Original Research Article

Solutions of Schrödinger and Klein-Gordon Equations with Hulthen plus Inversely Quadratic Exponential Mie-Type Potential

ABSTRACT

We have proposed a novel potential called Hulthen plus Inversely Quadratic Exponential Mie-Type potential (HIQEMP). The parametric Nikiforov-Uvarov method has been employed to study approximate solutions of Schrödinger and Klein-Gordon equations with our novel potential. We obtain bound state energies and the normalized wave function expressed in terms of Jacobi polynomial. The proposed potential is applicable in the field of vibrational and rotational spectroscopy. To ascertain the accuracy of our results, we apply the nonrelativistic limit to the Klein-Gordon equation to obtain the energy equation which is exactly the same to that obtain in Schrodinger equation. This is a proof that relativistic equation can be converted to nonrelativistic equation using a nonrelativistic limit with Greene-Aldrich approximation to the centrifugal term. The wave functions were normalized analytically using two infinite series of confluent hypergeometric functions. We implement MATLAB algorithm to obtain the numerical bound state energy eigenvalues for both Schrödinger and Klein-Gordon equations. Our potential reduces to many existing potentials and the result is in agreement with existing literature. The energy spectral diagrams were plotted using origin software. The energies from Schrodinger equation decreases with increase in quantum state while that of Klein-Gordon equation increases with increase in quantum state.

Keywords: Schrodinger equation, Klein-Gordon equation Nikiforov-Uvarov method, novel potential (HIQEMP).

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1. INTRODUCTION

The molecular, vibrational and rotational spectroscopy is one of the most recent research field that has practical applications in physical sciences especially in studying diatomic molecular interactions [1-7]. Bound state solutions of relativistic and non-relativistic wave equation arouse a lot of interest for decades. Schrodinger wave equations constitute non-relativistic wave equation while Klein-Gordon and Dirac equations constitute the relativistic wave equations. [8-12] Hulthen potential is one of the significant exponential potentials which behave like Coulomb potential [13]. This potential has a lot of applications in many branches of Physics specifically in atomic, solid state, chemical and Nuclear Physics. [14-17]. Mie-Type potential which belongs to a class of multi –parameter exponential potential has application in vibrational and rotational spectroscopy in physical sciences because its interaction model comprises of both repulsive and attractive terms for short and large intermolecular distances respectively for some diatomic molecules. [18]. The Klein-Gordon equation is the relativistic version of Schrodinger equation which describes spinless particles. This equation has attracted much attention in investigating the interaction of solitons in a collisionless plasma. [19-20]. The proposed novel potential is used in studying bound state energies of both Schrodinger and Klein-Gordon equations. Other potentials have been used to obtain bound state solutions suchs as Multi-parameter exponential type potential , Quantum interaction potential, Hulthen, Poschl-Teller, Eckart, Coulomb, Hyllearraas, Pseudoharmonic, Scarf II potentials and many others [21-28]. These potentials have been studied and investigated with some specific methods and techniques such as: Asymptotic iteration method, Nikiforov-Uvarov method, Supersymmetric quantum mechanics approach, formular method, exact quantisation and many more [29-40]. This article is divided into six sections. Section 1 is the introduction, section 2 is the brief introduction of parametric I Nikiforov-Uvarov method. In section 3, we present the radial solution to Schrodinger wave equation using the proposed potential and obtained both the energy eigenvalue and their corresponding normalized wave function. In section 4, we present the solution to one dimensional Klein-Gordon equation using the proposed potential and also present some deductions from the proposed potential and compare the result to that of existing literature. In section 5, we present analytical solution on normalizing the wave function using confluent hypergeometric function. Results and discussion of this work are presented in section 6. Section 7 gives the general conclusions to the article.

The proposed potential is given as

$$V(r)=-\left(V_0+\frac{V_1\chi_1}{\chi_2}\right)\frac{e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)}+\frac{A}{r^2}+\frac{\left(B-\eta\right)e^{-\alpha r}}{r}+C \qquad (1)$$

Equation (1) for the sake of clarity can be expressed as

$$V(r)=-\frac{V_0e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)}-\frac{V_1}{\chi_2}\left(\frac{\chi_1e^{-2\alpha r}}{1-e^{-2\alpha r}}\right)+\frac{A}{r^2}+\frac{\left(B-\eta\right)e^{-\alpha r}}{r}+C \qquad (2)$$

Where V_0 is the potential depth, α is the adjustable parameter known as the screening parameter. V_1, χ_1, χ_2 A, B C and η are all real constants. The variations of the HIQEMP with small and large values of alpha α (screening parameter) are presented in Figure 1 and Figure 2 respectively.

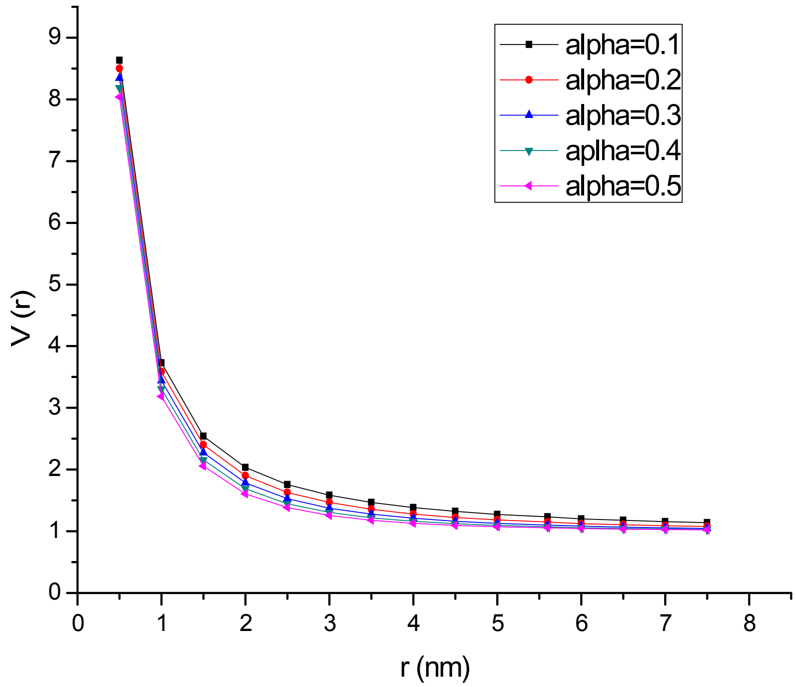

Figure 1: HIQEMP versus small values of α (screening parameter)

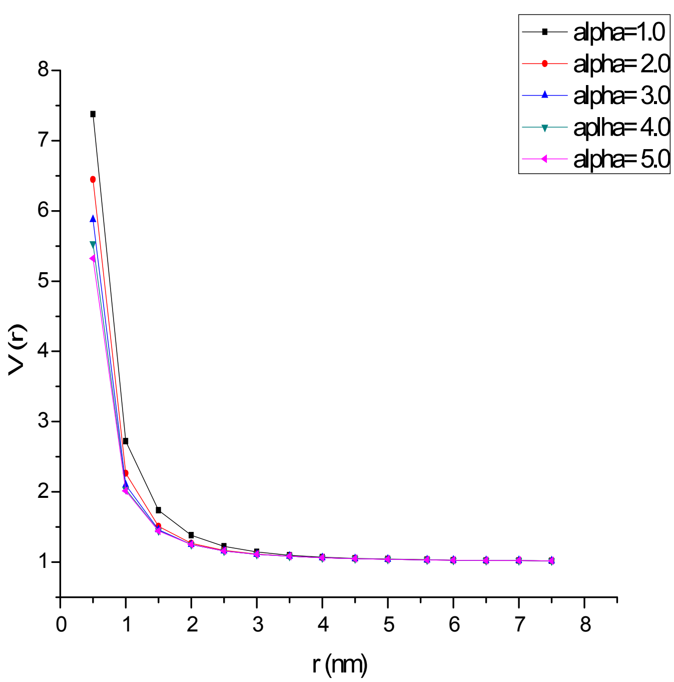

Figure 2: HIQEMP versus large values α (screening parameter).

2. NIKIFOROV-UVAROV METHOD: PARAMETRIC FORMULATION

The NU method is based on reducing second order linear differential equation to a generalized equation of hyper-geometric type [31-32] .This method provides exact solutions in terms of special orthogonal functions as well as corresponding energy eigen values. The NU method is applicable to both relativistic and non-relativistic equations. With appropriate coordinate transformation $S=S(x)$ the equation can be written as

$$\psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \tag{3}$$

where $\tilde{\tau}(s)$ is a polynomial of degree one while $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials of at most degree two.

The parametric formalization of NU is applicable and valid for both central and noncentral potential. Here the hypergeometric differential equation is given by

$$\Psi''(s)+\frac{c_1-c_2s}{s(1-c_3s)}\Psi'(s)+\frac{1}{s^2(1-c_3s)^2}\left[-\xi_1s^2+\xi_2s-\xi_3\right]\Psi(s)=o \tag{4}$$

Comparing equation (4) to (3) the following parametric polynomials can be obtain

$$\tilde{\tau}(s)=(c_1-c_2s) \tag{5}$$

$$\tilde{\sigma}(s)=-\xi_1s^2+\xi_2s-\xi_3 \tag{6}$$

$$\sigma(s)=s(1-c_3s) \tag{7}$$

Equation of the function $\pi(s)$ is given as

$$\pi(s)=c_4-c_5s\pm\sqrt{\left[\left(c_6-c_3k_\pm\right)s^2\left(c_7+k_\pm\right)s+c_8\right]} \tag{8}$$

Where
$$\begin{bmatrix} c_4=\frac{1}{2}(1-c_1), \\ c_5=\frac{1}{2}(c_2-2c_3) \\ c_6=c_5^2+\xi_1 \\ c_7=2c_4c_5-\xi_2 \\ c_8=\ c_4^2+\xi_3 \end{bmatrix} \tag{9}$$

From the condition that the function under the square root should be the square of polynomial, that is the discriminant $(b^2-4ac)=o$ then the parametric becomes

$$k_{\pm} = -(c_7 + 2c_3c_8) \pm 2\sqrt{c_8c_9} \qquad (10)$$

where

$$c_9 = c_3c_7 + c_3^2c_8 + c_6 \qquad (11)$$

The negative value of the parametric is obtained as

$$k_{-} = -(c_7 + 2c_3c_8) - 2\sqrt{c_8c_9} \qquad (12)$$

Then, the polynomial becomes

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - 2\left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)s - \sqrt{c_8}\right] \qquad (13)$$

For bound state condition to be satisfied, then the derivative of equation (13) will be negative. That is

$$\tau'(s) = -2c_3 - 2\left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)\right] < 0 \qquad (14)$$

The energy equation is given by

$$c_2n - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) + n(n-1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \qquad (15)$$

The weight function is obtained as

$$\rho(s) = s^{c_{10}}(1 - c_3s)^{c_{11}} \qquad (16)$$

with Rodrigue relation in equation, one part of the wave function can be obtain as

$$\chi_n(s) = P_n^{(c_{10},c_{11})}(1 - 2c_3s) \qquad (17)$$

where

$$\begin{aligned} c_9 &= c_3c_7 + c_3^2c_8 + c_6 \\ c_{10} &= c_1 + 2c_4 + 2\sqrt{c_8} \\ c_{11} &= c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}) \end{aligned} \qquad (18)$$

And $P_n^{(c_{10},c_{11})}$ is the Jacobi polynomial which in most cases reduces to Lagguerre polynomial for $c_3 = 0$.

