\left[\Delta_j \Delta'_k - \Delta'_k \Delta_j\right] f\,(\boldsymbol{n}; t) \equiv 0\,. \tag{3}$$

We also assume that $\partial_t^2 f\,(\boldsymbol{n}; t) := \frac{\partial^2}{\partial t^2} f\,(\boldsymbol{n}; t)$ is a continuous function of t.

§ 3. Quantum mechanics in $(1+1)$-dimensional phase space

This simple toy model of the time-independent quantum mechanics is discussed to introduce discrete phase space and relativistic quantum mechanics in the section § 5 later on.

The mathematics of such a model comprises of state vectors $\overrightarrow{\boldsymbol{\psi}}$ ~~of~~ a Hilbert space and linear operators $F(\boldsymbol{P}, \boldsymbol{Q})$ involving the momentum operator $\boldsymbol{P}$ and the position operator $\boldsymbol{Q}$. In the usual Schrödinger representation of quantum mechanics, these mathematical objects are identified as:

$$\overrightarrow{\boldsymbol{\psi}} := \psi(q)\ ,\ q \in \mathbb{R}\ ; \tag{4a}$$

$$\boldsymbol{P}\overrightarrow{\boldsymbol{\psi}} := -i\frac{d}{dq}\psi(q)\ , \tag{4b}$$

$$\boldsymbol{Q}\overrightarrow{\boldsymbol{\psi}} := q\psi(q)\ , \tag{4c}$$

$$[\boldsymbol{P}, \boldsymbol{Q}]\,\overrightarrow{\boldsymbol{\psi}} := [\boldsymbol{PQ} - \boldsymbol{QP}]\,\overrightarrow{\boldsymbol{\psi}} = -i\overrightarrow{\boldsymbol{\psi}} = -i\psi(q)\ . \tag{4d}$$

In the separable sector of the Hilbert space,[11] it is assumed that $\left\langle \overrightarrow{\boldsymbol{\psi}} | \overrightarrow{\boldsymbol{\psi}} \right\rangle := \int_{\mathbb{R}} \overline{\psi}(q)\psi(q)\,dq < \infty$. On the other hand, in the non-separable sector,[2]

$$\lim_{L\to\infty}\left\{(1/2L)\int_{-L}^{L}\overline{\psi}(q)\psi(q)\,dq\right\} < \infty$$

In the discrete phase space representation of quantum mechanics, we can try difference operators $\boldsymbol{P} := c_1\Delta + c_2\Delta'$ and $\boldsymbol{Q} := c_3\Delta + c_4\Delta'$, where $\overrightarrow{\boldsymbol{\psi}} := f(n)\ ,\ n \in \mathbb{N}$. Such a representation *fails* by the equation (3).

We define *two new difference operators* in the following:

$$\Delta^{\#} f(n) := \left(1/\sqrt{2}\right)\left[\sqrt{n+1}\,f(n+1) - \sqrt{n}\,f(n-1)\right]\ , \tag{5a}$$

$$\overset{\circ}{\Delta}\, f(n) := \left(1/\sqrt{2}\right)\left[\sqrt{n+1}\,f(n+1) + \sqrt{n}\,f(n-1)\right]\ . \tag{5b}$$

One possible discrete phase space representation of the quantum mechanics is furnished by:

$$\overrightarrow{\boldsymbol{\psi}} := \phi(n)\ ,\ n \in \mathbb{N}\ ; \tag{6a}$$

$$\boldsymbol{P}\overrightarrow{\boldsymbol{\psi}} := -i\Delta^{\#}\phi(n)\ , \tag{6b}$$

$$\boldsymbol{Q}\overrightarrow{\boldsymbol{\psi}} := \overset{\circ}{\Delta}\,\phi(n)\ , \tag{6c}$$

$$\boldsymbol{A}\overrightarrow{\boldsymbol{\psi}} := \left(1/\sqrt{2}\right)(\boldsymbol{Q} - i\boldsymbol{P})\,\overrightarrow{\boldsymbol{\psi}} = \sqrt{n}\,\phi(n-1)\ , \tag{6d}$$

$$\boldsymbol{A}^{\dagger}\overrightarrow{\boldsymbol{\psi}} := \left(1/\sqrt{2}\right)(\boldsymbol{Q} + i\boldsymbol{P})\,\overrightarrow{\boldsymbol{\psi}} = \sqrt{n+1}\,\phi(n+1)\ , \tag{6e}$$

$$\left[\boldsymbol{A}^{\dagger}, \boldsymbol{A}\right]\overrightarrow{\boldsymbol{\psi}} := \phi(n) = \overrightarrow{\boldsymbol{\psi}}\ . \tag{6f}$$

The mathematics in (6d, 6e, 6f) are analogous to the creation and annihilation operators in the standard quantum field theory.[12]

We shall now solve the energy eigenvalue problem for a one dimensional (idealized) harmonic oscillator by the finite difference representation in (6a, 6b, 6c).

$$(1/2)\left[(\boldsymbol{P})^2 + (\boldsymbol{Q})^2\right]\overrightarrow{\boldsymbol{\psi}}_{(N)} = \lambda_{(N)}\overrightarrow{\boldsymbol{\psi}}_{(N)}\ , \tag{7a}$$

$$\text{or,}\quad \left[-\left(\Delta^{\#}\right)^2 + \left(\overset{\circ}{\Delta}\right)^2\right]\phi_{(N)}(n) = 2\lambda_{(N)}\phi(n)\ , \tag{7b}$$

$$\text{or,}\quad \left[\left(n + \frac{1}{2}\right) - \lambda_{(N)}\right]\phi_{(N)}(n) = 0\ . \tag{7c}$$

Clearly, the eigenvalues and the real-valued normalized eigen functions are provided by:

$$\lambda_{(N)} = N + \left(\frac{1}{2}\right) \geq \frac{1}{2}\,, \quad N \in \mathbb{N}\,, \tag{8a}$$

$$\phi_{(N)}(n) = \delta_{(N)n}\ , \tag{8b}$$

$$\left\|\overrightarrow{\psi}_{(N)}\right\|^2 := \sum_{n=0}^{\infty} \left|\phi_{(N)}(n)\right|^2 \equiv 1\ . \tag{8c}$$

Consider the simple harmonic oscillator orbits in the $(1+1)$-dimensional phase plane with quantized energy levels:

$$(1/2)\left(p^2 + q^2\right) = N + \left(\frac{1}{2}\right)\ , \qquad N \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}\ . \tag{9}$$

The equation above yields concentric circles[4] of radii $\sqrt{2N+1}$ as depicted in fig. 1.

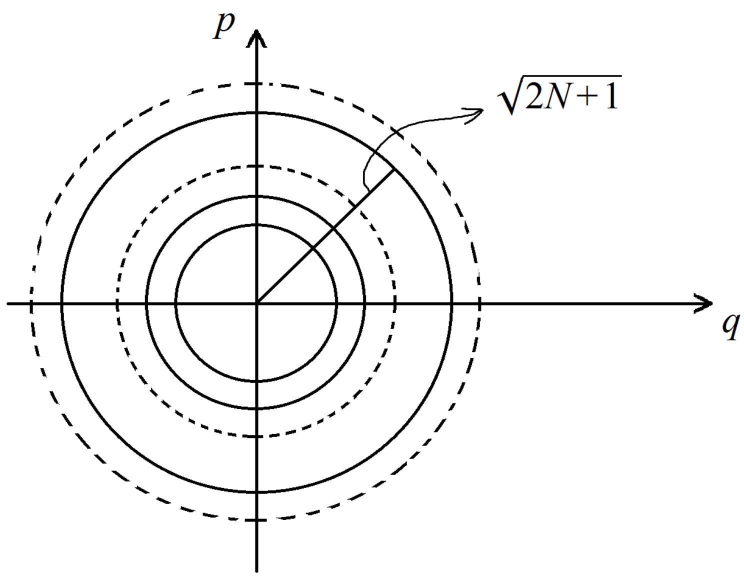

Figure-1: Discrete orbits for possible occupation of the oscillating particle.

