Original Research Article

Weak Competition and Ideally Distributed Populations in a Cooperative Diffusive Model with Crowding Effects

Abstract

We study a dynamic model describing the cooperation-competition between two species, where the first species diffuses along a smooth distribution function while the second is dispersive randomly. The analysis is designed for weak competition with corresponding coefficients and by considering different resource functions. It is shown that the directed diffusion population has evolutionary advantages to design its own niche. The higher carrying capacity is an important issue of persistence. If there is a combination of two strategies adopted by the two species then the ideal free distribution is attained and the coexistence steady state is a global attractor.

Keywords: Diffusion strategies; global attractor; competition; cooperation; coexistence

AMS subject classification: 92D25, 35K57 (primary), 35K50, 37N25

1 Introduction

Mathematical modeling is always an important tool in economy and ecology to describe the characteristic life of populations in nature to scientific structures. Instantaneously, for example, the mathematical model used to describe some favorite features to predict the dynamics of

- competition
- cooperation
- mutualistic relation, and
- predator-prey interactions.

At present time and in the past two decades, the reaction-diffusion model including standard dispersion was considered in the literature [3, 4, 7, 8, 9, 10, 17] and references therein. One essential and important observation is that the slowest diffuser is the sole winner in competition and become a justification of the standard dispersion and was established in [7]. Crowdiness effects are another important issue for Lotka type models and were studied in [8]. A nonlinear system of initial boundary value problem was considered in [9]; and they investigated the solutions of a predator-prey model that generally covered a wide class of reaction-diffusion equations.

In this paper, we consider a problem of two interacting species competing in a non-homogeneous habitat for the fundamental resources such as

- water and food
- shelter and territory
- light or any means to maintain life and reproduce.

The diffusion script of the system is dissimilar for each resident. The diffusion motion of one population is influenced by a distribution function introduced in [2] while the other one is dispersing classically.

The notion of the ideal free distribution from ecology distinguishes how animals optimally distribute themselves crosswise the habitats. Generally, regular dispersal strategies cannot conduct to achieve the ideal free distribution in a spatially heterogeneous environment. The environmental gradient corresponds to the reaction-diffusion-advection model with the combination of directed and regular movement and was studied in [1, 4, 5, 6]. For a particular dynamical system when the ratio of the diffusion and the advection coefficients tends to zero then for such problem the solutions tend to be ideally distributed. If there is any movement of ideally distributed populations, the system will decrease the total fitness of traveling polls. An ideal free distribution can be gained for a measurable rate of advection in the pattern considered by [6] and recently this result was explored and upgraded by [1]. One important target of this paper is to develop the ideal free solution by considering a system with divergent diffusion strategies. It is noted that the following diffusion model (1.1) was studied in my PhD thesis [1].

In the present work, the model is defined in the following way: out of two diffusion strategies, the first population is diffusing with the positive distribution $P(x)$ while the other species is diffusing classically (randomly). Considering different carrying capacities of the two species, the system is governing by the following equations with homogeneous Neumann boundary conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta \left(\dfrac{u(t,x)}{P(x)} \right) + u(t,x) \left(K_1(x) - u(t,x) - \mu v(t,x) \right), \, t > 0, \; x \in \Omega, \\ \dfrac{\partial v}{\partial t} = d_2 \nabla \cdot \frac{\nabla v(t,x)}{P(x)} + v(t,x) \left(K_2(x) - \nu u(t,x) - v(t,x) \right), \, t > 0, \; x \in \Omega, \\ \dfrac{\partial (u/P)}{\partial n} = \dfrac{\partial v}{\partial n} = 0, \; x \in \partial\Omega, \\ u(t_0, x) = u_0(x), \; v(t_0, x) = v_0(x), \; x \in \Omega. \end{cases} \tag{1.1}$$

For functions K_1 and K_2, we assume that either $K_1 \not\equiv K_2$ on a nonempty open domain or $K_1 \equiv K_2$ for any $x \in \Omega$ and both are positive. The functions $u(t,x)$ and $v(t,x)$ represent the densities of the two competing species with corresponding diffusion rates $d_1 > 0$ and $d_2 > 0$. Here Ω is a bounded smooth domain in $\mathbb{R}^n$ with boundary $\partial\Omega$. The constants μ, ν account for competitive interactions between two species. We also assume that $u(t_0,x)$ and $v(t_0,x)$ are smooth enough, non-negative and not identically zero in $\overline{\Omega}$. The function $P(x)$ is in the class of $C^{1+\alpha}(\overline{\Omega})$, $\alpha > 0$. We have two important assumptions throughout the paper:

[1] http://theses.ucalgary.ca/bitstream/11023/2771/1/ucalgary_2016_kamrujjaman_md.pdf

1. Distribution function, $P(x) \not\equiv const$ and is smooth enough.
2. Competition coefficients $\mu \in (0,1]$ and $\nu \in (0,1]$.

Shortly, we summarize some important results for this type of model (1.1) studied recently in the literatures [12, 13, 14].

1. If $\mu = \nu = 1$ and $P(x) \equiv K_1(x) \equiv K_2(x)$, the steady state $(K(x), 0)$ of (1.1) is globally asymptotically stable [12, 13], where a system of equation was investigated for logistic type growth [12] as well as for variety of growth laws [13]. In a heterogeneous environment, it is shown that the species distributed by the carrying capacity only survives.
2. If $\mu = \nu = 1$, $P(x) \equiv K_1(x) \not\equiv K_2(x)$ and $K_1(x) > K_2(x)$ in a nonempty open subdomain, the semi-trivial equilibrium $(K_1(x), 0)$ of (1.1) is globally asymptotically stable [14] for multiple growth functions. It is proven that the population assigned with higher carrying capacity is in advantageous situation.

When the movement of both species is affected according to the distribution functions then that type of model was considered in [15, 16]. They established the competitive exclusion of one species by the other and the cooperative scenarios between two populations due to the effect of competition coefficients, the influence of diffusion coefficients and intrinsic growth rates.

Out of many issues, the key ideas due to the following novelties in this paper:

- We consider the problem describe in (1.1) to show the effects of constant competition coefficients for rational and arbitrary functions.
- For unique competition coefficients, diversity of diffusion strategies provide ideal free solution for non-proportional carrying capacity and distribution function.
- In the reaction parts, the characterization $K_1(x) \not\equiv K_2(x)$ is referred to as the crowdiness effect, where the two species have similar physical attributes.
- At the end, we outline the effects of space-variable interactions for further research.

The paper is organized in the following way. In section 2, we establish some preliminary results for single species and for a couple of species and these will be used in the rest of the part. The global analysis of equilibria are investigated in section 3 for competition coefficients $\mu \le 1$ and $\nu \le 1$.