The other part of the wave function is given as

$$\phi(s) = s^{c_{12}}(1-c_3 s)^{c_{13}} \tag{19}$$

where

$$\begin{bmatrix} c_{12} = c_4 + \sqrt{c_8} \\ c_{13} = c_5 - \left(\sqrt{c_9} + c_3\sqrt{c_8}\right) \end{bmatrix} \tag{20}$$

The total wave function is the given by

$$\Psi(s) = \phi(s)\chi_n(s) = N_n s^{c_{12}}(1-c_3 s)^{c_{13}} P_n^{(c_{10},c_{11})}(1-2c_3 s)$$

(21)

3. RADIAL SOLUTIONS OF SCHRODINGER EQUATION

One dimensional radial Schrodinger wave equation is given as

$$\frac{d^2\psi(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{\hbar^2 l(l+1)}{2\mu r^2}\right]\psi(r) = 0 \tag{22}$$

Substituting equation (2) into (22) gives

$$\frac{d^2\psi(r)}{dr^2} + \frac{2\mu}{\hbar^2}\left[E + \frac{V_0 e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)} + \frac{V_1}{\chi_2}\left(\frac{\chi_1 e^{-2\alpha r}}{1-e^{-2\alpha r}}\right) - \frac{A}{r^2} - \frac{(B-\eta)e^{-\alpha r}}{r} - C - \frac{\hbar^2 l(l+1)}{2\mu r^2}\right]\psi(r) = 0 \tag{23}$$

Equation (23) can only be solved analytically to obtain exact solution if the angular orbital momentum number $l = 0$. However, for $l > 0$ equation (23) can only be solve by using some approximations to the centrifugal term. Greene Aldrich approximation is best suitable for equation (23).

Let's define Greene Aldrich approximation as[39]

$$\frac{1}{r^2} = \frac{4\alpha^2 e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2} \Rightarrow \frac{1}{r} = \frac{2\alpha e^{-\alpha r}}{\left(1-e^{-2\alpha r}\right)} \tag{24}$$

Substituting equation (24) into equation (23) with the transformation $s = e^{-2\alpha r}$ gives

$$\frac{d^2\psi(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{d\psi(s)}{ds}+\frac{1}{s^2(1-s)^2}\left[\begin{array}{l}\frac{2\mu E}{4\alpha^2\hbar^2}(1-s)^2+\frac{2\mu V_0}{4\alpha^2\hbar^2}(s-s^2)+\frac{2\mu V_1\sigma_1}{4\alpha^2\hbar^2}s(1-s)-\frac{2A\mu s}{\hbar^2}\\ -\frac{\mu(\beta-\eta)s}{\alpha\hbar^2}+\frac{\mu(\beta-\eta)s^2}{\alpha\hbar^2}-\frac{2\mu c}{4\alpha^2\hbar^2}(1-s)^2-\delta l(l+1)\end{array}\right]\psi(s)=0 \tag{25}$$

Equation (25) can further be reduced to

$$\frac{d^2\psi(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{d\psi(s)}{ds}+\frac{1}{s^2(1-s)^2}\left[\begin{array}{l}-(\varepsilon^2+\delta^2+\gamma_1\sigma_1-\gamma_4+\gamma_5)s^2\\ +(2\varepsilon^2+\delta^2+\gamma_1\sigma_1-\gamma_2-\gamma_3+2\gamma_5-l(l+1))s\\ -(\varepsilon^2+\gamma_5)\end{array}\right]\psi(s)=0$$

(26)

where

$$\varepsilon^2=-\frac{2\mu E}{4\hbar^2\alpha^2},\ \delta^2=\frac{2\mu V_0}{4\hbar^2\alpha^2},\sigma_1=\frac{\chi_1}{\chi_2},$$

$$\gamma_1=\frac{2\mu v_1}{4\alpha^2\hbar^2},\gamma_2=\frac{2A\mu}{\hbar^2},\gamma_3=\gamma_4=\frac{\mu(\beta-\eta)}{\alpha\hbar^2},\gamma_5=\frac{2\mu c}{4\alpha^2\hbar^2},\sigma_3=\frac{\mu c}{2\hbar^2\alpha^2}$$

(27)

Comparing equation (27) to the parametric NU equation (4) then gives

$$\begin{aligned}\xi_1&=\varepsilon^2+\delta^2+\gamma_1\sigma_1-\gamma_4+\gamma_5\\ \xi_2&=2\varepsilon^2+\delta^2+\gamma_1\sigma_1-\gamma_2-\gamma_3+2\gamma_5-l(l+1)\\ \xi_3&=\varepsilon^2+\gamma_5\end{aligned} \tag{28}$$

The parametric coefficient can be obtain as follows

$$\begin{aligned}&c_1=c_2=c_3=1\\ &c_4=0,c_5=-\frac{1}{2},\quad c_6=\frac{1}{4}+\varepsilon^2+\delta^2+\gamma_1\sigma_1-\gamma_4+\gamma_5\\ &c_7=-2\varepsilon^2-\delta^2-\gamma_1\sigma_1+\gamma_2+\gamma_3-2\gamma_5+l(l+1),\quad c_8=\varepsilon^2+\gamma_5\\ &c_9=\frac{1}{4}+\gamma_2+l(l+1),\quad c_{10}=1+2\sqrt{\varepsilon^2+\gamma_5}\\ &c_{11}=2+\sqrt{1+4\gamma_2+4l(l+1)}+2\sqrt{\varepsilon^2+\gamma_5},\quad c_{12}=\sqrt{\varepsilon^2+\gamma_5}\\ &c_{13}=-\frac{1}{2}-\frac{1}{2}\sqrt{1+4\gamma_2+4l(l+1)}\end{aligned} \tag{29}$$

Using equation (15), the energy eigenvalue equation can be calculated analytically with simple mathematical algebraic simplification as

$$\varepsilon^2 = \left\{ \frac{\left(n^2+n+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)\sqrt{1+4\gamma_2+4l(l+1)}+\gamma_2+\gamma_3-\delta^2-\gamma_1\sigma_1+l(l+1)}{(2n+1)+\sqrt{1+4\gamma_2+4l(l+1)}} \right\}^2 - \gamma_5$$

(30)

Substituting parameters of equation (27) into (30) gives the energy eigenvalue equation as

$$E_{nl} = -\frac{2\hbar^2\alpha^2}{\mu}\left\{ \frac{\left(n^2+n+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)\sqrt{1+\frac{8A\mu}{\hbar^2}+4l(l+1)}+\frac{2A\mu}{\hbar^2}+\frac{\mu(B-\eta)}{\alpha\hbar^2} -\frac{\mu V_0}{2\alpha^2\hbar^2}-\frac{\mu V_1}{2\alpha^2\hbar^2}\frac{\chi_1}{\chi_2}+l(l+1)}{(2n+1)+\sqrt{1+\frac{8A\mu}{\hbar^2}+4l(l+1)}} \right\}^2 + c$$

(31)

The wave function expressed in terms of Jacobi polynomial is calculated using equation (21). Hence

$$\Psi(s)=\phi(s)\chi_n(s)=N_n s^{\sqrt{\varepsilon^2+\gamma_5}}(1-s)^{-\frac{1}{2}-\frac{1}{2}\sqrt{1+4\gamma_2+4l(l+1)}} P_n^{\left[\left(1+2\sqrt{\varepsilon^2+\gamma_5}\right),\left(2+2\sqrt{1+4\gamma_2+4l(l+1)}+2\sqrt{\varepsilon^2+\gamma_5}\right)\right]}(1-2s)$$

(32)

4. RADIAL SOLUTIONS OF KLEIN-GORDON EQUATIONS

One dimensional Klein-Gordon equation for equal scalar and vector potential is given as

$$\frac{d^2R(r)}{dr^2}+\left[E_R^2-m^2(r)-2(E_R+m)V(r)-\frac{\lambda}{r^2}\right]R(r)=0 \qquad (33)$$

Substituting equation (2) into (33) gives

$$\frac{d^2R(r)}{dr^2}+\left[E_R^2-m^2(r)-2(E_R+m)\left(-\frac{V_0e^{-2\alpha r}}{(1-e^{-2\alpha r})}-\frac{V_1}{\chi_2}\left(\frac{\chi_1e^{-2\alpha r}}{1-e^{-2\alpha r}}\right)+\frac{A}{r^2}+\frac{(B-\eta)e^{-\alpha r}}{r}+C\right)-\frac{\lambda}{r^2}\right]R(r)=0$$

(34)