In the corresponding $(2+1)$-dimensional *state space*[13] $\mathbb{R}^2 \times \mathbb{R}$, one possible discrete orbit in the phase plane traces a vertical, 2-dimensional circular cylinder as the *world sheet.*[9] (See fig. 2.)

In case the oscillator absorbs extra energy through an external interaction, the world sheet suddenly jumps into a larger size. (See fig. 3.)

In the fig. 1 , discrete orbits in $(1+1)$-dimensional phase space resemble *closed strings* of the string theory.[9] Moreover, hollow circular cylinders in $(2+1)$-dimensional state space of fig. 2 resemble *world sheets* of the string theory.[9] We shall briefly compare and contrast discrete phase space orbits and circular cylinders in the state space with closed strings and world sheets of the string theory.

(1) Discrete circular orbits in phase space may or may not be occupied by a particle (or a quanta). However, a closed string has always a mass density and a tension.[9]

(2) Vertical hollow cylinders in the state space may or may not contain a world line of a particle. But a world sheet in string theory[9] has always a mass density associated with it.

(3) A particle or a quanta can jump from one vertical circular cylinder to another by interaction with an external agent. However, in string theory, one world sheet can bend or rupture into several world sheets.[9]

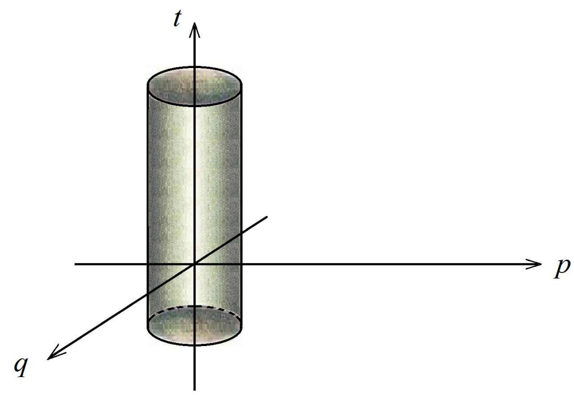

Figure-2 : The two-dimensional cylindrical world sheet.

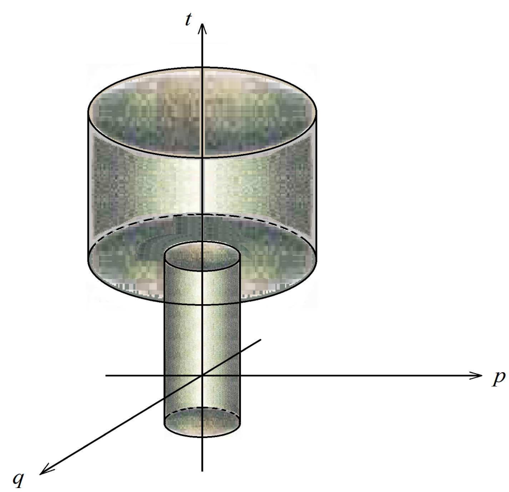

Figure-3 : World sheet associated with the oscillator jumping from one orbit to another.

We shall interpret in the appendix, the discrete orbits in ~~pahse~~ space as depicted in fig. 1 , as *degenerate phase cells*.

~~Now, we~~ shall discuss the transformation of the Schrödinger representation of quantum mechanics into the discrete phase representation of the same. The Schrödinger representation is provided in equations (4a, ..., 4d). For the discrete phase space representation, we need to introduce the Hermite polynomials[14] and the following equations :

$$H_n(q) := (-1)^n e^{q^2} \frac{d^n}{(dq)^n} \left(e^{-q^2} \right) \ , \tag{10a}$$

$$f_n(q) := \frac{e^{-(q^2/2)} H_n(q)}{\pi^{1/4} \cdot 2^{n/2} \cdot \sqrt{n!}} \ , \tag{10b}$$

$$\int_{-\infty}^{\infty} f_n(q) f_m(q) \, dq = \delta_{nm} \ . \tag{10c}$$

The transformation from the Schrödinger representation to the discrete phase space representation is furnished by the following :

$$\overrightarrow{\boldsymbol{\psi}} := \phi(n) \ , \tag{11a}$$

$$\phi(n) := \int_{-\infty}^{\infty} \psi(q) f_n(q) \, dq \ , \tag{11b}$$

$$\boldsymbol{P}\overrightarrow{\boldsymbol{\psi}} = \int_{-\infty}^{\infty} \left[-i \frac{d\psi(q)}{dq} \right] f_n(q) \, dq = -i\Delta^{\#} \phi(n) \ , \tag{11c}$$

$$\boldsymbol{Q}\overrightarrow{\boldsymbol{\psi}} = \int_{-\infty}^{\infty} [q\psi(q)] \, f_n(q) \, dq = \overset{\circ}{\Delta} \ \phi(n) \ . \tag{11d}$$

Here, we have assumed that $\lim\limits_{|q| \to \infty} |\psi(q)| = 0$.

~~Moreover, for~~ the derivation of (11c), we have utilized $\frac{dH_n(q)}{dq} = 2nH_{n-1}(q)$. Furthermore, to deduce (11d), we used $H_{n+1}(q) = 2qH_n(q) - 2nH_{n-1}(q)$. Thus, we have recovered equations (6a, 6b, 6c) .

§ 4. Finite difference – differential version of the Schrödinger equation

The wave function, position operators, and momentum operators in discrete phase space and continuous time are represented by[2,3]:

$$\overrightarrow{\boldsymbol{\psi}} := \phi(n^1, n^2, n^3; t) \equiv \phi\,(\boldsymbol{n}; t) \ , \tag{12a}$$

$$\boldsymbol{Q}^k \overrightarrow{\boldsymbol{\psi}} := \delta^{kj} \ \overset{\circ}{\Delta}_j \ \phi\,(\boldsymbol{n}; t) \ , \tag{12b}$$

$$\boldsymbol{P}_j \overrightarrow{\boldsymbol{\psi}} := -i \Delta_j^{\#} \phi\,(\boldsymbol{n}; t) \ . \tag{12c}$$

The time-dependent partial difference-differential version of the Schrödinger wave equation is represented[2] by :

$$\frac{1}{2m}\delta^{jk}\Delta_j^{\#}\Delta_k^{\#}\phi\left(\boldsymbol{n};t\right)-\left[V\left(\overset{\circ}{\Delta}_1,\overset{\circ}{\Delta}_2,\overset{\circ}{\Delta}_3\,;\,t\right)\right]\phi\left(\boldsymbol{n};t\right)=-i\partial_t\phi\left(\boldsymbol{n};t\right)\,. \tag{13}$$

In case of a conservative physical system, the wave function $\phi\left(\boldsymbol{n};t\right)$ and the Schrödinger equation (13) reduce to

$$\phi\left(\boldsymbol{n};t\right)=\chi\left(\boldsymbol{n}\right)\cdot\exp(-iEt)\;, \tag{14a}$$

$$\delta^{jk}\Delta_j^{\#}\Delta_k^{\#}\chi\left(\boldsymbol{n}\right)+2m\left[E-V\left(\overset{\circ}{\Delta}_1,\overset{\circ}{\Delta}_2,\overset{\circ}{\Delta}_3\right)\right]\chi\left(\boldsymbol{n}\right)=0\;. \tag{14b}$$

Here, the constant E stands for the eigenvalue of energy.

Consider an idealized three dimensional oscillator in the Hamiltonian mechanics[13] characterized by :

$$H\left(p_1,p_2,p_3\,;q^1,q^2,q^3\right):=\left(\frac{1}{2}\right)\left[\delta^{jk}p_jp_k+\delta_{jk}q^jq^k\right]=E>0\,. \tag{15}$$

The corresponding Schrödinger equation (14b) drastically reduces to *the algebraic equation*

$$\left[E-\left(n^1+n^2+n^3+\frac{3}{2}\right)\right]\chi\left(\boldsymbol{n}\right)=0\,. \tag{16}$$

(Compare the equation above with (7c).)