Here we construct the following results:

1. If $\mu = \nu = 1$ and $K(x) \equiv \alpha P(x) + \beta \int\limits_{\Omega} P(x)\, dx$ then there exists a unique ideal free distribution $(\alpha P(x), \beta \int\limits_{\Omega} P(x)\, dx)$ which is stable for arbitrary diffusion coefficients.
2. Including other assumptions as above, if $\mu, \nu \in (0,1)$ then the system (1.1) has a stable coexistence solution.
3. For arbitrary $P(x)$ and $K(x)$, if $\mu \in (0, \mu_*)$, $\nu \in (0, \nu_*)$ then the problem (1.1) has at least one coexistence solution (u_s, v_s) independently of the diffusion coefficients.

4. The system (1.1) has a stable coexistence solution if the deviation between two resource functions is bounded and very small.

Section 4 deals with the effects of crowding tolerance. This segment illustrates the dynamics for different distributions of $P(x)$, $K_1(x)$ and $K_2(x)$ with $\mu = \nu = 1$. If the ratio of $P(x)$ and $K_1(x)$ is a positive constant and $K_1(x) \geq K_2(x)$ in a nonempty open domain then the semi-trivial equilibrium $(u^*(x), 0)$ is globally asymptotically stable.

Finally, portion 5 presents summary of the results and we introduce the spatially distributed competition coefficients and edited the problem (1.1) for further research. In that case, we establish some results by considering $\mu(x) \equiv \frac{K(x)-P(x)}{Q(x)} > 0$ and $\nu(x) \equiv \frac{K(x)-Q(x)}{P(x)} > 0$ and $d_1 \neq d_2$.

2 Preliminaries

Several next results correspond to the stationary solution of the monotone dynamical system (1.1) considering the case of single-species.

We assume that the function $u^*(x)$ is the unique solution of the following single-species boundary value problem (BVP) when the species v is identically equal to zero in (1.1)

$$d_1 \Delta \left(\frac{u^*(x)}{P(x)} \right) + u^*(x) \left(K_1(x) - u^*(x) \right) = 0, \ x \in \Omega, \ \frac{\partial (u^*/P)}{\partial n} = 0, \ x \in \partial\Omega. \tag{2.1}$$

Proposition 1. *[12, 13, 16] Let $K_1(x) \not\equiv const$ and if $P(x)$ and $K_1(x)$ are linearly independent then*

$$\int_\Omega P(x) \left(u^*(x) - K_1(x) \right) dx = d_1 \int_\Omega \frac{|\nabla (u^*/P)|^2}{(u^*/P)^2} \, dx > 0. \tag{2.2}$$

and

$$\int_\Omega K_1(x)(K_1(x) - u^*(x)) \, dx > 0. \tag{2.3}$$

Similarly, for single-species v, let us assume that $v^*(x)$ is the unique positive solution of the equation

$$d_2 \nabla \cdot \frac{\nabla v^*(x)}{P(x)} + v^*(x) \left(K_2(x) - v^*(x) \right) = 0, \ x \in \Omega, \ \frac{\partial v^*}{\partial n} = 0, \ x \in \partial\Omega. \tag{2.4}$$

The result of the following can be justified similarly to proposition 1.

Proposition 2. *Suppose that $K_2(x) \not\equiv const$, $P(x)$ and $K_2(x)$ are linearly independent and $v^*(x)$ is a positive solution of (2.4) then*

$$\int_\Omega \left(v^*(x) - K_2(x) \right) dx = \int_\Omega \frac{d_2}{P(x)} \frac{|\nabla v^*(x)|^2}{v^{*2}(x)} \, dx > 0. \tag{2.5}$$

and

$$\int_\Omega K_2(x)(K_2(x) - v^*(x)) \, dx > 0. \tag{2.6}$$

Proof. Since $v^* > 0$, dividing the first equation of (2.4) by v^*, we obtain

$$d_2 \frac{\nabla \cdot \frac{\nabla v^*(x)}{P(x)}}{v^*} + (K_2(x) - v^*(x)) = 0,\ x \in \Omega,\ \frac{\partial v^*}{\partial n} = 0,\ x \in \partial\Omega. \tag{2.7}$$

Integrating (2.7) over the domain Ω using boundary conditions in (2.7), we have

$$d_2 \int\limits_\Omega \frac{|\nabla v^*|^2}{P(x) v^{*2}}\, dx + \int\limits_\Omega (K_2(x) - v^*(x))\, dx = 0. \tag{2.8}$$

Therefore,

$$\int\limits_\Omega (v^*(x) - K_2(x))\, dx = \int\limits_\Omega \frac{d_2}{P(x)} \frac{|\nabla v^*|^2}{v^{*2}}\, dx > 0,\ \text{unless } v^*(x) = const. \tag{2.9}$$

But $v^* = const$ is not a solution of (2.4) as long as $K_2(x) \not\equiv const$.
The second part of the proof can be established by directly integrating the equation of v in (2.4) and hence the details proof is omitted.

□

The instability of trivial equilibrium was shown in the following result.

Lemma 1. *[12, 13, 14] The zero solution $(0,0)$ of (1.1) is unstable and its a repeller.*

Proposition 3. *Assume that (u_s, v_s) is a strictly positive stationary solution of (1.1), $K_1 \equiv K(x) \equiv K_2$ and $\mu,\ \nu \in (0,1]$. Then*

$$\int\limits_\Omega K(x)\,(K(x) - \nu u_s - \mu v_s)\, dx \geq \int\limits_\Omega (\nu u_s + \mu v_s - K)^2\, dx, \tag{2.10}$$

where equality is attained in (2.10) only when $\mu = \nu = 1$. The inequality (2.10) is strictly positive unless $\nu u_s(x) + \mu v_s(x) \equiv K(x)$.

Proof. Assume that there exists a stationary positive solution $(u_s(x), v_s(x))$ and the equilibrium $(u_s(x), v_s(x))$ of (1.1) satisfies

$$\begin{cases} d_1 \Delta \left(\dfrac{u_s(x)}{P(x)} \right) + u_s(x)\,(K(x) - u_s(x) - \mu v_s(x)) = 0,\ x \in \Omega, \\ d_2 \nabla \cdot \left(\frac{\nabla v_s(x)}{P(x)} \right) + v_s(x)\,(K(x) - \nu u_s(x) - v_s(x)) = 0,\ x \in \Omega, \\ \dfrac{\partial (u_s/P)}{\partial n} = \dfrac{\partial v_s}{\partial n} = 0,\ x \in \partial\Omega. \end{cases} \tag{2.11}$$

Multiplying the first equation of (2.11) by ν, second by μ adding them and integrating over Ω using the boundary conditions, we obtain

$$\begin{aligned} 0 &= \int\limits_\Omega \left[K(x)(\nu u_s + \mu v_s) - \left(\nu u_s^2 + \mu v_s^2 + 2\nu\mu u_s v_s \right) \right] dx \\ &\leq \int\limits_\Omega \left[K(x)(\nu u_s + \mu v_s) - \left(\nu^2 u_s^2 + \mu^2 v_s^2 + 2\mu\nu u_s v_s \right) \right] dx,\ \text{since } \nu \leq 1,\ \mu \leq 1 \\ &= \int\limits_\Omega \left[K(x)(\nu u_s + \mu v_s) - (\nu u_s + \mu v_s)^2 \right] dx = \int\limits_\Omega (\nu u_s + \mu v_s)\,(K(x) - \nu u_s - \mu v_s)\, dx. \end{aligned}$$