Substituting the approximation to the centrifugal term of equation (24) into (34) gives

$$\frac{d^2R(r)}{dr^2}+\left[\begin{array}{l} E_R^2-m^2(r) \\ -2\left(E_R+m\right)\left(\begin{array}{l}-\dfrac{V_0e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)}-\dfrac{V_1}{\chi_2}\left(\dfrac{\chi_1 e^{-2\alpha r}}{1-e^{-2\alpha r}}\right)+\dfrac{4\alpha^2Ae^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2} \\ +\dfrac{2\alpha\left(B-\eta\right)e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)}+C\end{array}\right)-\dfrac{4\alpha^2\lambda e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}\end{array}\right]R(r)=0 \quad (35)$$

However, from the transformation $s=e^{-2\alpha r}$ then

$$\frac{d^2R}{dr^2}=4\alpha^2e^{-4\alpha r}\frac{d^2R}{ds^2}+4\alpha^2e^{-2\alpha r}\frac{dR}{ds}=4\alpha^2s^2\frac{d^2R}{ds^2}+4\alpha^2s\frac{dR}{ds} \quad (36)$$

Substituting equation (36) into (35) then gives

$$\frac{d^2R}{ds^2}+\frac{1}{s}\frac{dR}{ds}+\frac{1}{4\alpha^2s^2}\left[\begin{array}{l} E_R^2-m^2(r) \\ -2\left(E_R+m\right)\left(\begin{array}{l}-\dfrac{V_0s}{\left(1-s\right)}-\dfrac{V_1}{\chi_2}\left(\dfrac{\chi_1 s}{1-s}\right)+\dfrac{4\alpha^2As}{\left(1-s\right)^2} \\ +\dfrac{2\alpha\left(B-\eta\right)s}{\left(1-s\right)}+C\end{array}\right)-\dfrac{4\alpha^2\lambda s}{\left(1-s\right)^2}\end{array}\right]R(s)=0 \quad (37)$$

Assuming that $E_R^2-m^2(r)=\tilde{E}_{nl}$ then with simple mathematical algebraic simplification, equation (37) can be written as

$$\frac{d^2R}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dR}{ds}+\frac{1}{s^2(1-s)^2}\left\{\begin{array}{l}\left[-\frac{1}{4\alpha^2}\left(\begin{array}{l}2(E_R+m)V_0-\tilde{E}_{nl}+2(E_R+m)\frac{V_1\chi_1}{\chi_2}\\ -4\alpha(E_R+m)(B-\eta)+2c(E_R+m)\end{array}\right)\right]s^2\\ +\left[\frac{1}{4\alpha^2}\left(\begin{array}{l}2(E_R+m)V_0-2\tilde{E}_{nl}+2(E_R+m)\frac{V_1\chi_1}{\chi_2}\\ -8A\alpha^2(E_R+m)-\\ 4\alpha(E_R+m)(B-\eta)+4c(E_R+m)-4\alpha^2\lambda\end{array}\right)\right]\\ -\left[\frac{1}{4\alpha^2}\left(2c(E_R+m)-\tilde{E}_{nl}\right)\right]\end{array}\right\}R(s)=0$$

(38)

Comparing equation (38) to parametric NU equation (4) then the following parametric constants are obtained.

$c_1=c_2=c_3=1$

$$\xi_1 = \frac{1}{4\alpha^2}\begin{pmatrix} 2(E_R+m)V_0 - \tilde{E}_{nl} + 2(E_R+m)\dfrac{V_1\chi_1}{\chi_2} \\ -4\alpha(E_R+m)(B-\eta) + 2c(E_R+m) \end{pmatrix} \quad (39)$$

$$\xi_2 = \frac{1}{4\alpha^2}\begin{pmatrix} 2(E_R+m)V_0 - 2\tilde{E}_{nl} + 2(E_R+m)\dfrac{V_1\chi_1}{\chi_2} \\ -8A\alpha^2(E_R+m) - \\ 4\alpha(E_R+m)(B-\eta) + 4c(E_R+m) - 4\alpha^2\lambda \end{pmatrix} \quad (40)$$

$$\xi_3 = \frac{1}{4\alpha^2}\left(2c(E_R+m) - \tilde{E}_{nl}\right) \quad (41)$$

$$c_4 = 0$$

$$c_5 = -\frac{1}{2}$$

$$c_6 = \frac{1}{4} + \frac{1}{4\alpha^2}\left[\begin{pmatrix} 2(E_R+m)V_0 - \tilde{E}_{nl} + 2(E_R+m)\dfrac{V_1\chi_1}{\chi_2} \\ -4\alpha(E_R+m)(B-\eta) + 2c(E_R+m) \end{pmatrix}\right] \quad (42)$$

$$c_7 = -\frac{1}{4\alpha^2}\begin{pmatrix} 2(E_R+m)V_0 - 2\tilde{E}_{nl} + 2(E_R+m)\dfrac{V_1\chi_1}{\chi_2} \\ -8A\alpha^2(E_R+m) - \\ 4\alpha(E_R+m)(B-\eta) + 4c(E_R+m) - 4\alpha^2\lambda \end{pmatrix} \quad (43)$$

$$c_8 = \frac{1}{4\alpha^2}\left(2c(E_R+m) - \tilde{E}_{nl}\right) \quad (44)$$

$$c_9 = \frac{1}{4} + 2(E_R+m)A + \lambda \quad (45)$$

$$c_{10} = 1 + 2\sqrt{\frac{1}{4\alpha^2} + 2(E_R+m)A + \lambda} \quad (46)$$

$$c_{11} = 2 + 2\left[\sqrt{\frac{1}{4} + 2(E_R+m)A + \lambda} + \sqrt{\frac{1}{4\alpha^2}\left(2c(E_R+m) - \tilde{E}_{nl}\right)}\right] \quad (47)$$

$$c_{12} = \sqrt{\frac{1}{4\alpha^2}\left(2c(E_R+m) - \tilde{E}_{nl}\right)} \quad (48)$$

$$c_{13} = -\frac{1}{2} - \left[\sqrt{\frac{1}{4} + 2(E_R+m)A + \lambda} + \sqrt{\frac{1}{4\alpha^2}\left(2c(E_R+m) - \tilde{E}_{nl}\right)}\right] \quad (49)$$

Energy eigen equation of Klein-Gordon equation can be calculated using equation (15) bearing in mind that for equal scalar and vector potential, $V_s = 2V$ which then transform $2\left(E_R + m\right) \rightarrow \frac{2\mu}{\hbar^2}$. Substituting the parametric constants to equation (15) gives

$$\left(n^2+n+\frac{1}{2}\right)+\left(\left(2n+1\right)\sqrt{\frac{1}{4}+\frac{2\mu A}{\hbar^2}+l(l+1)}\right)+\left(2n+1\right)\sqrt{\frac{1}{4\alpha^2}\left(\frac{2\mu c}{\hbar^2}-\tilde{E}_{nl}\right)}$$
$$-\frac{1}{4\alpha^2}\left[\frac{2\mu V_0}{\hbar^2}-2\tilde{E}_{nl}+\frac{2\mu V_1}{\hbar^2}\frac{\chi_1}{\chi_2}-\frac{4\mu A\alpha^2}{\hbar^2}-\frac{4\alpha\mu}{\hbar^2}\left(B-\eta\right)+\frac{4\mu c}{\hbar^2}-4\alpha^2\lambda\right] \tag{50}$$
$$+\frac{2}{4\alpha^2}\left[\frac{2\mu c}{\hbar^2}-\tilde{E}_{nl}\right]+2\sqrt{\frac{1}{4\alpha^2}\left(\frac{2\mu c}{\hbar^2}-\tilde{E}_{nl}\right)\left(\frac{1}{4}+\frac{2\mu A}{\hbar^2}+l(l+1)\right)}=0$$

Equation (50) can be reduced to

$$\frac{1}{4\alpha^2}\left(\frac{2\mu c}{\hbar^2}-\tilde{E}_{nl}\right)=\left\{\frac{\left(n^2+n+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)\sqrt{1+\frac{8\mu A}{\hbar^2}+4l(l+1)}-\frac{\mu V_0}{2\alpha^2\hbar^2}-\frac{\mu V_1}{2\alpha^2\hbar^2}\frac{\chi_1}{\chi_2}+\frac{\mu\left(B-\eta\right)}{\alpha\hbar^2}+l(l+1)}{\left(2n+1\right)+\sqrt{1+\frac{8\mu A}{\hbar^2}+4l(l+1)}}\right\}^2$$

(51)

Equation (51) can further be reduce to

$$\tilde{E}_{nl}=-4\alpha^2\left\{\frac{\left(n^2+n+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)\sqrt{1+\frac{8\mu A}{\hbar^2}+4l(l+1)}-\frac{\mu V_0}{2\alpha^2\hbar^2}-\frac{\mu V_1}{2\alpha^2\hbar^2}\frac{\chi_1}{\chi_2}+\frac{\mu\left(B-\eta\right)}{\alpha\hbar^2}+l(l+1)}{\left(2n+1\right)+\sqrt{1+\frac{8\mu A}{\hbar^2}+4l(l+1)}}\right\}^2+\frac{2\mu c}{\hbar^2}$$

(52)

Recall that $\tilde{E}_{nl}=E_R^2-m^2=\left(E_R-m\right)\left(E_R+m\right)$ equation (52) finally reduce to

$$E_R^2 - m^2 = -4\alpha^2 \left\{ \frac{\left(n^2 + n + \frac{1}{2}\right) + \left(n + \frac{1}{2}\right)\sqrt{1 + \frac{8\mu A}{\hbar^2} + 4l(l+1)} - \frac{\mu V_0}{2\alpha^2\hbar^2} - \frac{\mu V_1}{2\alpha^2\hbar^2}\frac{\chi_1}{\chi_2} + \frac{\mu\left(B - \eta\right)}{\alpha\hbar^2} + l(l+1)}{\left(2n+1\right) + \sqrt{1 + \frac{8\mu A}{\hbar^2} + 4l(l+1)}} \right\}^2 + \frac{2\mu c}{\hbar^2}$$

(53)

Equation (53) is the energy eigen equation for Klein-Gordon equation.

The nonrelativistic limit usually abbreviated as NR limit, convert relativistic equation to nonrelativistic equation.