Therefore, the energy eigenvalues and the corresponding normalized eigenfunctions are furnished by :

$$E_{(N^1,N^2,N^3)}=N^1+N^2+N^3+\left(\frac{3}{2}\right)\geq\frac{3}{2}\;, \tag{17a}$$

$$\chi_{(N^1,N^2,N^3)}\left(n^1,n^2,n^3\right)=\delta_{(N^1)n^1}\cdot\delta_{(N^2)n^2}\cdot\delta_{(N^3)n^3}\;, \tag{17b}$$

$$\left\|\overrightarrow{\boldsymbol{\psi}}\right\|^2:=\sum_{n^1=0}^{\infty}\sum_{n^2=0}^{\infty}\sum_{n^3=0}^{\infty}\chi_{(N^1,N^2,N^3)}\left(n^1,n^2,n^3\right)\equiv 1\;. \tag{17c}$$

§ 5. Discrete phase space, continuous time, and relativistic Klein-Gordon equation

The abstract operator form of the Klein-Gordon equation is given by :

$$\left[\eta^{\mu\nu}\boldsymbol{P}_\mu\boldsymbol{P}_\nu+m^2\boldsymbol{I}\right]\overrightarrow{\boldsymbol{\psi}}=\overrightarrow{\boldsymbol{0}}\;, \tag{18a}$$

$$\text{or,}\quad\left[\delta^{jk}\boldsymbol{P}_j\boldsymbol{P}_k-\left(\boldsymbol{P}_4\right)+m^2\boldsymbol{I}\right]\overrightarrow{\boldsymbol{\psi}}=\overrightarrow{\boldsymbol{0}}\;. \tag{18b}$$

It is clear that the abstract Hilbert-vector equations (18a,18b) are relativistic invariant equations for any mass parameter $m\geq 0$. Therefore, the Klein-Gordon equations (18a,18b), in every representation of quantum mechanics must be relativistic. But we need to prove the last assertion in a mathematically rigorous way. We choose the *mixed* finite difference-differential representation[5,6] of the equation (18b) as

$$\left[\delta^{jk}\Delta_j^{\#}\Delta_k^{\#}-\left(\partial_t\right)^2-m^2\right]\phi\left(\boldsymbol{n}\,;t\right)=0\,. \tag{19}$$

The main reason for such a choice is to maintain micro-causality relations[15] in the corresponding second quantization[5] of the scalar field $\phi\left(\boldsymbol{n}\,;t\right)$.

The relativistic invariance and covariance are governed by the ten parameter, continuous, Poincaré group[12,16] $\mathcal{I}\,O(3;1)$ provided by:

$$\widehat{q}^{\,\mu} = c^{\mu} + l^{\mu}{}_{\nu} q^{\nu} \ , \tag{20a}$$

$$\eta_{\mu\nu} l^{\mu}{}_{\alpha} i^{\nu}{}_{\beta} = \eta_{\alpha\beta} \ , \tag{20b}$$

$$a^{\mu}{}_{\beta} l^{\beta}{}_{\nu} = l^{\mu}{}_{\beta} a^{\beta}{}_{\nu} = \delta^{\mu}{}_{\nu} \ . \tag{20c}$$

The four parameter Abelian subgroup of space-time translation is characterized by:

$$l^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} = a^{\mu}{}_{\nu} \ , \tag{21a}$$

$$\widehat{q}^{\,\mu} = c^{\mu} + q^{\mu} \ , \tag{21b}$$

$$q^{\,\mu} = -c^{\mu} + \widehat{q}^{\,\mu} \ . \tag{21c}$$

A scalar field $\phi\left(q^1, q^2, q^3, q^4\right)$ transforms tensorially[18] as

$$\widehat{\phi}\left(\widehat{q}^{\,1}, \widehat{q}^{\,2}, \widehat{q}^{\,3}, \widehat{q}^{\,4}\right) = \phi\left(q^1, q^2, q^3, q^4\right) \ , \tag{22a}$$

$$\text{or,} \quad \widehat{\phi}\left(q^1, q^2, q^3, q^4\right) = \phi\left(q^1 - c^1, q^2 - c^2, q^3 - c^3, q^4 - c^4\right) \ . \tag{22b}$$

Assuming that the function $\phi\left(q^1, q^2, q^3, q^4\right)$ admits a Taylor series expansion[19] in a star-shaped domain, we obtain from (22b),

$$\begin{aligned}
&\widehat{\phi}\left(q^1, q^2, q^3, q^4\right) = \phi\left(q^1, q^2, q^3, q^4\right) \\
&+ \sum_{j=1}^{\infty} \frac{(-1)^j}{j\,!} \left[\sum_{\substack{i_1=1 \\ (i_1+\cdots+i_j=j)}}^{4} \cdots \sum_{i_j=1}^{4} \left(c^{i_1} \ldots c^{i_j}\right) \cdot \frac{\partial^j}{\partial q^{i_1} \cdots \partial q^{i_j}} \phi\left(q^1, q^2, q^3, q^4\right) \right] \ ,
\end{aligned} \tag{23a}$$

$$\text{or,} \quad \widehat{\phi}\left(q^1, q^2, q^3, q^4\right) = \exp\left[-c^{\mu}\partial_{q\mu}\right] \phi\left(q^1, q^2, q^3, q^4\right) \ , \tag{23b}$$

$$\begin{aligned}
\text{or,} \quad &\eta^{\alpha\beta}\partial_{q\alpha}\partial_{q\beta}\widehat{\phi}\left(q^1, q^2, q^3, q^4\right) - m^2\widehat{\phi}\left(q^1, q^2, q^3, q^4\right) \\
&= \exp\left[-c^{\mu}\partial_{q\mu}\right] \cdot \left[\eta^{\alpha\beta}\phi\left(q^1, q^2, q^3, q^4\right) - m^2\phi\left(q^1, q^2, q^3, q^4\right)\right] \\
&= 0 \ .
\end{aligned} \tag{23c}$$

Thus, the invariance of the Klein-Gordon equation under the four parameter subgroup of space-time translation is proved in an unusual way. There is a quantum mechanical aspect to this proof. The Schrödinger representation of relativistic quantum mechanics is characterized by :

$$\overrightarrow{\boldsymbol{\psi}} := \psi\left(q^1, q^2, q^3, q^4\right) \equiv \psi\left(q^1, q^2, q^3\,; t\right) \;, \tag{24a}$$

$$\boldsymbol{P}_j \overrightarrow{\boldsymbol{\psi}} := -i\,\partial_{qj}\psi\left(q^1, q^2, q^3, q^4\right) \;, \tag{24b}$$

$$\boldsymbol{P}_4 \overrightarrow{\boldsymbol{\psi}} := i\,\partial_{q4}\psi\left(q^1, q^2, q^3, q^4\right) \;, \tag{24c}$$

$$\boldsymbol{Q}^\nu \overrightarrow{\boldsymbol{\psi}} := q^\nu \psi\left(q^1, q^2, q^3, q^4\right) = \eta^{\nu\mu} q_\mu \psi\left(q^1, q^2, q^3, q^4\right) \;. \tag{24d}$$

The equation (23b) can be expressed as

$$\widehat{\overrightarrow{\boldsymbol{\psi}}} = \exp\left[-ic^\mu \boldsymbol{P}_\mu\right] \overrightarrow{\boldsymbol{\psi}} := \boldsymbol{U}\left(c^1, c^2, c^3, c^4\right) \overrightarrow{\boldsymbol{\psi}}\,. \tag{25}$$

Here, $\boldsymbol{U}\left(c^1, c^2, c^3, c^4\right)$ is *a unitary transformation* involving four real parameters c^μ .