It follows that

$$\int_{\Omega} (\nu u_s + \mu v_s) (K(x) - \nu u_s - \mu v_s) \, dx \geq 0. \tag{2.12}$$

Integrating both sides of $(\nu u_s+\mu v_s)(K(x) - \nu u_s - \mu v_s) = (\nu u_s+\mu v_s - K(x))(K(x) - \nu u_s - \mu v_s)$ $+ K (K(x) - \nu u_s - \mu v_s)$ over Ω yields the following integral

$$\int_{\Omega} K(x) (K(x) - \nu u_s - \mu v_s) \, dx \geq \int_{\Omega} (\nu u_s + \mu v_s - K)^2 \, dx > 0. \tag{2.13}$$

Here equality holds if $\mu = \nu = 1$ and then (2.13) is valid unless $\nu u_s(x) + \mu v_s(x) \equiv K(x)$. □

3 Global Analysis of Steady States

When only one population survives, we will state the results on stability of two semi-trivial equilibrium of (1.1), which are $(u^*(x), 0)$, $(0, v^*(x))$. The stationary solution $(u_s(x), v_s(x))$, if it exists, that is neither a trivial nor a semi-trivial equilibrium and satisfy the positivity $u_s > 0, \; v_s > 0$, then we have a coexistence solution.

Let us nominating

$$I_c := \alpha \int_{\Omega} P(x) \, dx > 0, \; \alpha > 0, \tag{3.1}$$

and we will use this notation in further study.

If $(u_s(x), v_s(x))$ is any stationary coexistence solution of (1.1) and $K_2(x) \equiv K(x)$ then the eigenvalue problem of the second equation of (1.1) around $(u^*(x), 0)$ is

$$d_2 \nabla \cdot \frac{\nabla \phi(x)}{P(x)} + \phi(x) (K(x) - \nu u^*(x)) = \sigma \phi(x), \; x \in \Omega, \; \frac{\partial \phi}{\partial n} = 0, \; x \in \partial\Omega. \tag{3.2}$$

3.1 Case of Identical Resource Function: $K_1 \equiv K(x) \equiv K_2$

Let us now explore the results for the case of equivalent carrying capacity. If the distribution function $P(x)$ is proportional to the carrying capacity $K(x)$ then we have the following result as a remark.

Remark 1. *Suppose that $K_1(x) \equiv K(x) \equiv K_2(x) \not\equiv const$, $P(x)/K(x) \equiv const$ and $\mu = \nu = 1$. Then the steady state $(u^*(x), 0)$ of (1.1) is globally asymptotically stable.*

In the following portion, we consider the arbitrary functions $P(x)$ and $K(x) \equiv K_1(x) \equiv K_2(x)$.

Lemma 2. *Assume that $K_1(x) \equiv K(x) \equiv K_2(x) \not\equiv const$, and $K(x) \equiv \alpha P(x) + \beta \int_{\Omega} P(x) \, dx$, $\alpha > 0, \; \beta > 0$. Then the semi-trivial steady state $(u^*(x), 0)$ of (1.1) is unstable if $\nu \leq 1$.*

Proof. The principal eigenvalue [4] of (3.2) around $(u^*(x), 0)$ is defined as

$$\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} \left[-d_2 \int_{\Omega} \frac{|\nabla \phi|^2}{P(x)} \, dx + \int_{\Omega} \phi^2 (K(x) - \nu u^*(x)) \, dx \right] \Bigg/ \int_{\Omega} \phi^2 \, dx.$$

Choosing the eigenfunction $\phi(x) = \sqrt{K(x) - \alpha P(x)} = const$ and denoting $I_p = \beta \int\limits_{\Omega} \int\limits_{\Omega} P(x)\, dx\, dx$, the principal eigenvalue σ_1 is given by

$$\begin{aligned}
\sigma_1 &\geq \frac{1}{I_p} \int\limits_{\Omega} (K(x) - \alpha P(x))\,(K(x) - \nu u^*(x))\; dx \\
&\geq \frac{1}{I_p} \int\limits_{\Omega} (K(x) - \alpha P(x))\,(K(x) - u^*(x))\; dx, \text{ if } \nu \leq 1 \\
&= \frac{1}{I_p} \int\limits_{\Omega} K(x)\,(K(x) - u^*(x))\; dx + \frac{\alpha}{I_p} \int\limits_{\Omega} P(x)\,(u^*(x) - K(x))\; dx.
\end{aligned}$$

While $K(x) \equiv K_1(x) \equiv K_2(x)$, from Proposition 1, we have $\int\limits_{\Omega} P(x)\,(u^*(x) - K(x))\; dx > 0$ and $\int\limits_{\Omega} K(x)\,(K(x) - u^*(x))\; dx > 0$. Therefore, σ_1 is strictly positive. □

Lemma 3. *Suppose that* $K_1(x) \equiv K(x) \equiv K_2(x) \not\equiv const$, *and* $K(x) \equiv \alpha P(x) + \beta \int\limits_{\Omega} P(x)\, dx$, $\alpha > 0,\ \beta > 0$. *Then the semi-trivial steady state* $(0, v^*(x))$ *of (1.1) is unstable if* $\mu \leq 1$.

Proof. Consider the associated eigenvalue problem of the first equation of (1.1) around $(0, v^*(x))$

$$d_1 \Delta \left(\frac{\phi(x)}{P(x)} \right) + \phi(x)\,(K(x) - \mu v^*(x)) = \sigma \phi(x),\ x \in \Omega,\ \frac{\partial(\phi/P)}{\partial n} = 0,\ x \in \partial\Omega. \tag{3.3}$$

The principal eigenvalue of (3.3) is

$$\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} \left[-d_1 \int\limits_{\Omega} |\nabla(\phi/P)|^2\, dx + \int\limits_{\Omega} \frac{\phi^2}{P}\,(K(x) - \mu v^*)\; dx \right] \Bigg/ \int\limits_{\Omega} \frac{\phi^2}{P}\, dx.$$

Taking positive eigenfunction $\phi(x) = \sqrt{\alpha} P(x)$, addressing I_c defined in (3.1) and if $K(x) \equiv \alpha P(x) + \beta \int\limits_{\Omega} P(x)\, dx$, we obtain

$$\begin{aligned}
\sigma_1 &\geq \frac{1}{I_c} \int\limits_{\Omega} \alpha P(x)\,(K(x) - \mu v^*(x))\; dx \\
&\geq \frac{1}{I_c} \int\limits_{\Omega} \alpha P(x)\,(K(x) - v^*(x))\; dx, \text{ since } \mu \leq 1 \\
&= \frac{1}{I_c} \int\limits_{\Omega} (\alpha P(x) - K(x) + K(x))\,(K(x) - v^*(x))\; dx \\
&= \frac{1}{I_c} \int\limits_{\Omega} K(x)\,(K(x) - v^*(x))\; dx + \frac{1}{I_c} \int\limits_{\Omega} (K(x) - \alpha P(x))\,(v^*(x) - K(x))\; dx \\
&= \frac{1}{I_c} \int\limits_{\Omega} K(x)\,(K(x) - v^*(x))\; dx + \frac{\beta}{I_c} \int\limits_{\Omega} \int\limits_{\Omega} P(x)\, dx\,(v^*(x) - K(x))\; dx.
\end{aligned}$$