Here $m + E_R = \frac{2\mu}{\hbar^2}$ and $m - E_R = -E_{nl} \Rightarrow E_R - m = E_{nl}$, Hence

$$E_R^2 - m^2 = \frac{2\mu E_{nl}}{\hbar^2} \quad (54)$$

Substituting equation (54) into (53) gives

$$\frac{2\mu E_{nl}}{\hbar^2} = -4\alpha^2 \left\{ \frac{\left(n^2 + n + \frac{1}{2}\right) + \left(n + \frac{1}{2}\right)\sqrt{1 + \frac{8\mu A}{\hbar^2} + 4l(l+1)} - \frac{\mu V_0}{2\alpha^2\hbar^2} - \frac{\mu V_1}{2\alpha^2\hbar^2}\frac{\chi_1}{\chi_2} + \frac{\mu\left(B - \eta\right)}{\alpha\hbar^2} + l(l+1)}{\left(2n+1\right) + \sqrt{1 + \frac{8\mu A}{\hbar^2} + 4l(l+1)}} \right\}^2 + \frac{2\mu c}{\hbar^2}$$

(55)

Equation (55) finally reduce to

$$E_{nl} = -\frac{2\hbar^2\alpha^2}{\mu}\left\{\frac{\left(n^2+n+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)\sqrt{1+\frac{8\mu A}{\hbar^2}+4l(l+1)}-\frac{\mu V_0}{2\alpha^2\hbar^2}-\frac{\mu V_1}{2\alpha^2\hbar^2}\frac{\chi_1}{\chi_2}+\frac{\mu(B-\eta)}{\alpha\hbar^2}+l(l+1)}{(2n+1)+\sqrt{1+\frac{8\mu A}{\hbar^2}+4l(l+1)}}\right\}^2+c$$

(56)

It can be observe that equation with high level of analytical mathematical accuracy, equation (56) is exactly the same as equation (31). This affirms the fact that the relativistic equation (Klein-Gordon) can be converted to nonrelativistic equation (Schrödinger) with application of nonrelativistic limit.

5. NORMALISING THE WAVE FUNCTION OF THE POTENTIAL USING CONFLUENT HYPERGEOMETRIC FUNCTION

The wave function for this system is given in equation (32). Basically to normalize a wave function we get the integral of wave function and its complex conjugate to be equal to one. That is

$$\int_0^\infty \Psi(r)\Psi^*(r)dr = 1 \qquad (57)$$

In a situation where $\Psi(r)$ and its complex conjugate are real function, then equation (57) can be expressed as

$$\int_0^\infty |\Psi(r)|^2 dr = 1 \qquad (58)$$

Considering the fact that $s = e^{-2\alpha r}$ then when $r = 0, s = 1$ and when $r = \infty, s = 0$,

Hence the wave function will be physically valid for $s \in [0,1]$ and $r \in (0,\infty)$

However from equation (32) let

$$\kappa_1 = \sqrt{\varepsilon^2+\gamma_5} \text{ and } \kappa_2 = \sqrt{1+4\gamma_2+4l(l+1)} \qquad (59)$$

Equation (32) can then be expressed as

$$\Psi_n(s) = N_n(s)(s)^{\kappa_1}[1-s]^{\left(-\frac{1}{2}+\frac{1}{2}\kappa_2\right)} \times P_n^{[(1+2\kappa_1),(2+\kappa_2+2\kappa_1)]}(1-2s) \qquad (60)$$

Substituting equation (60) into equation (58) gives

$$-\frac{N_n^2}{2\alpha}\int_0^1 (s)^{2\kappa_1-1}\left[1-s\right]^{-(1+\kappa_2)}\times\left|P_n^{\left[(1+2\kappa_1),(2+\kappa_2+2\kappa_1)\right]}(1-2s)\right|^2 ds=1$$

(61)

Jacobi polynomial $P_n^{(\rho,\nu)}(\upsilon_1)$ can be expressed in two different hypergeometric functions by

$$P_n^{(\rho,\nu)}(\upsilon_1)=2^{-n}\sum_{p=0}^{n}(-1)^{n-p}\binom{n+\rho}{p}\binom{n+\nu}{p}(1-\upsilon_1)^{n-p}(1+\upsilon_1)^{p} \quad (62)$$

$$P_n^{(\rho,\nu)}(\upsilon_1)=\frac{\Gamma(n+\rho+1)}{n!\Gamma(n+\rho+\nu+1)}\sum_{r=0}^{n}\binom{n}{r}\frac{\Gamma(n+\rho+\nu+r+1)}{\Gamma(r+\rho+1)}\left(\frac{\upsilon_1-1}{2}\right)^{r} \quad (63)$$

Where

$$\binom{n}{r}={}^{n}c_r=\frac{n!}{(n-r)!r!}=\frac{\Gamma(n+1)}{\Gamma(n-r+1)\Gamma(r+1)} \quad (64)$$

Equations (62) and (63) are used simultaneously in evaluating the Jacobi polynomial.

Considering the Jacobi polynomial of equation (61)

$$P_n^{\left[(1+2\kappa_1),(2+\kappa_2+2\kappa_1)\right]}(1-2s)\Rightarrow\rho=(1+2\kappa_1),\nu=(2+\kappa_2+2\kappa_1),\upsilon_1=(1-2s)$$

Using equation (62) then the Jacobi polynomial become

$$P_n^{\left[(1+2\kappa_1),(2+\kappa_2+2\kappa_1)\right]}(1-2s)=2^{-n}\sum_{p=0}^{n}(-1)^{n-p}\binom{n+1+2\kappa_1}{p}\binom{n+2+\kappa_2+2\kappa_1}{n-p}(1-1+2s)^{n-p}(1+1-2s)^{p}$$

$$\Rightarrow P_n^{\left[(1+2\kappa_1),(2+\kappa_2+2\kappa_1)\right]}(1-2s)=2^{-n}\sum_{p=0}^{n}(-1)^{n-p}\binom{n+1+2\kappa_1}{p}\binom{n+2+\kappa_2+2\kappa_1}{n-p}(2s)^{n-p}(-2s)^{p}$$

(65)

The summation sign in equation (65) can be evaluated simultaneously for p=0 and p=0, n as a partial sum.

Evaluating it for p=0

$$\sum_{p=0}^{n}(-1)^{n-p}\binom{n+1+2\kappa_1}{p}\binom{n+2+\kappa_2+2\kappa_1}{n-p}=\sum_{p=0}^{n}(-1)^{n}\binom{n+1+2\kappa_1}{0}\binom{n+2+\kappa_2+2\kappa_1}{n}$$

$$=(-1)^n\frac{(n+1+2\kappa_1)!}{\left[(n+1+2\kappa_1)-0\right]!0!}\frac{(n+2+\kappa_2+2\kappa_1)!}{\left[(n+2+\kappa_2+2\kappa_1)\right]!n!}=(-1)^n\frac{\Gamma(n+3+\kappa_2+2\kappa_1)}{\Gamma(n+1)\Gamma(n+3+\kappa_2+2\kappa_1)}$$

(66)

For p=0, n

$$\sum_{p=0}^{n}(-1)^{n-p}\binom{n+1+2\kappa_1}{p}\binom{n+2+\kappa_2+2\kappa_1}{n-p}=$$

$$(-1)^{p}\frac{(n+2\kappa_1+1)!}{\left[(n+2\kappa_1+1-p)\right]!\,p!}\frac{(n+2+\kappa_1+2\kappa_1)!}{\left[(n+2+\kappa_2+2\kappa_1)-(n-p)\right]!(n-p)!}$$

$$=(-1)^{p}\frac{(n+2\kappa_1+1)!}{p!(n-p)!\left[(n+2\kappa_1+1-p)\right]!}\frac{(n+2+\kappa_2+2\kappa_1)!}{\left[(n+2+\kappa_2+2\kappa_1+p)\right]!}$$

$$\Rightarrow(-1)^{p}\frac{\Gamma(n+2\kappa_1+2)}{p!(n-p)!\Gamma\left[(n+2\kappa_1+2-p)\right]}\frac{\Gamma(n+3+\kappa_2+2\kappa_1)}{\Gamma\left[(n+3+\kappa_2+2\kappa_1+p)\right]}$$

(67)

Substituting (66) and (67) into (65) gives

$$P_{n}^{\left[(1+2\kappa_1),(2+\kappa_2+2\kappa_1)\right]}(1-2s)=(-1)^{n}\frac{\Gamma(n+3+\kappa_2+2\kappa_1)}{\Gamma(n+1)\Gamma(n+3+\xi_2+2\xi_1)}\times$$
$$\sum_{p=0}^{n}(-1)^{p}\frac{\Gamma(n+2\kappa_1+2)}{p!(n-p)!\Gamma\left[(n+2\kappa_1+2-p)\right]}\frac{\Gamma(n+3+\kappa_2+2\kappa_1)2^{-n}(2s)^{n-p}(-2s)^{p}}{\Gamma\left[(n+3+\kappa_2+2\kappa_1+p)\right]} \quad (68)$$

Using the second expression for the Jacobi polynomial that is equation (63) then

$$P_{n}^{\left[(1+2\kappa_1),(2+\kappa_2+2\kappa_1)\right]}(1-2s)=\frac{\Gamma(n+2\kappa_1+2)}{\Gamma(4\kappa_1+\kappa_2+3)}\sum_{r=0}^{n}\frac{(-1)^{r}\Gamma(n+4+4\kappa_1+\kappa_2)}{r!(n-r)!(r+1+2\kappa_1)}s^{r}$$

(69)