In relativistic quantum mechanics and relativistic quantum field theories[4–6] , the generalization of the equation (25) to the ten parameter Poincaré group $\mathcal{I}\,O(3,1)$ is furnished by :

$$\begin{aligned} \widehat{\overrightarrow{\boldsymbol{\psi}}} &= \boldsymbol{U}\left[c^\mu, l^\alpha{}_\beta\right] \cdot \overrightarrow{\boldsymbol{\psi}} \\ &:= \exp\left[-ic^\mu \boldsymbol{P}_\mu + \left(\frac{i}{4}\right) \omega^{\alpha\beta}\left(\boldsymbol{Q}_\alpha \boldsymbol{P}_\beta - \boldsymbol{Q}_\beta \boldsymbol{P}_\alpha + \boldsymbol{P}_\beta \boldsymbol{Q}_\alpha - \boldsymbol{P}_\alpha \boldsymbol{Q}_\beta\right)\right] \cdot \overrightarrow{\boldsymbol{\psi}} \;, \end{aligned} \tag{26a}$$

$$\omega^{\beta\alpha} = -\omega^{\alpha\beta} \;. \tag{26b}$$

The six parameters $\omega^{\alpha\beta}$ are related to parameters $l^\alpha{}_\beta$ of the equations (20a, 20b).

The Schrödinger type of covariance is characterized by :

$$\widehat{\boldsymbol{P}_\mu} = \boldsymbol{P}_\mu \;, \quad \widehat{\boldsymbol{Q}_\mu} = \boldsymbol{Q}_\mu \;, \tag{27a}$$

$$\widehat{\overrightarrow{\boldsymbol{\psi}}} = \boldsymbol{U}\left[c^\mu, l^\alpha{}_\beta\right] \cdot \overrightarrow{\boldsymbol{\psi}} \;. \tag{27b}$$

It is well known[15,19] that the operator $\eta^{\mu\nu}\boldsymbol{P}_\mu \boldsymbol{P}_\nu$, which is one of the Casimir operators of the Poincaré group $\underline{\mathcal{I}}\,O(3,1)$, commutes with all ten generators $\boldsymbol{P}_\mu$ and $[\boldsymbol{Q}_\alpha \boldsymbol{P}_\beta \boldsymbol{Q}_\beta \boldsymbol{P}_\alpha + \boldsymbol{P}_\beta \boldsymbol{Q}_\alpha - \boldsymbol{P}_\alpha \boldsymbol{Q}_\beta]$. Therefore, we obtain from (18a, 18b), (26a, 26b), and (27a, 27b) that

$$\begin{aligned} \left[\eta^{\mu\nu}\widehat{\boldsymbol{P}}_\mu \widehat{\boldsymbol{P}}_\nu + m^2 \boldsymbol{I}\right] &= \overrightarrow{\boldsymbol{\psi}}\left[\eta^{\mu\nu}\widehat{\boldsymbol{P}}_\mu \widehat{\boldsymbol{P}}_\nu + m^2 \boldsymbol{I}\right] \boldsymbol{U}[\ldots] \cdot \overrightarrow{\boldsymbol{\psi}} \\ &= \boldsymbol{U}[\ldots] \cdot \left[\eta^{\mu\nu}\widehat{\boldsymbol{P}}_\mu \widehat{\boldsymbol{P}}_\nu + m^2 \boldsymbol{I}\right] \overrightarrow{\boldsymbol{\psi}} = \overrightarrow{\boldsymbol{O}}\,. \end{aligned} \tag{28}$$

Therefore, the above Hilbert-vector equation demonstrates the exact proof for the invariance of the Klein-Gordon Hilbert-vector equations (18a, 18b).

Now, every representation of quantum mechanics satisfies every operator and Hilbert-vector equations in (18a, 18b) , (26a, 26b) , and (27a, 27b) . Thus, we can conclude that the transformed scalar field is given by :

$$\begin{aligned} \widehat{\phi}\,(\boldsymbol{n}\,; t) &= \boldsymbol{U}[\ldots]\,\phi\,(\boldsymbol{n}\,; t) \\ := \exp\Bigg[-c^j \Delta_j^{\#} + c^4 \partial_t &+ \left(\frac{1}{4}\right)\omega^{jk}\left(\Delta_j^{\circ}\Delta_k^{\#} - \Delta_k^{\circ}\Delta_j^{\#} + \Delta_k^{\#}\Delta_j^{\circ} - \Delta_j^{\#}\Delta_k^{\circ}\right) \\ &+ \omega^{j4}\left(t\Delta_j^{\#} - \Delta_j^{\circ}\partial_t\right)\Bigg]\,\phi\,(\boldsymbol{n}\,; t) \end{aligned} \tag{29}$$

The transformed function $\widehat{\phi}\,(\boldsymbol{n}\,; t)$ in (29) must satisfy the Klein-Gordon equation (19) , namely

$$\left[\delta^{jk}\Delta_j^{\#}\Delta_k^{\#} - (\partial_t)^2 - m^2\right]\widehat{\phi}\,(\boldsymbol{n}\,; t) = 0\,. \tag{30}$$

The above equation concludes *the proof for the exact relativistic invariance* of the finite difference-differential version of the Klein-Gordon equation as expressed in (19).

In the Schrödinger representation of quantum mechanics, the usual Klein-Gordon equation is given by :

$$\delta^{jk}\partial_{qj}\partial_{qk}\psi\left(q^1,q^2,q^3;t\right)-(\partial_t)^2\psi\left(q^1,q^2,q^3;t\right)$$
$$-m^2\psi\left(q^1,q^2,q^3;t\right)=0\,. \tag{31}$$

On the other hand, the mixed partial difference-differential version of the Klein-Gordon equation from the equation (19) is provided by :

$$\delta^{jk}\Delta_j^{\#}\Delta_k^{\#}\phi\left(n^1,n^2,n^3;t\right)-(\partial_t)^2\phi\left(n^1,n^2,n^3;t\right)$$
$$-m^2\phi\left(n^1,n^2,n^3;t\right)=0\,. \tag{32}$$

Now, we shall *compare and contrast* various Green's functions arising out of (31) and (32).

The relevant Green's functions of the Klein-Gordon equations (31) *in the continuous space-time* are expressed as one of the integral representations.[20]

$$\Delta_{(a)}\left(\boldsymbol{q},q^4;\widehat{\boldsymbol{q}},\widehat{q}^4;m\right):=\frac{1}{(2\pi)^4}\cdot\int_{\mathbb{R}^3}\left\{\int_{C_{(a)}}\frac{\exp\left[ip_\mu\left(q^\mu-\widehat{q}^\mu\right)\right]}{\left[\eta^{\alpha\beta}p_\alpha p_\beta+m^2\right]}\cdot dp^4\right\}\cdot dp_1dp_2dp_3\,. \tag{33}$$

Here, $q^4=t$, $p^4=-p_4$, and $C_{(a)}$ is a contour in the complex p^4-plane. The integrand in (33) has two simple poles on the real line at

$$p^4=\pm\omega:=\pm\sqrt{(p_1)^2+(p_2)^2+(p_3)^2+m^2}\,. \tag{34}$$

We shall restrict contour integration to the four contours C_+,C_-,C and $C_{(\mathbb{R})}$ as depicted in the fig. 4.