Now for arbitrary positive and smooth function $P(x)$, the fact is $\int_\Omega P(x)\,dx = c > 0$, where c is a constant and we obtain

$$\sigma_1 \geq \frac{1}{I_c}\int_\Omega K(x)\,(K(x) - v^*(x))\;dx + \frac{c\beta}{I_c}\int_\Omega (v^*(x) - K(x))\;dx.$$

Next, proposition 2 implies that both integrals $\int_\Omega K(x)\,(K(x) - v^*(x))\;dx$ and $\int_\Omega (v^*(x) - K(x))\;dx$ are strictly positive as long as $K_1(x) \equiv K(x) \equiv K_2(x) \not\equiv const$. Therefore, $\sigma_1 > 0$ and the proof follows. □

Lemma 4. *Let $K_1(x) \equiv K(x) \equiv K_2(x) \not\equiv const$, $K(x) \equiv \alpha P(x) + \beta \int_\Omega P(x)\,dx$, $\alpha > 0,\ \beta > 0$, and $\mu = \nu = 1$ then the system (1.1) has a unique positive coexistence equilibrium $(u_s, v_s) \equiv (\alpha P(x), \beta \int_\Omega P(x)\,dx)$.*

Proof. For a stationary solution (u_s, v_s) under the assumption $K_1(x) \equiv K(x) \equiv K_2(x)$, the problem (1.1) can be written as

$$\begin{cases} d_1\Delta\left(\dfrac{u_s(x)}{P(x)}\right) + u_s(x)\,(K(x) - u_s(x) - v_s(x)) = 0,\ x \in \Omega, \\ d_2\nabla\cdot\left(\frac{1}{P(x)}\nabla v_s(x)\right) + v_s(x)\,(K(x) - u_s(x) - v_s(x)) = 0,\ x \in \Omega, \\ \dfrac{\partial(u_s/P)}{\partial n} = \dfrac{\partial v_s}{\partial n} = 0,\ x \in \partial\Omega. \end{cases} \tag{3.4}$$

By direct substitution, one can verify that $(\alpha P(x), \beta \int_\Omega P(x)\,dx)$ is a coexistence solution of (3.4). To prove the uniqueness, assume that there is an another solution (u_s, v_s) of (3.4) except $(\alpha P(x), \beta \int_\Omega P(x)\,dx)$.

The following result comes from the equation (3.4) for $v_s > 0$ and we obtain

$$\int_\Omega (u_s + v_s - K(x))\;dx = \int_\Omega \frac{d_2}{P}\frac{|\nabla v_s|^2}{v_s^2}\,dx \geq 0. \tag{3.5}$$

The equality is attained in (3.5) only when $v_s \equiv const$.

We have to show that $u_s(x) + v_s(x) \equiv K(x)$.

Let us define the eigenvalue problem by assuming to the contrary that $u_s(x)+v_s(x) \not\equiv K(x)$

$$d_1\Delta\left(\frac{\phi(x)}{P(x)}\right) + \phi(x)\,(K(x) - u_s - v_s) = \sigma\phi(x),\ x \in \Omega,\ \frac{\partial(\phi/P)}{\partial n} = 0,\ x \in \partial\Omega. \tag{3.6}$$

The principal eigenvalue of (3.6) is

$$\sigma_1 = \sup_{\phi\neq 0,\phi\in W^{1,2}} \left[-d_1\int_\Omega |\nabla(\phi/P)|^2\,dx + \int_\Omega \frac{\phi^2}{P}\,(K(x) - u_s - v_s)\;dx\right]\Bigg/\int_\Omega \frac{\phi^2}{P}\,dx.$$

Substituting $\phi(x) = \sqrt{\alpha}P(x)$, using I_c defined in (3.1) and using the notation as declared in Lemma 3, $\int\limits_\Omega P(x)\,dx = c > 0$, we have

$$\begin{aligned}
\sigma_1 &\geq \frac{1}{I_c}\int\limits_\Omega (\alpha P(x) - K(x) + K(x))\,(K(x) - u_s - v_s)\;dx \\
&= \frac{\beta}{I_c}\int\limits_\Omega\int\limits_\Omega P(x)\,dx\,(u_s + v_s - K(x))\;dx + \frac{1}{I_c}\int\limits_\Omega K(x)\,(K(x) - u_s - v_s)\;dx \\
&= \frac{c\beta}{I_c}\int\limits_\Omega (u_s + v_s - K(x))\;dx + \frac{1}{I_c}\int\limits_\Omega K(x)\,(K(x) - u_s - v_s)\;dx.
\end{aligned}$$

If $\mu = \nu = 1$, proposition 3 becomes $\int\limits_\Omega K(x)\,(K(x) - u_s - v_s)\;dx > 0$. Thus, $\sigma_1 > 0$ using (3.5) and by Proposition 3. The zero principal eigenvalue of (3.6) contradicts the positivity of $\sigma_1 > 0$ and thus $u_s(x) + v_s(x) \equiv K(x)$.

Next, if $u_s(x) + v_s(x) \equiv K(x)$ then by the Maximum Principle [11], $w_s = const$ and $v_s = const$ in (1.1), where $u_s/P = w_s$. So we must have $P(x)w_s + v_s \equiv \alpha P(x) + \beta\int\limits_\Omega P(x)\,dx$ and this implicity implies that $w_s = \alpha$ and $v_s = \beta\int\limits_\Omega P(x)\,dx$. Hence the unique solution of (1.1) is $(u_s, v_s) = (\alpha P(x), \beta\int\limits_\Omega P(x)\,dx)$. $\square$

For the monotone dynamical system (1.1), if all equilibriums are unstable except one then we can conclude that the remaining steady state is globally asymptotically stable. The next theorem shows that the coexistence equilibrium (u_s, v_s) of (1.1) remains globally asymptotically stable regardless of the initial functions.