Then the square of the Jacobi polynomial in equation (61) then become

$$\left[P_{n}^{\left[(1+2\kappa_1),(2+\kappa_2+2\kappa_1)\right]}(1-2s)\right]^{2}=\text{Equation (68) multiplied by (69)}$$

$$\left[P_{n}^{\left[(1+2\kappa_1),(2+\kappa_2+2\kappa_1)\right]}(1-2s)\right]^{2}=\frac{\Gamma(n+2\kappa_1+2)}{\Gamma(4\kappa_1+\kappa_2+3)}\sum_{r=0}^{n}\frac{(-1)^{r}\Gamma(n+4+4\kappa_1+\kappa_2)}{r!(n-r)!(r+1+2\xi_1)}s^{r}$$
$$\times(-1)^{n}\frac{\Gamma(n+3+\kappa_2+2\kappa_1)}{\Gamma(n+1)\Gamma(n+3+\kappa_2+2\kappa_1)}\times$$
$$\sum_{p=0}^{n}(-1)^{p}\frac{\Gamma(n+2\kappa_1+2)}{p!(n-p)!\Gamma\left[(n+2\kappa_1+2-p)\right]}\frac{\Gamma(n+3+\kappa_2+2\kappa_1)2^{-n}(2s)^{n-p}(-2s)^{p}}{\Gamma\left[(n+3+\kappa_2+2\kappa_1+p)\right]}$$

(70)

Equation (70) can be further simplified to

$$\left[P_n^{\left[(1+2\kappa_1),(2+\kappa_2+2\kappa_1)\right]}(1-2s)\right]^2=\frac{(-1)^{n+2p+r}2^{-2n}s^{p+r}\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(n+2\kappa_1+2\right)}{\Gamma\left(n+1\right)\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(4\kappa_1+\kappa_2+3\right)}$$
$$\sum_{p=0}^{n}\sum_{r=0}^{n}\frac{\Gamma\left(n+2+2\kappa_1\right)\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(n+4+4\kappa_1+\kappa_2\right)}{p!\left(n-p\right)!r!\left(n-r\right)!\left(r+1+2\kappa_1\right)\Gamma\left(P+3+\kappa_2+2\kappa_1\right)}$$

(71)

Substituting equation (71) into (61) gives

$$-\frac{N_n^2}{2\alpha}\frac{(-1)^{n+2p+r}2^{-2n}s^{p+r}\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(n+2\kappa_1+2\right)}{\Gamma\left(n+1\right)\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(4\kappa_1+\kappa_2+3\right)}$$
$$\sum_{p=0}^{n}\sum_{r=0}^{n}\frac{\Gamma\left(n+2+2\kappa_1\right)\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(n+4+4\kappa_1+\kappa_2\right)}{p!\left(n-p\right)!r!\left(n-r\right)!\left(r+1+2\kappa_1\right)\Gamma\left(P+3+\kappa_2+2\kappa_1\right)}\int_0^1(s)^{2\kappa_1-1}\left[1-s\right]^{-(1+\kappa_2)}ds=1$$

(72)

Confluent hypergeometric function can be define as follows:

$${}_2F_1\left(\alpha_0,\beta_0:\alpha_0+1;1\right)=\alpha_0\int_0^1(s)^{\alpha_0-1}\left[1-s\right]^{-\beta_0}ds=1 \qquad (73)$$

Assuming that

$\gamma_0=\alpha_0+1$, then ${}_2F_1\left(\alpha_0,\beta_0:\gamma_0;1\right)=\alpha_0\int_0^1(s)^{\alpha_0-1}\left[1-s\right]^{-\beta_0}ds=1$

(74)

However,

$${}_2F_1\left(\alpha_0,\beta_0:\gamma_0;1\right)=\frac{\Gamma(\gamma_0)\Gamma\left(\gamma_0-\alpha_0-\beta_0\right)}{\Gamma\left(\gamma_0-\alpha_0\right)\Gamma\left(\gamma_0-\beta_0\right)}$$

(75)

Considering

$\int_0^1(s)^{2\kappa_1-1}\left[1-s\right]^{-(1+\kappa_2)}ds$, $\alpha_0=2\kappa_1$, $\beta_0=\left(1+\kappa_2\right)$, $\gamma_0=\alpha_0+1$

(76)

Therefore

$$\int_0^1 (s)^{2\kappa_1-1}\left[1-s\right]^{-(1+\kappa_2)} ds = \frac{\Gamma\left(\alpha_0+1\right)\Gamma\left(\alpha_0+1-\alpha_0-\beta_0\right)}{\alpha_0\Gamma\left(\gamma_0-\alpha_0\right)\Gamma\left(\gamma_0-\beta_0\right)}$$

$$\Rightarrow \int_0^1 (s)^{2\kappa_1-1}\left[1-s\right]^{-(1+\kappa_2)} ds = \frac{\Gamma\left(\alpha_0+1\right)\Gamma\left(1-\beta_0\right)}{\alpha_0\Gamma\left(\alpha_0+1-\alpha_0\right)\Gamma\left(\alpha_0+1-\beta_0\right)} = \frac{\Gamma\left(2\kappa_1+1\right)\Gamma\left(1-\left(1+\kappa_2\right)\right)}{\alpha_0\Gamma\left(2\kappa_1-\left(1+\kappa_2\right)\right)}$$

$$\Rightarrow \int_0^1 (s)^{2\kappa_1-1}\left[1-s\right]^{-(1+\kappa_2)} ds = \frac{\Gamma\left(2\kappa_1+1\right)\Gamma\left(-\kappa_2\right)}{\alpha_0\Gamma\left(2\kappa_1-\kappa_2-1\right)}$$

(77)

Substituting equation (77) into (72) gives

$$-\frac{N_n^2}{2\alpha}\frac{(-1)^{n+2p+r}2^{-2n}s^{p+r}\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(n+2\kappa_1+2\right)}{\Gamma\left(n+1\right)\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(4\kappa_1+\kappa_2+3\right)}$$
$$\sum_{p=0}^{n}\sum_{r=0}^{n}\frac{\Gamma\left(n+2+2\kappa_1\right)\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(n+4+4\kappa_1+\kappa_2\right)}{p!(n-p)!r!(n-r)!\left(r+1+2\kappa_1\right)\Gamma\left(P+3+\kappa_2+2\kappa_1\right)}\frac{\Gamma\left(2\kappa_1+1\right)\Gamma\left(-\kappa_2\right)}{\alpha_0\Gamma\left(2\kappa_1-\kappa_2-1\right)}=1 \quad (78)$$

Let

$$M_1 = \frac{(-1)^{n+2p+r}2^{-2n}s^{p+r}\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(n+2\kappa_1+2\right)}{2\alpha\Gamma\left(n+1\right)\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(4\kappa_1+\kappa_2+3\right)}$$
$$\sum_{p=0}^{n}\sum_{r=0}^{n}\frac{\Gamma\left(n+2+2\kappa_1\right)\Gamma\left(n+3+\kappa_2+2\kappa_1\right)\Gamma\left(n+4+4\kappa_1+\kappa_2\right)}{p!(n-p)!r!(n-r)!\left(r+1+2\kappa_1\right)\Gamma\left(P+3+\kappa_2+2\kappa_1\right)}\frac{\Gamma\left(2\kappa_1+1\right)\Gamma\left(-\kappa_2\right)}{\alpha_0\Gamma\left(2\kappa_1-\kappa_2-1\right)} \quad (79)$$

However,

$$N_n^2 M_1 = 1 \Rightarrow N_n(s) = \frac{1}{\sqrt{M_1}} \quad (80)$$

Hence, the normalized wave function then become

$$\frac{1}{\sqrt{M_1}} s^{\sqrt{\varepsilon^2+\gamma_5}}\left(1-s\right)^{-\frac{1}{2}-\frac{1}{2}\sqrt{1+4\gamma_2+4l(l+1)}} P_n^{\left[\left(1+2\sqrt{\varepsilon^2+\gamma_5}\right),\left(2+2\sqrt{1+4\gamma_2+4l(l+1)}+2\sqrt{\varepsilon^2+\gamma_5}\right)\right]}\left(1-2s\right) \quad (81)$$

Equation (81) is the normalized wave function for the proposed potential.

6. RESULTS AND DISCUSSION

In this section the numerical computation of energy eigenvalues of Schrodinger and Klein-Gordon equations are presented. Using equation (31) we implemented MATLAB algorithm to calculate the numerical bound state energies of Schrodinger equation with the proposed potential using the following real constants. In tables 1, 2, 3, 4 and 5, the numerical values for the energy particles in Schrodinger equations for $\alpha = 0.1, 0.2, 0.3, 0.4$ and 0.5 are discussed respectively. These tables show negative energies which satisfies bound state condition. However, the numerical bound state energies decreases with an increase in quantum state.

The energy eigenvalues for the Klein-Gordon particles for $\alpha = 0.1, 0.2, 0.3, 0.4$ and 0.5 are discussed and presented in tables 6, 7, 8, 9 and 10 respectively. The bound state energies in this case increases with an increase in quantum state with respect to orbital angular quantum number.

$$\chi_1 = 0.1, \quad \chi_2 = 0.2, \quad V_0 = 0.01, \quad V_1 = 0.02, \quad A = \hbar = \mu = 1.0$$
$$\eta = 0.03, \quad B = 2.0, \quad 0.1 \le \alpha \le 0.5$$

Table 1 Numerical bound state energy for Schrodinger Equation for $\alpha = 0.1$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	-1.64465136	**0**	**1**	-1.46103113	**0**	**2**	-1.31165417	**0**	**3**	-1.22436776
1	**0**	-1.43664026	**1**	**1**	-1.37624632	**1**	**2**	-1.31051024	**1**	**3**	-1.26269160
2	**0**	-1.38809186	**2**	**1**	-1.36818112	**2**	**2**	-1.33954161	**2**	**3**	-1.31527438
3	**0**	-1.39422175	**3**	**1**	-1.39398112	**3**	**2**	-1.38722481	**3**	**3**	-1.37995459
4	**0**	-1.42825952	**4**	**1**	-1.44039760	**4**	**2**	-1.44922720	**4**	**3**	-1.45578639
5	**0**	-1.48085112	**5**	**1**	-1.50224603	**5**	**2**	-1.52362416	**5**	**3**	-1.54230271
6	**0**	-1.54803903	**6**	**1**	-1.57715975	**6**	**2**	-1.60945488	**6**	**3**	-1.63925131
7	**0**	-1.62791544	**7**	**1**	-1.66393353	**7**	**2**	-1.70619621	**7**	**3**	-1.74648628
8	**0**	-1.71946872	**8**	**1**	-1.76190112	**8**	**2**	-1.81354355	**8**	**3**	-1.86391847