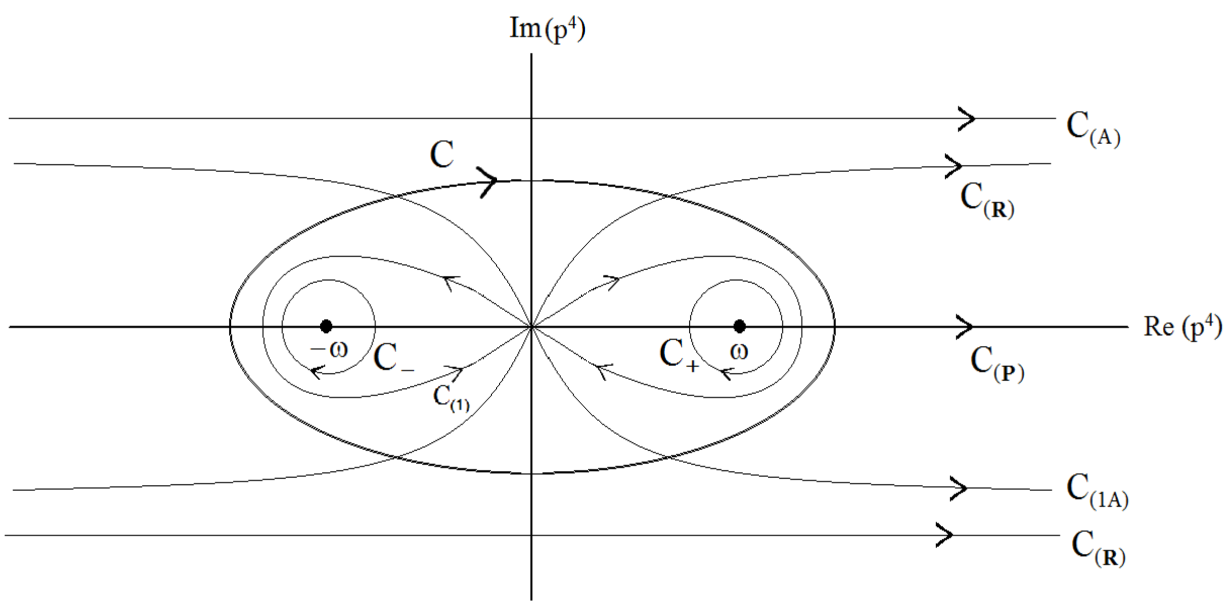

Figure-4 : The complex p^4-plane and contour $C_{(a)}$.

We define

$$\begin{aligned} s \quad &:= \quad -\eta_{\mu\nu}\left(q^\mu - \widehat{q}^\mu\right)\left(q^\nu - \widehat{q}^\nu\right) \\ &= \quad \left(q^4 - \widehat{q}^4\right)^2 - \delta_{jk}\left(q^j - \widehat{q}^j\right)\left(q^k - \widehat{q}^k\right) . \end{aligned} \tag{35}$$

Note that $s < 0$ for a spacelike separation and $s > 0$ for a timelike separation.

We also recall step functions by :

$$\theta(x) \;:=\; \begin{cases} 1 & \text{for } x > 0\,, \\ 0 & \text{for } x < 0\,. \end{cases} \tag{36a}$$

$$\varepsilon(x) \;:=\; \left(\frac{x}{|x|}\right) \quad \text{for } x \neq 0\,. \tag{36b}$$

Now, we shall provide explicitly four of the Green's functions (33) and contours exhibited in the fig. 4. Denoting the Dirac delta function by $\delta(s)$, the explicit expressions are furnished in the following[15,20] :

$$\begin{aligned} \Delta_+\left(\boldsymbol{q}, q^4; \boldsymbol{0}, 0; m\right) = {}& \frac{1}{4\pi}\varepsilon(q^4)\delta(s) - \frac{m}{8\pi}\frac{\varepsilon(q^4)\theta(s)}{\sqrt{s}} J_1\left(m\sqrt{(s)}\right) + \frac{im}{8\pi}\frac{\theta(s)}{\sqrt{s}} N_1\left(m\sqrt{(s)}\right) \\ & + \frac{im}{4\pi^2}\frac{\theta(-s)}{\sqrt{-s}} K_1\left(m\sqrt{(-s)}\right) , \end{aligned} \tag{37a}$$

$$\begin{aligned} \Delta_-\left(\boldsymbol{q}, q^4; \boldsymbol{0}, 0; m\right) = {}& \frac{1}{4\pi}\varepsilon(q^4)\delta(s) - \frac{m}{8\pi}\frac{\varepsilon(q^4)\theta(s)}{\sqrt{s}} J_1\left(m\sqrt{(s)}\right) - \frac{im}{8\pi}\frac{\theta(s)}{\sqrt{s}} N_1\left(m\sqrt{(s)}\right) \\ & - \frac{im}{4\pi^2}\frac{\theta(-s)}{\sqrt{-s}} K_1\left(m\sqrt{(-s)}\right) , \end{aligned} \tag{37b}$$

$$\Delta(\ldots) = \Delta_+(\ldots) + \Delta_-(\ldots) = \frac{1}{2\pi}\varepsilon(q^4)\delta(s) - \frac{m}{4\pi}\frac{\varepsilon(q^4)\theta(s)}{\sqrt{s}} J_1(m\sqrt{(s)}) , \tag{37c}$$

$$\begin{aligned} \Delta_{(\mathbb{R})}(\ldots) &= \theta(q^4)\Delta_+(\ldots) - \theta(-q^4)\Delta_-(\ldots) \\ &= \frac{1}{4\pi}\delta(s) - \frac{m}{8\pi}\frac{\theta(s)}{\sqrt{s}}\left[J_1\left(m\sqrt{(s)}\right) - iN_1\left(m\sqrt{(s)}\right)\right] + \frac{im}{4\pi^2}\frac{\theta(-s)}{\sqrt{-s}} K_1\left(m\sqrt{(-s)}\right) . \end{aligned} \tag{37d}$$

Here, $J_1(\cdots)$, $N_1(\cdots)$ and $K_1(\cdots)$ are various Bessel functions.[21,22] Every Greens function $\Delta_{(a)}(\ldots)$ has *singularity* on the light cone $s = 0$ and contributes to divergence difficulties of the S-matrix. (The Greens function $\Delta_{(\mathbb{R})}(\ldots) = \left(\frac{i}{2}\right)\Delta_{(\mathbb{F})}(\ldots)$ of the Feynman-Dyson notation.)

Now, we shall investigate the corresponding Greens functions of the finite difference-differential version of the Klein-Gordon equation (31,32). The required Greens functions[5] are furnished by the improper integrals :

$$\begin{aligned} \Delta^{\#}_{(a)}\left(\boldsymbol{n}, t; \widehat{\boldsymbol{n}}, \widehat{t}; m\right) := {}& \frac{1}{(2\pi)}\int_{\mathbb{R}^3}\left\{\left[\prod_{j=1}^{3}\xi_{n^j}(p_j)\cdot\overline{\xi_{\widehat{n}^j}(p_j)}\right]\cdot\left[\int_{C_{(a)}}\frac{\exp\left[-ip^4(t-\hat{t})\right]}{\left[\delta^{kl}p_kp_l - (p^4)^2 + m^2\right]}dp^4\right]\right\} \\ & dp_1dp_2dp_3 , \end{aligned} \tag{38a}$$

$$\xi_{n^j}(p_j) := (i)^{n^j}\cdot f_{n^j}(p_j) = \frac{(i)^{n^j}\cdot e^{-(p_j/2)}\cdot H_{n^j}(p_j)}{\pi^{1/4}\cdot 2^{n^j/2}\cdot\sqrt{(n^j)!}} , \tag{38b}$$

Here, $H_{n^j}(p_j)$ are Hermite polynomials as mentioned in the equation (10a). The contours $C_{(a)}$ are identical to those given in the fig. 4. We introduce spherical polar coordinates by

$$p_1 = p\sin\theta\cos\phi\ , \quad p_2 = p\sin\theta\sin\phi\ , \quad p_3 = p\cos\theta\,. \tag{39}$$

Using the above equation (39) , we obtain from (38a, 38b) ,

$$\Delta^{\#}_{(a)}\left(\boldsymbol{n}, t; \widehat{\boldsymbol{n}}, \widehat{t}; m\right) := \frac{(i)^{n^1+n^2+n^3}}{(2\pi)\cdot\pi^{3/2}\cdot 2^{(n^1+n^2+n^3)/2}\cdot\sqrt{(n^1)!(n^2)!(n^3)!}}\cdot$$

$$\cdot\int_0^{\infty}\int_0^{\pi}\int_{-\pi}^{\pi}\left\{\left[e^{-p^2}\cdot H_{n^1}(p\sin\theta\cos\phi)\cdot H_{n^2}(p\sin\theta\sin\phi)\cdot H_{n^3}(p\cos\theta)\right]\right.$$

$$\left.\cdot\left[\int_{C_{(a)}}\frac{\exp\left[-ip^4 t\right]}{\left[p^2-(p^4)^2+m^2\right]}dp^4\right]\right\}p^2\sin\theta dp d\theta d\phi\ . \tag{40}$$