Theorem 1. *Let $K_1(x) \equiv K(x) \equiv K_2(x) \not\equiv const$, $K(x) \equiv \alpha P(x) + \beta\int\limits_\Omega P(x)\,dx$, $\alpha > 0$, $\beta > 0$ and $\mu = \nu = 1$. Then there exists a unique coexistence solution $(u_s, v_s) \equiv (\alpha P(x), \beta\int\limits_\Omega P(x)\,dx)$ of (1.1) which is globally asymptotically stable. Moreover, if $\mu < 1$ and $\nu < 1$, the system (1.1) has a stable coexistence solution (u_s, v_s).*

Theorem 2. *Assume that $K_1(x) \equiv K(x) \equiv K_2(x)$ and let $P(x)$ and $K(x)$ are non-constant and arbitrary. Then there exists positive μ_* and ν_* such that for $\mu \in (0, \mu_*)$, $\nu \in (0, \nu_*)$, the problem (1.1) has a stable coexistence solution (u_s, v_s).*

Proof. It is sufficient to show that two semi-trivial equilibria $(u^*, 0)$ and $(0, v^*)$ are unstable. Let us define

$$a = \int\limits_\Omega P(x)K(x)\,dx > 0, \quad b = \int\limits_\Omega K(x)\,dx > 0$$

such that

$$\mu_* = \min\left\{\frac{a}{\int\limits_\Omega P(x)v^*(x)\,dx}, 1\right\}, \quad \nu_* = \min\left\{\frac{b}{\int\limits_\Omega u^*(x)\,dx}, 1\right\}. \tag{3.7}$$

If $\mu < \mu_*$ then $\int\limits_\Omega P(x)K(x)\,dx > \mu \int\limits_\Omega P(x)v^*(x)\,dx$ which implies that

$$\int\limits_\Omega P(x)\left(K(x) - \mu v^*(x)\right)\,dx > 0. \tag{3.8}$$

In a similar fashion, for $\nu < \nu_*$, we have

$$\int\limits_\Omega \left(K(x) - \nu u^*(x)\right)\,dx > 0. \tag{3.9}$$

Linearize the second equation in (1.1) around $(u^*, 0)$ and consider the associated eigenvalue problem

$$d_2 \nabla \cdot \frac{\nabla \phi(x)}{P(x)} + \phi(x)\left(K(x) - \nu u^*(x)\right) = \sigma \phi(x),\ x \in \Omega,\ \frac{\partial \phi}{\partial n} = 0,\ x \in \partial\Omega. \tag{3.10}$$

The principal eigenvalue of (3.10) can be designated by

$$\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} [-d_2 \int\limits_\Omega \frac{|\nabla \phi|^2}{P(x)}\,dx + \int\limits_\Omega \phi^2\left(K(x) - \nu u^*(x)\right)\,dx] / \int\limits_\Omega \phi^2\,dx.$$

Sorting constant $\phi(x)$ and using (3.9), we have $\sigma_1 = \frac{1}{|\Omega|}\int\limits_\Omega \left(K(x) - \nu u^*(x)\right)\,dx > 0$. Therefore, $(u^*, 0)$ is not stable. For $\mu < \mu_*$, instability of $(0, v^*)$ is verified similarly. $\square$

3.2 Small Deviations between Two Carrying Capacities

This section continues the development of equilibrium analysis in case of $K_1(x) \equiv K_2(x) \pm \epsilon$, $\epsilon > 0$ as we did in previous sections for function $K_1(x) \equiv K_2(x)$ and directed distribution $P(x)$ while $\mu = \nu = 1$. The problem is linearized about the equilibria to determine the behavior of the models near the equilibria.

Theorem 3. *Let $K_1(x)$ and $K_2(x)$ be non-constant, $\mu = \nu = 1$, $K_1(x) \equiv P(x) + b$, and $K_2(x) \equiv P(x) + c$, where b and c are positive constants. Then the system (1.1) has a stable coexistence solution if $|K_2(x) - K_1(x)| < \epsilon$ for any $x \in \Omega$, where ϵ is very small and positive.*

Proof. It is simple to observe that all spatial functions $P(x)$, $K_1(x)$ and $K_2(x)$ are arbitrary. For small deviations between $K_1(x)$ and $K_2(x)$, we have either $K_1 - \epsilon < K_1 \leq K_2 < K_1 + \epsilon$ or $K_2 - \epsilon < K_2 \leq K_1 < K_2 + \epsilon$, where ϵ is small enough and positive. Also it is noted that either $K_1 \equiv K_2 + b^*$ or $K_2 \equiv K_1 + c^*$, where $b^* = b - c > 0$ and $c^* = c - b > 0$.

First, let us consider the case $K_1(x) \equiv P(x) + b$, $b > 0$, and $|K_2(x) - K_1(x)| < \epsilon$ with $K_2(x) \geq K_1(x)$. Then the principal eigenvalue of (1.1) around $(u^*(x), 0)$ is defined as

$$\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} \left[-d_2 \int\limits_\Omega \frac{|\nabla \phi|^2}{P(x)}\,dx + \int\limits_\Omega \phi^2\left(K_2(x) - u^*(x)\right)\,dx\right] \Bigg/ \int\limits_\Omega \phi^2\,dx. \tag{3.11}$$

It is noted that by construction, $K_1(x) > P(x)$ for any $x \in \Omega$. Considering $\phi(x) = \sqrt{K_1(x) - P(x)} = \sqrt{b} = b_* > 0$, the principal eigenvalue is

$$\begin{aligned}
\sigma_1 &\geq \frac{1}{b_*^2|\Omega|} \int_\Omega (K_1(x) - P(x))\,(K_2(x) - u^*(x))\,dx \\
&\geq \frac{1}{b_*^2|\Omega|} \int_\Omega (K_1(x) - P(x))\,(K_1(x) - u^*(x))\,dx,\ K_1 \leq K_2 < K_1 + \epsilon \\
&= \frac{1}{b_*^2|\Omega|} \left[\int_\Omega K_1(x)\,(K_1(x) - u^*(x))\,dx + \int_\Omega P(x)\,(u^*(x) - K_1(x))\,dx \right].
\end{aligned}$$

Hence, σ_1 is positive by Proposition 1.

The instability of $(0, v^*)$ is computed similarly if $K_2(x) \equiv P(x) + c$ and $K_1(x) \geq K_2(x)$ such that $|K_2(x) - K_1(x)| < \epsilon$ which due to $K_2 - \epsilon < K_2 \leq K_1 < K_2 + \epsilon$.

Since both semi-trivial equilibria are unstable and the trivial solution is unstable by Lemma 1, the proof follows in case of strong monotone dynamical system (1.1). □

4 Effects of Crowding Tolerance: $\mu = \nu = 1$

In this section, our study is exploring the analysis for non-symmetric growth functions due to $K_1(x) \not\equiv K_2(x)$.