Table 2 Numerical bound state energy for Schrodinger Equation for $\alpha = 0.2$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	-1.88975995	**0**	**1**	-1.69219460	**0**	**2**	-1.50811426	**0**	**3**	-1.39348029
1	**0**	-1.76400003	**1**	**1**	-1.71576175	**1**	**2**	-1.64352235	**1**	**3**	-1.58568452
2	**0**	-1.82462372	**2**	**1**	-1.83758627	**2**	**2**	-1.82954958	**2**	**3**	-1.81793731
3	**0**	-1.96900525	**3**	**1**	-2.02024911	**3**	**2**	-2.06013064	**3**	**3**	-2.09021402
4	**0**	-2.17092576	**4**	**1**	-2.25218191	**4**	**2**	-2.33298508	**4**	**3**	-2.40250386
5	**0**	-2.42122386	**5**	**1**	-2.52885771	**5**	**2**	-2.64709988	**5**	**3**	-2.75480151
6	**0**	-2.71602359	**6**	**1**	-2.84821002	**6**	**2**	-3.00196934	**6**	**3**	-3.14710408
7	**0**	-3.05345644	**7**	**1**	-3.20918647	**7**	**2**	-3.39731808	**7**	**3**	-3.57940992

8	**0**	-3.43253160	**8**	**1**	-3.61120532	**8**	**2**	-3.83298579	**8**	**3**	-4.05171801

Table 3 Numerical bound state energy for Schrodinger Equation for $\alpha = 0.3$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	-2.16930505	**0**	**1**	-1.96658237	**0**	**2**	-1.74992797	**0**	**3**	-1.60805107
1	**0**	-2.17896947	**1**	**1**	-2.16067328	**1**	**2**	-2.09428411	**1**	**3**	-2.03462651
2	**0**	-2.42094736	**2**	**1**	-2.49470577	**2**	**2**	-2.53266378	**2**	**3**	-2.55388015
3	**0**	-2.79546643	**3**	**1**	-2.93660208	**3**	**2**	-3.06276886	**3**	**3**	-3.16445273
4	**0**	-3.27703681	**4**	**1**	-3.47644512	**4**	**2**	-3.68373531	**4**	**3**	-3.86574948
5	**0**	-3.85675185	**5**	**1**	-4.11035415	**5**	**2**	-4.39517931	**5**	**3**	-4.65747668
6	**0**	-4.53084335	**6**	**1**	-4.83655763	**6**	**2**	-5.19690924	**6**	**3**	-5.53947571
7	**0**	-5.29749476	**7**	**1**	-5.65415342	**7**	**2**	-6.08882077	**7**	**3**	-6.51165483
8	**0**	-6.15574282	**8**	**1**	-6.56264281	**8**	**2**	-7.07085315	**8**	**3**	-7.57395797

Table 4 Numerical bound state energy for Schrodinger Equation for $\alpha = 0.4$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	-2.48761162	**0**	**1**	-2.28714049	**0**	**2**	-2.03898155	**0**	**3**	-1.86936584
1	**0**	-2.68405529	**1**	**1**	-2.71299165	**1**	**2**	-2.66431826	**1**	**3**	-2.61069940
2	**0**	-3.17891352	**2**	**1**	-3.34115303	**2**	**2**	-3.45022542	**2**	**3**	-3.52422616
3	**0**	-3.87514745	**3**	**1**	-4.14444851	**3**	**2**	-4.39637721	**3**	**3**	-4.60375779
4	**0**	-4.74796553	**4**	**1**	-5.11447643	**4**	**2**	-5.50265113	**4**	**3**	-5.84658638
5	**0**	-5.78870515	**5**	**1**	-6.24794902	**5**	**2**	-6.76899275	**5**	**3**	-7.25137470
6	**0**	-6.99370132	**6**	**1**	-7.54336546	**6**	**2**	-8.19537492	**6**	**3**	-8.81740064
7	**0**	-8.36118727	**7**	**1**	-8.99996145	**7**	**2**	-9.78178283	**7**	**3**	-10.5442465
8	**0**	-9.89022613	**8**	**1**	-10.6173145	**8**	**2**	-11.5282079	**8**	**3**	-12.4316569

Table 5 Numerical bound state energy for Schrodinger Equation for $\alpha = 0.5$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	-2.84442445	**0**	**1**	-2.65370233	**0**	**2**	-2.37517286	**0**	**3**	-2.17735777
1	**0**	-3.27912704	**1**	**1**	-3.37261728	**1**	**2**	-3.35355331	**1**	**3**	-3.31385034
2	**0**	-4.09843675	**2**	**1**	-4.37685692	**2**	**2**	-4.58217834	**2**	**3**	-4.72893041
3	**0**	-5.20798403	**3**	**1**	-5.64373199	**3**	**2**	-6.06090826	**3**	**3**	-6.40808912
4	**0**	-6.58365927	**4**	**1**	-7.16622796	**4**	**2**	-7.78969054	**4**	**3**	-8.34497772
5	**0**	-8.21703815	**5**	**1**	-8.94159987	**5**	**2**	-9.76850184	**5**	**3**	-10.5364610
6	**0**	-10.1045565	**6**	**1**	-10.9685947	**6**	**2**	-11.9973305	**6**	**3**	-12.9808459
7	**0**	-12.2444961	**7**	**1**	-13.2465743	**7**	**2**	-14.4761703	**7**	**3**	-15.6771531
8	**0**	-14.6359459	**8**	**1**	-15.7751860	**8**	**2**	-17.2050173	**8**	**3**	-18.6247840

Table 6 Numerical bound state energy for Klein-Gordon Equation for $\alpha = 0.1$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	1.56078821	**0**	**1**	1.44326494	**0**	**2**	1.33976433	**0**	**3**	1.27530075
1	**0**	1.42800751	**1**	**1**	1.38689113	**1**	**2**	1.34046127	**1**	**3**	1.30557436
2	**0**	1.39607455	**2**	**1**	1.38260427	**2**	**2**	1.36270848	**2**	**3**	1.34555806
3	**0**	1.40161936	**3**	**1**	1.40192383	**3**	**2**	1.39768515	**3**	**3**	1.39298392
4	**0**	1.42632300	**4**	**1**	1.43509648	**4**	**2**	1.44161928	**4**	**3**	1.44651644
5	**0**	1.46310089	**5**	**1**	1.47783895	**5**	**2**	1.49249815	**5**	**3**	1.50521768
6	**0**	1.50855790	**6**	**1**	1.52786770	**6**	**2**	1.54904970	**6**	**3**	1.56835886
7	**0**	1.56076465	**7**	**1**	1.58376477	**7**	**2**	1.61036844	**7**	**3**	1.63534336
8	**0**	1.61846900	**8**	**1**	1.64454446	**8**	**2**	1.67575880	**8**	**3**	1.70567016

Table 7 Numerical bound state energy for Klein-Gordon Equation for $\alpha = 0.2$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	1.71580710	**0**	**1**	1.59890838	**0**	**2**	1.48137321	**0**	**3**	1.40312221
1	**0**	1.64377856	**1**	**1**	1.61503846	**1**	**2**	1.57060386	**1**	**3**	1.53402539

2	**0**	1.68127778	**2**	**1**	1.68937796	**2**	**2**	1.68511330	**2**	**3**	1.67862470
3	**0**	1.76552428	**3**	**1**	1.79453470	**3**	**2**	1.81692043	**3**	**3**	1.83367388
4	**0**	1.87664722	**4**	**1**	1.91958870	**4**	**2**	1.96142228	**4**	**3**	1.99673647
5	**0**	2.00572680	**5**	**1**	2.05877449	**5**	**2**	2.11555801	**5**	**3**	2.16600175
6	**0**	2.14776420	**6**	**1**	2.20850914	**6**	**2**	2.27715820	**6**	**3**	2.34012334
7	**0**	2.29956491	**7**	**1**	2.36635705	**7**	**2**	2.44463445	**7**	**3**	2.51809358
8	**0**	2.45892715	**8**	**1**	2.53057604	**8**	**2**	2.61679975	**8**	**3**	2.69915104

Table 8 Numerical bound state energy for Klein-Gordon Equation for $\alpha = 0.3$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	1.87458509	**0**	**1**	1.76427509	**0**	**2**	1.63791262	**0**	**3**	1.54952699
1	**0**	1.88097630	**1**	**1**	1.87163444	**1**	**2**	1.83628236	**1**	**3**	1.80385290
2	**0**	2.00591970	**2**	**1**	2.04254769	**2**	**2**	2.06129168	**2**	**3**	2.07176828
3	**0**	2.18484443	**3**	**1**	2.24861478	**3**	**2**	2.30418430	**3**	**3**	2.34803420
4	**0**	2.39522883	**4**	**1**	2.47714169	**4**	**2**	2.55954690	**4**	**3**	2.62978880
5	**0**	2.62617888	**5**	**1**	2.72107037	**5**	**2**	2.82386668	**5**	**3**	2.91533798
6	**0**	2.87144056	**6**	**1**	2.97602900	**6**	**2**	3.09478934	**6**	**3**	3.20361654
7	**0**	3.12707408	**7**	**1**	3.23913926	**7**	**2**	3.37069308	**7**	**3**	3.49392204
8	**0**	3.39045330	**8**	**1**	3.50842736	**8**	**2**	3.65043249	**8**	**3**	3.78577306

Table 9: Numerical bound state energy for Klein-Gordon Equation for $\alpha = 0.4$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	2.03895248	**0**	**1**	1.93867203	**0**	**2**	1.80663818	**0**	**3**	1.71050135
1	**0**	2.13364352	**1**	**1**	2.14735009	**1**	**2**	2.12478201	**1**	**3**	2.09956567
2	**0**	2.35434749	**2**	**1**	2.42236317	**2**	**2**	2.46709439	**2**	**3**	2.49700964
3	**0**	2.63359008	**3**	**1**	2.73397967	**3**	**2**	2.82469151	**3**	**3**	2.89724245
4	**0**	2.94646222	**4**	**1**	3.06835839	**4**	**2**	3.19240423	**4**	**3**	3.29842395
5	**0**	3.28073839	**5**	**1**	3.41787156	**5**	**2**	3.56709096	**5**	**3**	3.69988218