There exist nine distinct contours $C_{(a)}$ in the fig. 4. In case Green's function $\Delta^{\#}_{+}(\ldots)$ and $\Delta^{\#}_{-}(\ldots)$ are investigated, the seven other Green's functions out of $\Delta^{\#}_{(a)}(\ldots)$ can be dealt with linear combinations[20] of $\Delta^{\#}_{+}(\ldots)$ and $\Delta^{\#}_{-}(\ldots)$. Therefore, we carry out the contour integration C_+ and C_- from the equation (40). In that case, we derive that

$$\Delta^{\#}_{+}\left(\boldsymbol{n}, t; \boldsymbol{0}, 0; m\right) = \frac{(i)^{n^1+n^2+n^3+1}}{2\pi^{3/2}\cdot 2^{(n^1+n^2+n^3)/2}\cdot\sqrt{(n^1)!(n^2)!(n^3)!}}\cdot$$

$$\int_0^{\infty}\int_0^{\pi}\int_{-\pi}^{\pi}\left\{e^{-p^2}\cdot H_{n^1}(\cdots)\cdot H_{n^2}(\cdots)\cdot H_{n^3}(\cdots)\cdot\left[\frac{e^{-i\omega t}}{\omega}\right]\right\}p^2\sin\theta\, dp\, d\theta\, d\phi\ , \tag{41a}$$

$$\Delta^{\#}_{-}\left(\boldsymbol{n}, t; \boldsymbol{0}, 0; m\right) = \frac{(i)^{n^1+n^2+n^3-1}}{2\pi^{3/2}\cdot 2^{(n^1+n^2+n^3)/2}\cdot\sqrt{(n^1)!(n^2)!(n^3)!}}\cdot$$

$$\int_0^{\infty}\int_0^{\pi}\int_{-\pi}^{\pi}\left\{e^{-p^2}\cdot H_{n^1}(\cdots)\cdot H_{n^2}(\cdots)\cdot H_{n^3}(\cdots)\cdot\left[\frac{e^{i\omega t}}{\omega}\right]\right\}p^2\sin\theta\, dp\, d\theta\, d\phi\ . \tag{41b}$$

Therefore, we deduce that

$$\lim_{t\to 0}\left[\Delta^{\#}(\cdots)\right] = \lim_{t\to 0}\left[\Delta^{\#}_{+}(\cdots)+\Delta^{\#}_{-}(\cdots)\right]$$

$$= \lim_{t\to 0}\left\{\cdots\int_0^{\infty}\int_0^{\pi}\int_{-\pi}^{\pi}\left\{\cdots\left[\frac{\sin\omega t}{\omega}\right]\right\}p^2\sin\theta\, dp\, d\theta\, d\phi\right\} = 0\ . \tag{42}$$

Thus, in the second quantization[5] of a scalar field $\phi(\boldsymbol{n})$, the semblance of microcausality is still preserved !

Now, we shall investigate the convergence of improper integrals contained in the equation (40) defining Green's functions. The task is considerably simpler if we set the constant $m = 0$. Thus, we obtain from (41a, 41b) the following :

$$\Delta^{\#}_{\pm}\left(\boldsymbol{n}, t; \boldsymbol{0}, 0; 0\right) = \frac{(i)^{n^1+n^2+n^3\pm 1}}{2\pi^{3/2}\cdot 2^{(n^1+n^2+n^3)/2}\cdot\sqrt{(n^1)!(n^2)!(n^3)!}}\cdot$$

$$\int_0^{\infty}\int_0^{\pi}\int_{-\pi}^{\pi}\left\{e^{-p^2}\cdot H_{n^1}(p\sin\theta\cos\phi)\cdot H_{n^2}(p\sin\theta\sin\phi)\cdot H_{n^3}(p\cos\theta)\cdot\left[e^{\mp ipt}\right]\right\}$$

$$\cdot p\sin\theta\, dp\, d\theta\, d\phi\ . \tag{43}$$

Now, we consider the two dimensional integral :

$$I_{(0)} := \int_0^{\pi}\int_{-\pi}^{\pi}\left\{e^{-p^2}\cdot p\cdot H_{n^1}(p\sin\theta\cos\phi)\cdot H_{n^2}(p\sin\theta\sin\phi)\cdot H_{n^3}(p\cos\theta)\right.$$

$$\left.\cdot[\cos pt]\right\}\sin\theta\, d\theta\, d\phi\ . \tag{44}$$

By the mean value theorem of integration[23] , there exists a point (θ_0, ϕ_0) such that

$$I_{(0)} = (2\pi^2) \cdot e^{-p^2} \cdot p \cdot [\cos pt] \cdot H_{n^1}(p \sin\theta_0 \cos\phi_0) \cdot H_{n^2}(p \sin\theta_0 \sin\phi_0) \cdot H_{n^3}(p \cos\theta_0)\, \sin\theta_0 \ . \tag{45}$$

Similarly, the integral

$$\begin{aligned} I_{(1)} = \int_0^\pi \int_{-\pi}^\pi & \left\{ e^{-p^2} \cdot p \cdot H_{n^1}(p \sin\theta \cos\phi) \cdot H_{n^2}(p \sin\theta \sin\phi) \cdot H_{n^3}(p \cos\theta) \cdot [\sin pt] \right\} \\ & \cdot \sin\theta \, d\theta \, d\phi \\ = (2\pi^2) \cdot e^{-p^2} \cdot p \cdot [\sin pt] & \cdot H_{n^1}(p \sin\theta_1 \cos\phi_1) \cdot H_{n^2}(p \sin\theta_1 \sin\phi_1) \cdot H_{n^3}(p \cos\theta_1) \cdot \sin\theta_1 \ . \end{aligned} \tag{46}$$

Therefore, improper integrals

$$\int_0^\infty \int_0^\pi \int_{-\pi}^\pi \left\{ e^{-p^2} \cdot H_{n^1}(p \sin\theta \cos\phi) \cdot H_{n^2}(p \sin\theta \sin\phi) \cdot H_{n^3}(p\cos\theta) \cdot \left[e^{\mp ipt} \right] \right\}$$

$$p \sin\theta \, dp \, d\theta \, d\phi$$

$$= (2\pi^2) \int_0^\infty \left\{ \cdot e^{-p^2} \cdot p \cdot [\cos pt] \cdot H_{n^1}(p \sin\theta_0 \cos\phi_0) \cdot H_{n^2}(p \sin\theta_0 \sin\phi_0) \cdot H_{n^3}(p \cos\theta_0) \right.$$

$$\left. \sin\theta_0 \right\} dp$$

$$\mp i(2\pi^2) \cdot \int_0^\infty \left\{ e^{-p^2} \cdot p \cdot [\sin pt] \cdot H_{n^1}(p \sin\theta_1 \cos\phi_1) \cdot H_{n^2}(p \sin\theta_1 \sin\phi_1) \cdot H_{n^3}(p \cos\theta_1) \right.$$

$$\left. \cdot \sin\theta_1 \right\} dp \ . \tag{47}$$

Since $H_{n^j}(\cancel{edots})$ are *polynomial functions*, the improper integrals in (47) *converge*.

Therefore, from the equation (43), Green's functions $\Delta^{\#}_{\pm}(\boldsymbol{n}, t; \mathbf{0}, 0; 0)$ are *non-singular*. By the linear combinations[20] of $\Delta^{\#}_{+}(\cdots)$ and $\Delta^{\#}_{-}(\cdots)$, other *seven* Green's functions obtainable from the fig. 4 are also *non-singular*.