Lemma 5. *Let $P(x)$, $K_1(x)$ and $K_2(x)$ be non-constant, $\mu = \nu = 1$, $K_2(x) \equiv P(x)+c$, $c > 0$, and $K_1(x) \geq K_2(x)$ for any $x \in \Omega$ and $K_1(x) > K_2(x)$ in a nonempty open domain. Then the equilibrium $(0, v^*(x))$ of (1.1) is unstable. Moreover, if $K_1 \equiv \alpha P$, $\alpha > 0$ and $K_1 \geq K_2$ on Ω then $(0, v^*(x))$ is also unstable.*

Proof. Let us study the eigenvalue problem of (1.1) around $(0, v^*(x))$ and we obtain

$$d_1 \Delta \left(\frac{\phi(x)}{P(x)} \right) + \phi(x)\,(K_1(x) - v^*(x)) = \sigma \phi(x),\ x \in \Omega,\ \frac{\partial(\phi/P)}{\partial n} = 0,\ x \in \partial\Omega. \tag{4.1}$$

Considering $\phi(x) = \sqrt{\alpha} P(x)$ and inviting I_c drafted in (3.1), the principal eigenvalue of (4.1) is given by

$$\begin{aligned}
\sigma_1 &\geq \frac{\alpha}{I_c} \int_\Omega P(x)\,(K_1(x) - v^*(x))\,dx \\
&\geq \frac{\alpha}{I_c} \int_\Omega P(x)\,(K_2(x) - v^*(x))\,dx,\ \text{when}\ \ K_1 \geq K_2 \\
&= \frac{\alpha}{I_c} \int_\Omega (K_2(x) - c)\,(K_2(x) - v^*(x))\,dx,\ \text{while}\ \ P(x) \equiv K_2(x) - c \\
&= \frac{\alpha}{I_c} \int_\Omega K_2(x)\,(K_2(x) - v^*(x))\,dx + \frac{c\alpha}{I_c} \int_\Omega (v^*(x) - K_2(x))\,dx.
\end{aligned}$$

Thus, $\sigma_1 > 0$ by Proposition 2.

Next, if $K_1 \equiv \alpha P$, $\alpha > 0$ then we have

$$\sigma_1 \geq \frac{\alpha}{I_c} \int_\Omega P(x) \left(K_1(x) - v^*(x)\right) dx = \frac{1}{I_c} \int_\Omega K_1(x) \left(K_1(x) - v^*(x)\right) dx.$$

In absence of species u, $v^*(x)$ is the solution of (2.4) and after integrating the equation (2.4) over Ω using boundary conditions, we have $\int_\Omega v^* \left(K_2(x) - v^*(x)\right) dx = 0$ such that

$$0 = \int_\Omega v^* \left(K_2(x) - v^*(x)\right) dx \leq \int_\Omega v^* \left(K_1(x) - v^*(x)\right) dx, \text{ while } K_1 \geq K_2.$$

Therefore, $\int_\Omega K_1(x) \left(K_1(x) - v^*(x)\right) dx \geq 0$ and the inequality is strict since $K_1 \not\equiv const \not\equiv v^*$ and hence $\sigma_1 > 0$. $\square$

Lemma 6. *Assume that $P(x)$, $K_1(x)$ and $K_2(x)$ are non-constant and $\mu = \nu = 1$. If $K_1(x) \equiv P(x)+b$, $b > 0$, and $K_2(x) > K_1(x)$ for any $x \in \Omega$ then the semi-trivial equilibrium $(u^*(x), 0)$ of (1.1) is unstable.*

Proof. The principal eigenvalue of (1.1) around $(u^*(x), 0)$ is defined as

$$\begin{aligned}
\sigma_1 &= \sup_{\phi\neq 0, \phi\in W^{1,2}} \left[-d_2 \int_\Omega \frac{1}{P(x)} |\nabla\phi|^2 \, dx + \int_\Omega \phi^2 \left(K_2(x) - u^*(x)\right) dx \right] \Big/ \int_\Omega \phi^2 \, dx \\
&\geq \sup_{\phi\neq 0, \phi\in W^{1,2}} \left[-d_2 \int_\Omega \frac{1}{P(x)} |\nabla\phi|^2 \, dx + \int_\Omega \phi^2 \left(K_1(x) - u^*(x)\right) dx \right] \Big/ \int_\Omega \phi^2 \, dx
\end{aligned}$$

where $K_2(x) > K_1(x)$ in a nonempty open domain.

For eigenfunction $\phi(x) = \sqrt{K_1(x) - P(x)} = \sqrt{a}$, and designating $I_a = \int_\Omega a \, dx$, the principal eigenvalue becomes

$$\sigma_1 \geq \frac{1}{I_a} \int_\Omega K_1(x) \left(K_1(x) - u^*(x)\right) dx + \frac{1}{I_a} \int_\Omega P(x) \left(u^*(x) - K_1(x)\right) dx. \tag{4.2}$$

Hence, σ_1 is positive by Proposition 1. $\square$

Lemma 7. *Let $P(x)$, $K_1(x)$ and $K_2(x)$ be non-constant and $\mu = \nu = 1$. If $K_1(x) \equiv \alpha P(x)$, $\alpha > 0$ and $K_1(x) \geq K_2(x)$ in some nonempty open domain then (1.1) has no coexistence solution $(u_s(x), v_s(x))$.*

Proof. Let us assume that there is a stationary solution $(u_s(x), v_s(x))$ and at the end we will show the contradictory results. For $(u_s(x), v_s(x))$, the problem (1.1) is as follows:

$$\begin{cases}
d_1 \Delta \left(\dfrac{u_s(x)}{P(x)} \right) + u_s(x) \left(K_1(x) - u_s(x) - v_s(x)\right) = 0, \ x \in \Omega, \\
d_2 \nabla \cdot \left(\frac{1}{P(x)} \nabla v_s(x) \right) + v_s(x) \left(K_2(x) - u_s(x) - v_s(x)\right) = 0, \ x \in \Omega, \\
\dfrac{\partial (u_s/P)}{\partial n} = \dfrac{\partial v_s}{\partial n} = 0, \ x \in \partial\Omega.
\end{cases} \tag{4.3}$$

Adding the equations of u_s and v_s in (4.3), integrating over Ω and employing $K_1(x) \geq K_2(x)$, we obtain

$$
\begin{aligned}
0 &= \int_\Omega u_s(x)\left(K_1(x) - u_s(x) - v_s(x)\right) dx + \int_\Omega v_s(x)\left(K_2 - u_s(x) - v_s(x)\right) dx \\
&\leq \int_\Omega \left(u_s(x) + v_s(x)\right)\left(K_1(x) - u_s(x) - v_s(x)\right),
\end{aligned}
$$

since $\Delta\left(\dfrac{u_s(x)}{P(x)}\right) = 0$ and $\nabla \cdot \left(\frac{1}{P(x)}\nabla v_s(x)\right) = 0$ due to the boundary conditions. Thus

$$
\int_\Omega \left(u_s(x) + v_s(x)\right)\left(K_1(x) - u_s(x) - v_s(x)\right) \geq 0
$$

which yields

$$
\int_\Omega K_1(x)\left(K_1(x) - u_s - v_s\right) dx \geq \int_\Omega \left(K_1(x) - u_s - v_s\right)^2 dx > 0 \tag{4.4}
$$

unless $u_s + v_s \equiv K_1$. The equality holds only for $K_1 \equiv K_2$. Hence, we have two cases:

Case 1: If $u_s + v_s \not\equiv K_1$, we consider the principal eigenvalue and obtain

$$
\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} \left[-d_1 \int_\Omega |\nabla(\phi/P)|^2 \, dx + \int_\Omega \frac{\phi^2}{P}\left(K_1(x) - u_s - v_s\right) dx \right] \Big/ \int_\Omega \frac{\phi^2}{P} \, dx.
$$

For a suitable selection of positive eigenfunction $\phi(x)$, the principal eigenvalue becomes

$$
\sigma_1 \geq \frac{1}{I_c} \int_\Omega \alpha P(x)\left(K_1(x) - u_s - v_s\right) dx = \frac{1}{I_c} \int_\Omega K_1(x)\left(K_1(x) - u_s - v_s\right) dx > 0 \tag{4.5}
$$

by inequality (4.4) and using the primary condition $K_1(x) \equiv \alpha P(x)$; a contradiction, $\sigma_1 > 0$ with the zero principal eigenvalue.