6	**0**	3.62950666	**6**	**1**	3.77792682	**6**	**2**	3.94675932	**6**	**3**	4.10135887
7	**0**	3.98852690	**7**	**1**	4.14559523	**7**	**2**	4.33009584	**7**	**3**	4.50275580
8	**0**	4.35505052	**8**	**1**	4.51892607	**8**	**2**	4.71620439	**8**	**3**	4.90404021

Table 10 Numerical bound state energy for Klein-Gordon Equation for $\alpha = 0.5$

n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$	n	l	$E_n(eV)$
0	**0**	2.20816652	**0**	**1**	2.12020339	**0**	**2**	1.98465271	**0**	**3**	1.88244936
1	**0**	2.39710542	**1**	**1**	2.43585243	**1**	**2**	2.42808346	**1**	**3**	2.41173156
2	**0**	2.71753379	**2**	**1**	2.81815139	**2**	**2**	2.89012636	**2**	**3**	2.94049721
3	**0**	3.09906517	**3**	**1**	3.23663248	**3**	**2**	3.36307673	**3**	**3**	3.46479270
4	**0**	3.51505640	**4**	**1**	3.67706650	**4**	**2**	3.84289558	**4**	**3**	3.98478741
5	**0**	3.95252064	**5**	**1**	4.13177744	**5**	**2**	4.32729326	**5**	**3**	4.50127459
6	**0**	4.40425729	**6**	**1**	4.59625826	**6**	**2**	4.81488555	**6**	**3**	5.01500002
7	**0**	4.86594077	**7**	**1**	5.06769771	**7**	**2**	5.30479040	**7**	**3**	5.52655647
8	**0**	5.33481885	**8**	**1**	5.54425752	**8**	**2**	5.79642084	**8**	**3**	6.03639613

With the help of origin software, we obtain numerical bound state energy diagrams plots for both Schrodinger and Klein-Gordon equations using their respective numerical bound state energy values. Figures 3, 4, 5, 6 and 7 show the variation of energy eigen values with quantum state (n) with various orbital quantum number (*l*) for $\alpha = 0.1, 0.2, 0.3, 0.4$ and 0.5 respectively for Schrodinger particles. These graph show unique quantization of the energy levels with respect to quantum state. Also, the same plots are carried out for the Klein-Gordon particles and are discussed in figures 8, 9,10, 11 and 12 for $\alpha = 0.1, 0.2, 0.3, 0.4$ and 0.5 respectively.

It can be observed that this graph is direct opposite to that obtained from Schrodinger equation. This implies that while the negatives energies from Schrodinger equations describe the particle constituting the state of the system, that of the Klein-Gordon equation described both spinless particle and anti-particle state of the system.