Divergence-free Green's functions are necessary (but not sufficient) to remove divergence difficulties of the S-matrix theory. Thus, non-singular Green's functions in (38a, 38b) are obviously important.[5,6]

Now we evaluate explicitly some important Green's functions in the equation (40) at *the coincident points*. These are provided by

$$\Delta^{\#}_{+}(\mathbf{0}, 0; \mathbf{0}, 0; 0) = \left(\frac{i}{\sqrt{\pi}} \right) \ , \tag{48a}$$

$$\Delta^{\#}_{-}(\mathbf{0}, 0; \mathbf{0}, 0; 0) = -\left(\frac{i}{\sqrt{\pi}} \right) \ , \tag{48b}$$

$$\Delta(\mathbf{0}, 0; \mathbf{0}, 0; 0) = 0 \ , \tag{48c}$$

$$\lim_{t \to 0_+} \left[\Delta^{\#}_{\mathbb{R}}(\mathbf{0}, t; \mathbf{0}, 0; 0) \right] = \left(\frac{i}{\sqrt{\pi}} \right) \ . \tag{48d}$$

§ Appendix : Peano curves and degenerate string-like phase cells

The purpose of this appendix is to elaborate the meaning of circular orbits in fig. 1 as *degenerate phase cells* and also one possible random movement of a particle inside such a cell.

Consider a parametrized curve f_1 into a plane as depicted in the fig. A1.

Here, f_1 represents a continuous, piecewise linear curve defined over nine closed intervals $\left[\frac{j-1}{9}, \frac{j}{9}\right]$ of $\mathbb{R}$, with $j \in \{1, 2, \ldots, 9\}$. The image of the function f_1 is exhibited in the continuous, piecewise zigzag oriented curve inside a square of unit area of x-y plane.

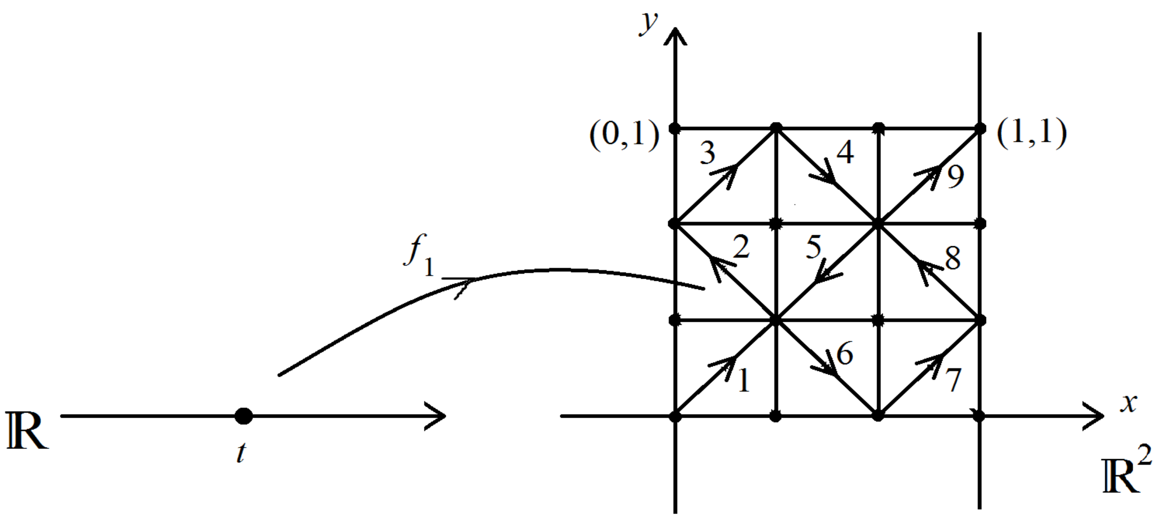

Figure-A1: The graph of the curve f_1 .

The continuous, piecewise linear parametrized curve f_2 has $9^2 = 81$ linear segments as shown in the fig. A2 below.

The continuous, piecewise linear parametrized curve f_n has 9^n oriented line segments. the sequence of functions $\{f_n\}_1^\infty$ possesses the limiting function $f := \lim\limits_{n\to\infty} f_n$. It can be rigorously proved that the graph of *the limiting function* f *fully covers*[8] *the area of the square* $\overline{D}$ with Area$\,(\overline{D}) = 1$. Such an example of f constitutes an example for *Peano curves.*[8]

Now, we define a sequence of functions $\{h_1, h_2, \ldots, h_M, \ldots\}$ from the domain $\overline{D}$ into the sequence of closed co-domains $\{\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_M, \ldots\}$ such that each of $\overline{D}_M$ is a subset inside $\mathbb{R}^2$. (Consult the fig. A3 .)

The linear transformation h_M is explicitly specified by :

$$\rho = \left(\frac{1}{2M\pi}\right) x + \left(\frac{1}{2}\right) \ , \tag{49a}$$

$$\phi = (2M\pi)y - M\pi \ ; \quad M \in \{1, 2, \ldots\} \ . \tag{49b}$$

The Jacobian of each of the transformations h_M is furnished by :

$$\frac{\partial(\rho, \phi)}{\partial(x, y)} \equiv 1. \tag{50}$$

Therefore, the area of $\overline{D}_M$ is provided by the double integral :

$$\text{Area}\,(\overline{D}_M) = \int_{1/2}^{1/2+1/2M\pi} \int_{-M\pi}^{M\pi} d\rho d\phi \equiv 1\,. \tag{51}$$

We can physically interpret both the x-y plane $\mathbb{R}^2$ and ρ-ϕ plane $\mathbb{R}^2$ as two dimensional phase planes.[13] Thus, the closed regions $\overline{D}$ and $\overline{D}_M$ can both be physically interpreted as phase cells. Each of $\overline{D}$ and $\overline{D}_M$ is endowed with area Area$(\overline{D}_M)$ =Area$(\overline{D})$ $= 1$ permitted by *the uncertainty principle* $|\Delta x \cdot \Delta y| = |\Delta\rho \cdot \Delta\phi| = 1$. Moreover, the mapping h_M is *a canonical mapping* of the Hamiltonian mechanics[13] and quantum mechanics. In the limiting case $\lim\limits_{M\to\infty}$ Area$(\overline{D}_M) = 1$. In the same limiting case, the sequence of closed co-domains $\{\overline{D}_M\}_1^\infty$ collapses into *the infinite straight line* given by $\rho = \frac{1}{2}$

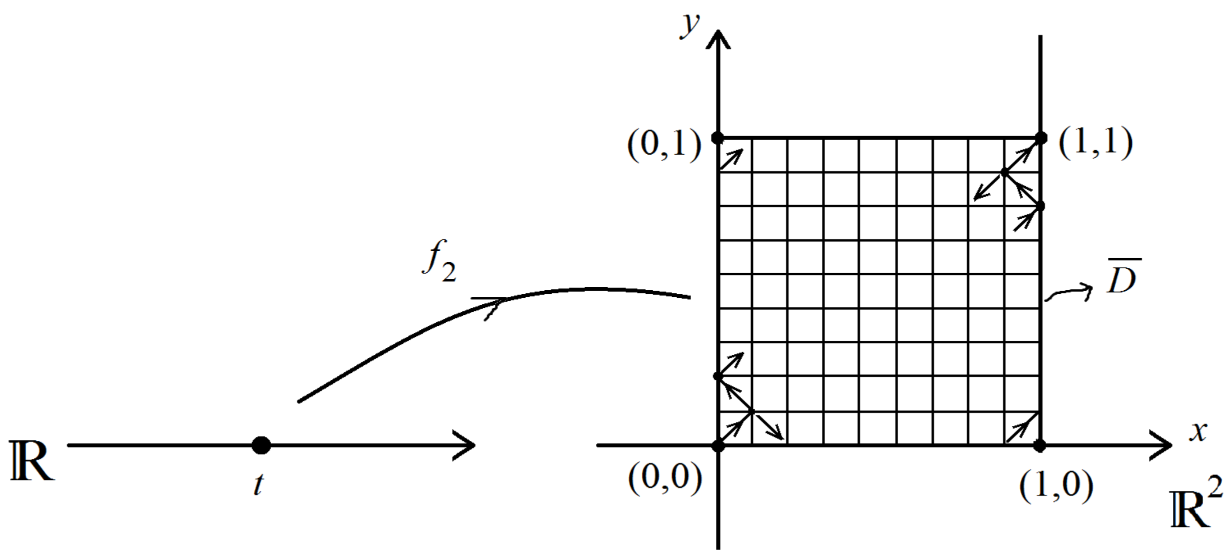

Figure-A2 : The graph of the curve f_2 .

and $\phi \in (-\infty, \infty)$. Thus, the limiting infinite straight line (with unit area) in the ρ-ϕ phase plane represents an *open* string-like phase cell.