Case 2: If $u_s(x)+v_s(x) \equiv K_1(x) \equiv K_2(x)$, by the Maximum Principle [11] and introducing a new variable $u_s/P = w_s$, the solutions of (1.1) are $w_s = const$ and $v_s = const$. Then we must have $P(x)w_s + v_s \equiv K_1(\equiv K_2)$, which implies that $w_s = 1/\alpha$ and $v_s = 0$. A contradiction of zero solution follows the proof. □

Once again, for a monotone problem (1.1), the rest equilibrium $(u^*, 0)$ is the global attractor and the result is implemented in the following Theorem.

Theorem 4. *Let $P(x)$, $K_1(x)$ and $K_2(x)$ be non-constant and $\mu = \nu = 1$. If $K_1(x) \equiv \alpha P(x)$, $\alpha > 0$ and $K_1(x) \geq K_2(x)$ in some nonempty open domain then the semi-trivial equilibrium $(u^*(x), 0)$ of (1.1) is globally asymptotically stable.*

If the second species are in homogeneous environment while the functions $P(x)$ and $K_1(x)$ are arbitrary, we can explore the next few results.

Lemma 8. *Suppose that $P(x)$, $K_1(x)$ are non-constant, $K_2 \equiv const$ and $\mu = \nu = 1$. If $K_1(x) \equiv P(x) + b$, $b > 0$, and K_2 is the upper bound of $K_1(x)$ in a nonempty open domain then the semi-trivial steady state $(u^*(x), 0)$ of (1.1) is unstable.*

Lemma 6 is still valid for this case and the proof of Lemma 8 is omitted.

Lemma 9. *Let $P(x)$, $K_1(x)$ be non-constant, $K_2 \equiv const$ and $\mu = \nu = 1$. If $K_1(x) \equiv P(x) + b$, $b > 0$, and K_2 is the upper bound of $K_1(x)$ in a nonempty open domain then the system (1.1) has no coexistence solution.*

Proof. This result is proven by the method of contradiction. Assume that there is a stationary coexistence solution $(u_s(x), v_s(x))$, and the system (1.1) can be written as

$$\begin{cases} d_1 \Delta \left(\dfrac{u_s(x)}{P(x)} \right) + u_s(x) \left(K_1(x) - u_s(x) - v_s(x) \right) = 0, \ x \in \Omega, \\ d_2 \nabla \cdot \left(\frac{1}{P(x)} \nabla v_s(x) \right) + v_s(x) \left(K_2(x) - u_s(x) - v_s(x) \right) = 0, \ x \in \Omega, \\ \dfrac{\partial (u_s/P)}{\partial n} = \dfrac{\partial v_s}{\partial n} = 0, \ x \in \partial\Omega. \end{cases} \tag{4.6}$$

Adding first two equations of (4.6) and integrating over Ω, we obtain

$$\int_\Omega u_s(x) \left(K_1(x) - u_s(x) - v_s(x) \right) dx + \int_\Omega v_s(x) \left(K_2 - u_s(x) - v_s(x) \right) dx = 0, \tag{4.7}$$

since diffusion terms are vanishes due to the boundary conditions. For the upper bound of $K_1(x)$, we have $(K_2 - u_s(x) - v_s(x)) > (K_1(x) - u_s(x) - v_s(x))$ such that

$$\int_\Omega \left(u_s(x) + v_s(x) \right) \left(K_2 - u_s(x) - v_s(x) \right) > 0.$$

After few steps and notifying $c^* = K_2^{-1}$, we obtain

$$\int_\Omega \left(K_2 - u_s - v_s \right) dx > c^* \int_\Omega \left(K_2 - u_s - v_s \right)^2 dx > 0 \tag{4.8}$$

which excludes the possibility of $u_s + v_s \equiv K_2$. Then we consider the principal eigenvalue for $u_s + v_s \not\equiv K_2$ and obtain

$$\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} \left[-d_2 \int_\Omega \frac{|\nabla \phi|^2}{P(x)} dx + \int_\Omega \phi^2 \left(K_2 - u_s - v_s \right) dx \right] \Bigg/ \int_\Omega \phi^2 \, dx. \tag{4.9}$$

Choosing constant eigenfunction ϕ, the principal eigenvalue becomes

$$\sigma_1 \geq \frac{1}{|\Omega|} \int_\Omega \left(K_2 - u_s(x) - v_s(x) \right) dx > 0 \tag{4.10}$$

by inequality (4.8); a contradiction of the positivity of σ_1 follows the proof. □

Lemmata 8 and 9 follow due to the following result pertaining to the problem (1.1).

Theorem 5. *Suppose that $P(x)$, $K_1(x)$ are non-constant, $K_2 \equiv const$ and $\mu = \nu = 1$. If $K_1(x) \equiv P(x) + b$, $b > 0$, and K_2 is the upper bound of $K_1(x)$ in a nonempty open domain, the semi-trivial equilibrium $(0, v^*(x))$ of (1.1) is globally asymptotically stable.*

5 Summary and Further Work

We investigated a Lotka-Volterra type reaction-diffusion model that describes two species cooperative-competitive dynamics with different dispersal strategies. By considering non-homogeneous environment, we established several results. If the growth functions are symmetric and the carrying capacity is allotted in terms of distribution function, there is a unique coexistence solution. For weak competition with common resource area of both populations, there exists at least one stable coexistence solution.

If there is a small difference between two resource functions then, once again, the coexistence solution is stable. By setting competition coefficients equal to 1, if the resource and distribution functions changes rationally then the global stability for the species operated by the distribution $P(x)$, is guaranteed. If the second species is in homogeneous environment while the rest one is in heterogeneous niche, only the second species survives.

To expand the current model for further research, we can modify the problem (1.1) to introduce the new idea, where the competition coefficients are spatially distributed. In this problem, we introduce non-constant competition coefficients $\mu(x) > 0,\ \nu(x) > 0$ with a common carrying capacity $K(x)$ of both species in (1.1). Thus the problem (1.1) can be rewritten as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta \left(\dfrac{u(t,x)}{P(x)} \right) + u(t,x) \left(K(x) - u(t,x) - \mu(x) v(t,x) \right), \ t > 0, \ x \in \Omega, \\ \dfrac{\partial v}{\partial t} = d_2 \nabla \cdot \dfrac{1}{P(x)} \nabla v(t,x) + v(t,x) \left(K(x) - \nu(x) u(t,x) - v(t,x) \right), \ t > 0, \ x \in \Omega, \\ \dfrac{\partial (u/P)}{\partial n} = \dfrac{\partial v}{\partial n} = 0, \ x \in \partial\Omega, \\ u(0,x) = u_0(x), \ v(0,x) = v_0(x), \ x \in \Omega, \end{cases} \tag{5.1}$$

where

$$\mu(x) \equiv \frac{K(x) - P(x)}{Q(x)} > 0 \ \text{ and } \ \nu(x) \equiv \frac{K(x) - Q(x)}{P(x)} > 0 \tag{5.2}$$

such that $0 < P(x) < K(x)$ and $0 < Q(x) < K(x)$ for any x over the domain Ω. The function $Q(x)$ is in the class of $C^{1+\alpha}(\overline{\Omega}),\ \alpha > 0$ and is positive.