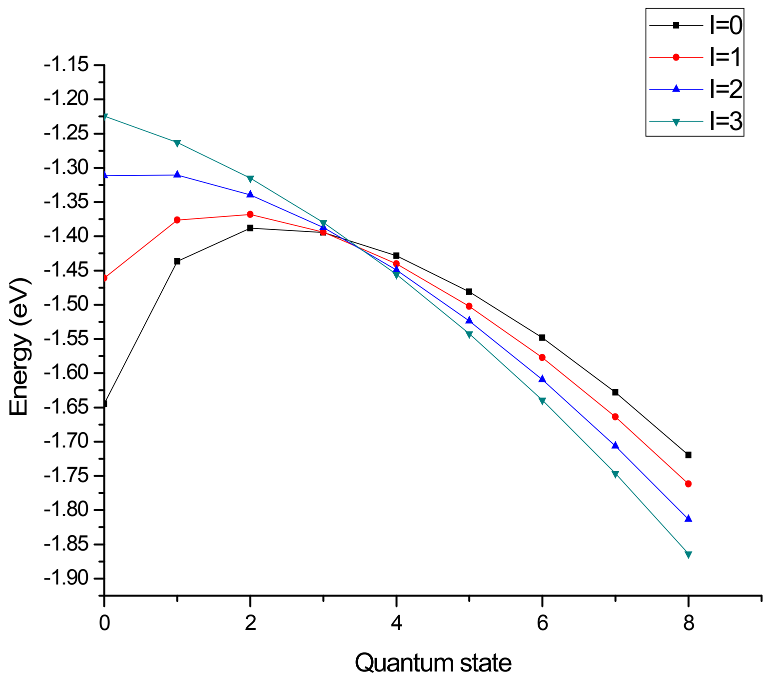

Figure 3: Energy spectral diagram of Schrodinger equation for $\alpha = 0.1$

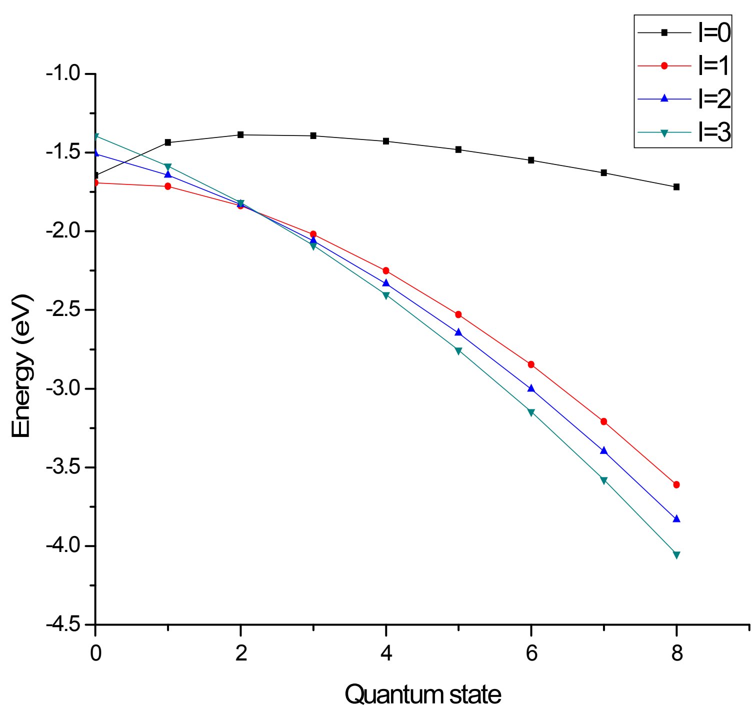

Figure 4: Energy spectral diagram of Schrodinger equation for $\alpha = 0.2$

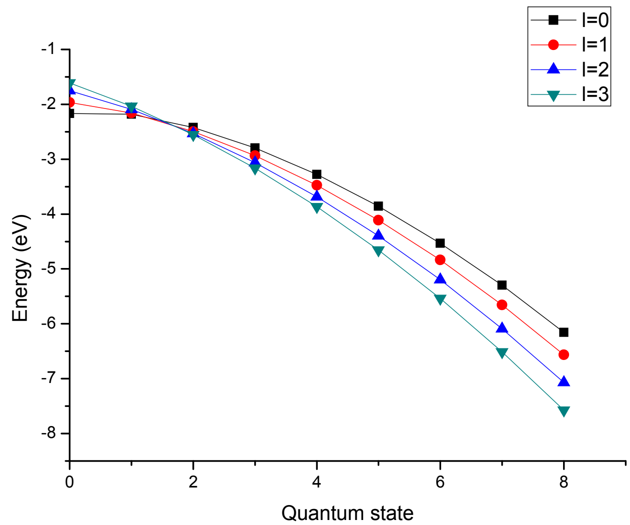

Figure 5: Energy spectral diagram of Schrodinger equation for $\alpha = 0.3$

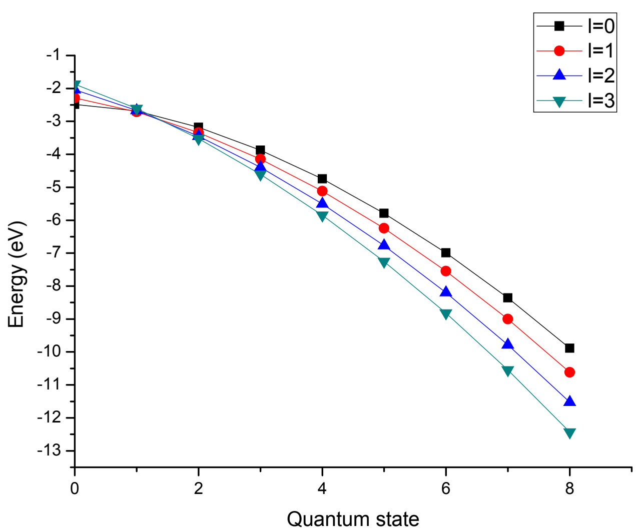

Figure 6: Energy spectral diagram of Schrodinger equation for $\alpha = 0.4$

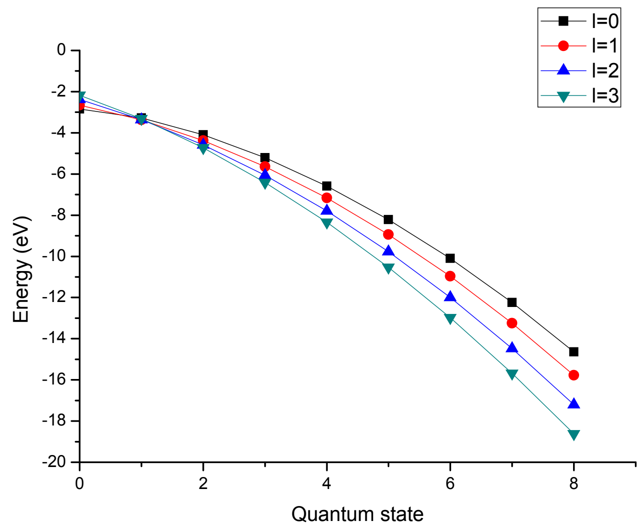

Figure 7: Energy spectral diagram of Schrodinger equation for $\alpha = 0.5$

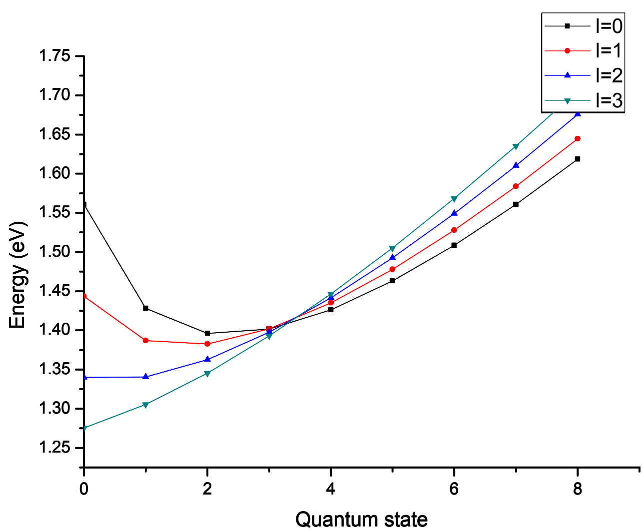

Figure 8: Energy spectral diagram of Klein-Gordon equation for $\alpha = 0.1$

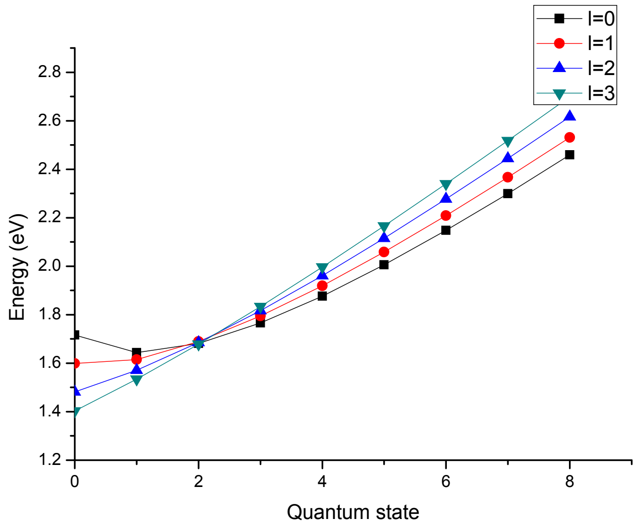

Figure 9: Energy spectral diagram of Klein-Gordon equation for $\alpha = 0.2$

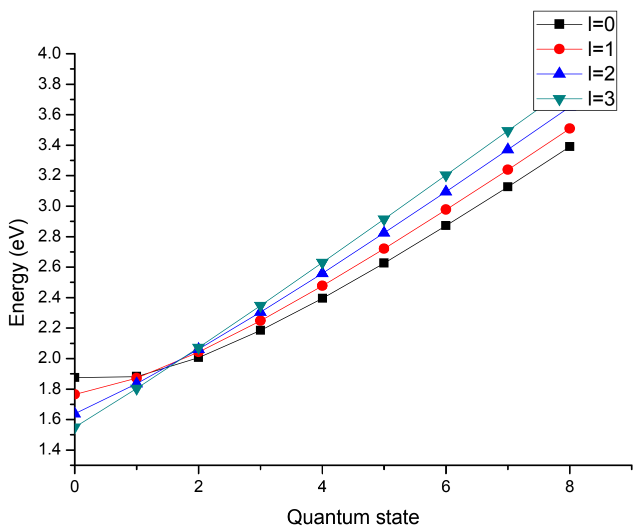

Figure 10: Energy spectral diagram of Klein-Gordon equation for $\alpha = 0.3$

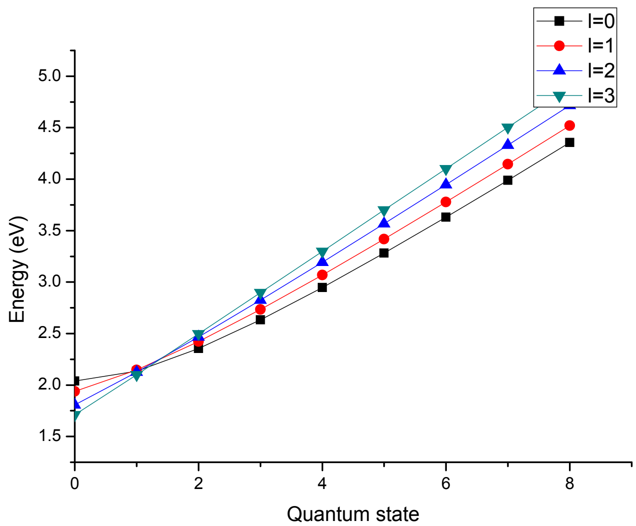

Figure 11: Energy spectral diagram of Klein-Gordon equation for $\alpha = 0.4$

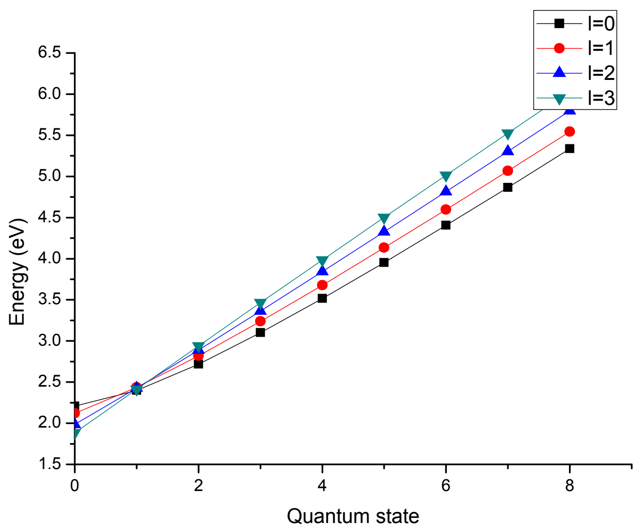

Figure 12: Energy spectral diagram of Klein-Gordon equation for $\alpha = 0.5$

Furthermore our novel potential could be deduced to some well known potentials by adjusting some potential parameters.

(i) Hulthen potential

Setting $A = B = c = \eta = V_1 = 0$ in equation (2) result to Hulthen potential given as

$$V(r)=\frac{-V_0e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)} \quad (82)$$

The energy of this potential is given as

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$$E_{nl}=-\frac{2\hbar^2\alpha^2}{\mu}\left[\frac{\left(-\frac{\mu V_0}{2\hbar^2\alpha^2}+l(l+1)\right)+(n^2+n+\frac{1}{2})+(n+\frac{1}{2})\sqrt{(4l(l+1)+1)}}{\left(1+2n+\sqrt{(4l(l+1)+1)}\right)}\right]^2 \quad (83)$$

However $\sqrt{(4l(l+1)+1)}=2l+1$, then equation (83) becomes

$$E_{nl}=-\frac{2\hbar^2\alpha^2}{\mu}\left[\frac{\left(-\frac{\mu V_0}{2\hbar^2\alpha^2}\right)+(n+l)(n+l+2)+1}{2\left(n+l+1\right)}\right]^2 \quad (84)$$

Equation (84) is in agreement to that obtained by Okon *et.al,* 2017

(ii) Yukawa potential

Setting $V_0=V_1=B=C=0$ in equation (2) then the potential reduced to Yukawa potential.

$$V(r)=-\frac{\eta e^{-\alpha r}}{r} \quad (85)$$

By substituting those constants to energy eigen value equation (31), then, the corresponding energy equation for Yukawa potential is given as

$$E_{nl}=-\frac{2\hbar^2\alpha^2}{\mu}\left[\frac{(n^2+n+\frac{1}{2})+(n+\frac{1}{2})\sqrt{(4l(l+1)+1)}-\frac{\mu\eta}{\hbar^2\alpha}+l(l+1)}{\left(1+2n+\sqrt{(4l(l+1)+1)}\right)}\right]^2$$

(86)

However, Okon *et. al*, 2017 obtain the energy-eigen value equation for Yukawa potential as

$$E_{nl}=-\frac{2\hbar^2\alpha^2}{\mu}\left[\frac{\left(-\frac{\mu A}{\hbar^2\alpha}+l(l+1)\right)+(n^2+n+\frac{1}{2})+(n+\frac{1}{2})\sqrt{(4l(l+1)+1)}}{\left(1+2n+\sqrt{(4l(l+1)+1)}\right)}\right]^2$$

(87)

It can be observe that equation (86) is exactly the same as equation (87) which shows that the result agrees to that of existing literature.

(iii) Exponential Mie-type potential

Setting $V_0 = V_1 = 0$ in equation (2) then the potential reduced to exponential Mie-type potential.

$$V(r) = \frac{A}{r^2} + \frac{(B-\eta)e^{-\alpha r}}{r} + C \tag{88}$$

Substituting the same constants to equation (31) gives the energy eigen equation for exponential Mie-Type potential as

$$E_{nl} = -\frac{2\hbar^2\alpha^2}{\mu}\left\{\frac{\left(n^2+n+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)\sqrt{1+\frac{8\mu A}{\hbar^2}+4l(l+1)}+\frac{\mu(B-\eta)}{\alpha\hbar^2}+l(l+1)}{(2n+1)+\sqrt{1+\frac{8\mu A}{\hbar^2}+4l(l+1)}}\right\}^2 + c \tag{89}$$

7. CONCLUSION

In this paper, we have obtained an approximate analytical solutions of Schrodinger and Klein-Gordon equations with a new proposed potential model called Hulthen plus inversely quadratic exponential Mie-Type potential (HIQEMP) via parametric Nikiforov-Uvarov method . We obtained numerical solutions by implementing MATLAB algorithm to obtain bound state energies for both Schrodinger and Klein-Gordon equations. Numerical bound state energies increases with an increase in quantum state with respect to the adjustable parameter. With application of nonrelativistic limit, the energy eigen equation of Klein-Gordon equation is converted to that of Schrodinger equation. The proposed potential reduces to three potentials namely: Hulthen, Yukawa and exponential Mie-Type potential. The results for some of deduced potential are in agreement to that of existing literature. The bound state energy spectral diagram for both cases shows quantization of distinct energy levels. The negative energies in Schrodinger equation ascertain bound state condition describing the particle states (negative energy) of the system while the bound state energies from Klein-Gordon equation described anti-particles (positive energy).

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