Now, we shall introduce another canonical transformation g_M from the phase space region $\overline{D}_M$ into the *annular* phase space region $\overline{A}_M$ as depicted in the following fig. A4 .

The canonical transformation g_M is furnished by :

$$q = \sqrt{2\rho}\cos\phi \ , \tag{52a}$$

$$p = \sqrt{2\rho}\sin\phi \ , \tag{52b}$$

$$\frac{\partial(q,p)}{\partial(\rho,\phi)} \equiv 1 \ , \tag{52c}$$

$$\text{Area}\,(\overline{A}_M) \equiv 1 \ . \tag{52d}$$

In the limiting case of $M \to \infty$, the outer circular boundary of the annular region $\overline{A}_M$ collapses into the inner circular boundary of the unit radius. However, in this limiting process, the unit area of $\overline{A}_M$ is *still preserved* by the equation (52d) . This collapsed inner circle of unit area, possessing infinite winding number, is now identified with the smallest of *closed, circular string-like phase cells* depicted in the fig. 1.

In case of a closed, circular phase cell of radius $\sqrt{2N+1}$ in the fig. 1, the function $g_M^{(N)}$ and the closed domain $\overline{D}_M^{(N)}$ have to be defined as follows :

$$\overline{D}_M^{(N)} := \left\{ (\rho,\phi) \ : \ N + \frac{1}{2} \leq \rho \leq N + \frac{1}{2} + \frac{1}{2M\pi} \ , \ \ -M\pi \leq \phi \leq M\pi \right\} \ . \tag{53}$$

The mapping $g_M^{(N)}$ is exactly the same as given in (52a, 52b, 52c) . The corresponding closed co-domain $\overline{A}_M^{(N)}$ is *an annular region* in the *q*-*p* phase plane $\mathbb{R}^2$.

Now, we shall discuss the physical meaning of a Peano curve exemplified in fig. A1, fig. A2, and fig. A4 . In fig. A1, fig. A2, and fig. A3, the region $\overline{D}$ of unit area is interpreted as a phase cell inside

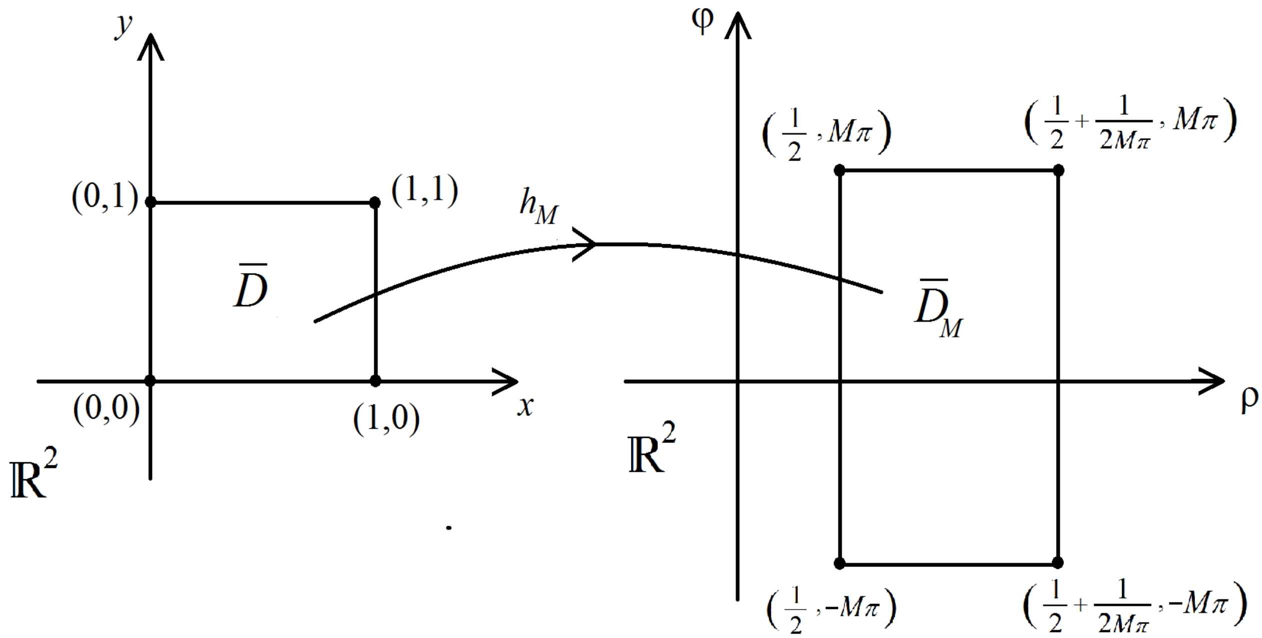

Figure-A3 : The graph of the function h_M .

the x-y phase plane $\mathbb{R}^2$. Graphs of the mapping $\{f_n\}_1^\infty$ yield continuous zig-zag tracks of a particle *hidden* from external observations. Specially, the graph of the limiting mapping $f := \lim_{n\to\infty} f_n$ *covers completely* the phase cell $\overline{D}$. Therefore, the graph of the mapping $g_M^{(N)} \circ h_M^{(N)} \circ f$ from $\mathbb{R}$ into $\mathbb{R}^2$ is a continuous zig-zag curve completely covering the annular region $\overline{A}_M^{(N)}$ in the q-p phase plane. This Peano curve represents a possible particle trajectory inside a phase cell of unit area. Moreover, in the limit $M \to \infty$, the annular region $\overline{A}_M^{(N)}$, containing the Peano curve,[8] *completely collapses* to the circle of radius $\sqrt{2N+1}$ as shown in the fig. 1.

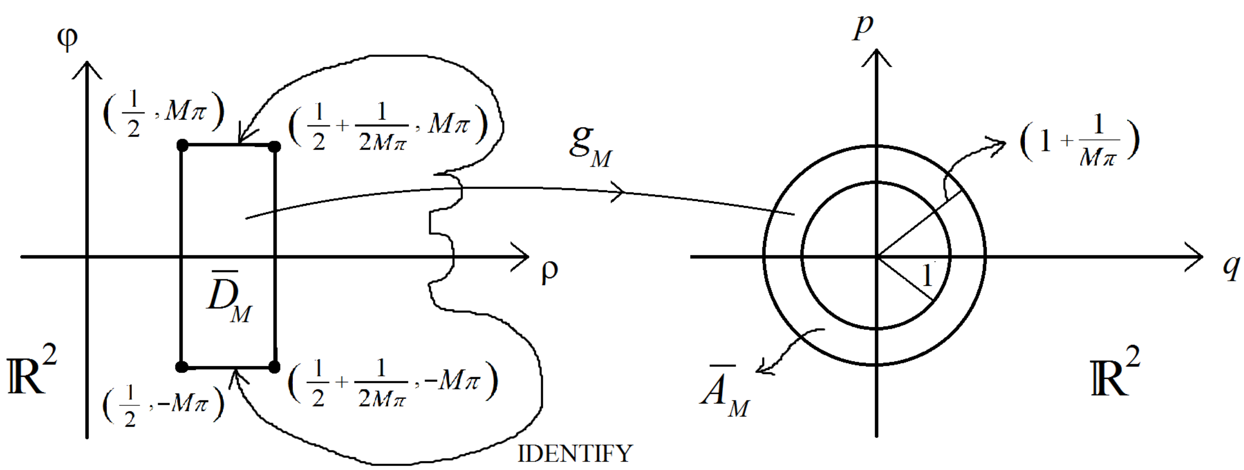

Figure-A4 : The canonical transformation g_M .

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