Let us explore some instant results of the system (5.1). For constant Q, the system has a coexistence solution and the solution is attractive globally under certain conditions of $P(x)$ and $K(x)$.

Theorem 6. *Let $Q \equiv const$ and $K(x) \equiv P(x) + Q$ for any $x \in \Omega$. Then the system (5.1) has a unique solution $(u_s, v_s) \equiv (P(x), Q)$, which is globally asymptotically stable.*

Proof. Following Lemma 2, σ_1, the principle eigenvalue of the equation of v in (5.1) around $(u^*, 0)$ is expressed as

$$\sigma_1 = \sup_{\phi \neq 0, \phi \in W^{1,2}} \left[-d_2 \int_\Omega \frac{1}{P(x)} |\nabla \phi|^2 \, dx + \int_\Omega \phi^2 \left(K(x) - \nu(x) u^*(x) \right) dx \right] \Big/ \int_\Omega \phi^2 \, dx$$

Selecting $\phi(x) = \sqrt{K(x) - P(x)} = \sqrt{Q} = const$, and denoting $I_Q = \int\limits_\Omega Q\,dx$, σ_1 is given by

$$\begin{aligned}\sigma_1 &\geq \frac{1}{I_Q}\int\limits_\Omega (K(x) - P(x))\left(K(x) - \frac{(K(x)-Q)u^*(x)}{P(x)}\right)dx \\ &\geq \frac{1}{I_Q}\int\limits_\Omega (K(x) - P(x))\,(K(x) - u^*(x))\;dx,\ \text{if } K(x) \equiv P(x) + Q \\ &= \frac{1}{I_Q}\int\limits_\Omega K(x)\,(K(x) - u^*(x))\;dx + \frac{1}{I_Q}\int\limits_\Omega P(x)\,(u^*(x) - K(x))\;dx.\end{aligned}$$

Therefore, σ_1 is strictly positive by Propositions 1.

The instability of $(0, v^*)$ is evaluated similarly. By extending the proof of Lemma 4, it is easy to establish that the system (5.1) has a unique coexistence solution as long as $K(x) \equiv P(x) + Q$.

□

Theorem 7. *Assume that $K(x) \equiv P(x) + c$, $c > 0$ and $K(x) \leq P(x) + Q(x)$ for any $x \in \Omega$. Then the system (5.1) has a stable coexistence solution (u_s, v_s).*

Proof. It can be checked that there exists some non-constant functions $P(x)$, $Q(x)$ and $K(x)$ such that $\frac{K(x)-P(x)}{Q(x)} \leq 1$ and $\frac{K(x)-Q(x)}{P(x)} \leq 1$, for all x in some nonempty open domain.

If $K(x) \equiv P(x)+c$ and $\frac{K(x)-Q(x)}{P(x)} \leq 1$, then by Propositions 1, we can prove that the semi-trivial equilibrium $(u^*, 0)$ is unstable. Similarly, it is also possible to show that the semi-trivial equilibrium $(0, v^*)$ is unstable by Proposition 2 while $K(x) \equiv P(x)+c$ and $\frac{K(x)-P(x)}{Q(x)} \leq 1$. □

References

[1] I. Averill, Y. Lou, and D. Munther, On several conjectures from evolution of dispersal, *J. Biol. Dyn.* **6** (2012), No. 2, 117–130.

[2] E. Braverman and L. Braverman, Optimal harvesting of diffusive models in a non-homogeneous environment, *Nonlin. Anal. Theory Meth. Appl.* **71** (2009), e2173–e2181.

[3] Y. Morita and K. Tachibana, An entire solution to the Lotka-Volterra competition-diffusion equations, *SIAM J. Math. Anal.*, **40** (2009), No. 6, 2217–2240.

[4] R. S. Cantrell and C. Cosner, Spatial Ecology via Reaction-diffusion Equations, *Wiley Series in Mathematical and Computational Biology*, John Wiley & Sons, Chichester, 2003.

[5] R. S. Cantrell, C. Cosner,Y. Lou, Approximating the ideal free distribution via reaction-diffusion-advection equations, *J. Differential Equations* **245** (2008), no. 12, 3687-3703.

[6] R. S. Cantrell, C. Cosner, Y. Lou, Evolution of dispersal and the ideal free distribution, *Math. Biosci. Eng.* **7** (2010), 17–36.

[7] J. Dockery, V. Hutson, K. Mischaikow, M. Pernarowski, The evolution of slow dispersal rates: a reaction diffusion model, *J. Math. Biol.* **37** (1) (1998), 61–83.

[8] A. Leung, Limiting behaviour for a prey-predator model with diffusion and crowding effects, *J. Math. Biol.*, **6** (1978), 87–93.

[9] S. Williams and P. Chow, Nonlinear reaction-diffusion models for interacting populations, *J. Math. Anal. Appl.*, **62** (1978), 157–159.

[10] X. Q. He and W. M. Ni, The effects of diffusion and spatial variation in Lotka-Volterra competition-diffusion system I: Heterogeneity vs. homogeneity, *J. Differential Equations* **254** (2013), no. 2, 528-546.

[11] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, second edition, Springer-Verlag, Berlin, 1983.

[12] L. Korobenko and E. Braverman, On logistic models with a carrying capacity dependent diffusion: stability of equilibria and coexistence with a regularly diffusing population, *Nonlinear Anal. B: Real World Appl.* **13** (2012), no. 6, 2648–2658.

[13] L. Korobenko and E. Braverman, On evolutionary stability of carrying capacity driven dispersal in competition with regularly diffusing populations, *J. Math. Biol.* **69** (2014), No. 5, 1181.

[14] E. Braverman, Md. Kamrujjaman and L. Korobenko, Competitive spatially distributed population dynamics models: does diversity in diffusion strategies promote coexistence? *Math. Biosci.* **264** (2015), 6373.

[15] E. Braverman and Md. Kamrujjaman, Lotka systems with directed dispersal dynamics: Competition and influence of diffusion strategies *Math. Biosci.* **279** (2016), 1–12.

[16] E. Braverman and Md. Kamrujjaman, Competitive–cooperative models with various diffusion strategies *Comp. Math. with Appl.* **72** (2016), 653–662.

[17] K. Y. Lam and W. M. Ni, Uniqueness and complete dynamics in heterogeneous competition-diffusion systems, *SIAM J. Appl. Math.* **72** (2012), 1695–1712.