Hilbert scheme and multiplet matter content

ABSTRACT

Development of the concept of Euler characteristic, from the Euclidean geometry to the algebraic geometry is considered. A singular toric variety is studied within the framework of the algebraic geometry. Procedure of the blowing up of its singularities in terms of cones is represented by Hilbert scheme. Special cases of the blowing up of orbifold singularities of C^3/Z_n using Nakamura's algorithm are performed. Hilbert schemes and their physical interpretation in terms of Euler characteristic are presented.

Keywords: Euler characteristic, Hilbert scheme, toric variety, orbifold, singularities, Nakamura's algorithm

1. INTRODUCTION

In the article [1], Atiyah presented the current researches in mathematics which are related to the global study and become important in the applications to topology that was predicted by Poincare. He lists a number of areas of mathematics - complex analysis, differential equations, number theory, where the global properties were additional to the local approach. Thus, implicit solutions of differential equations could not be resolved by the usual methods. Global solutions were associated with singularities of the space. The transition to such solutions is associated with the increasing role of the topological approach.

Similar changes in the approaches for solving the problems were observed in physics, where the locality was associated with differential equations, and the transition to high-energy physics was connected with non-linear equations. The solution of non-linear equations became impossible by usual methods. The appearance of solitonic solutions in the form of D-branes [2] - objects in multidimensional space-time, gave the powerful impetus to the development of geometric methods in high energy physics, confirming Wheeler statement: "Physics is geometry". Due to the use of topological and algebraic-geometric methods in physics it has become possible to find solutions to physical problems in terms of topological invariants.

The theory of superstrings and D-branes as the modern version of the unified theory of fundamental interactions, gives answer to the question, what happens in a short interval of time from the Big Bang. Among the many properties of the theory of D-branes are of particular importance the following three. First, gravity and quantum mechanics as essential principles of the Universe should be united. Secondly, the investigations over the last century have shown that there are key concepts for understanding the Universe: the generations of particles, gauge symmetry, symmetry breaking, supersymmetry. All these ideas are naturally flowing from the theory of D-branes. Third, in contrast to the Standard model with 19 free parameters, D-brane theory is free of parameters.

Since we are dealing with solitonic objects - D-branes, the space-time manifold is endowed with a certain structure. For a principal bundle representing D-brane it is possible to construct vector bundle, which plays an important role for calculations of topological invariants characterizing the D-branes. The bases of such bundles are manifolds of extra dimensions such as Calabi-Yau or orbifolds.

At every stage of researches in D-brane theory physicists searched for experimentally observable consequences of the theory. In this aspect, it was observed that the number of generations of quarks and leptons is connected with the structure of the manifold of extra dimensions. Thus, the number of generations is a topological invariant, associated with the structure of Calabi-Yau or orbifolds.

The article is devoted to the studying of the properties of such manifold of extra dimensions as orbifold. For its description complex differential forms $\omega^{p,q}$ and Dolbeault cohomology group $H^{p,q}(M)$ defined by differential forms of degree (p,q) on the manifold M are introduced. As $\dim H^{p,q}(M) = h^{p,q}$, where $h^{p,q}$ are Hodge numbers and the Euler characteristic is connected with Hodge numbers $\chi = \sum_{p,q} (-1)^{(p+q)} h^{p,q}$, we can determine

The number of generations $= \frac{1}{2}|\chi|$.

The purpose of our paper is the studying of orbifold C^3/Z_n which is carried out on the basis of Nakamura's algorithm. This algorithm makes it possible to receive the Hilbert scheme. Hilbert scheme is common mathematical object that is very actively studied by mathematicians and physicists. The last of such papers are, for example, PhD thesis of Ádám Gyenge "Hilbert schemes of points on some classes surface singularities" [3] and the article of Zheng [4]. As Hilbert scheme is the blowing up of orbifold singularity, we can apply to it the technique of differential forms and can give an adequate interpretation of particle generation, characterizing orbifold. The task of the paper is not only the application of the Nakamura algorithm, but also a deeper understanding of the physical consequences from the mathematical structure of the space of extra dimensions such as orbifolds.

2. EULER CHARACTERISTIC IN EUCLIDEAN GEOMETRY

Coxeter [5] considered new type of geometry, called elliptical geometry, where the lines and planes are replaced by circles and spheres. Since the elliptical geometry is a kind of non-Euclidean or projective geometry, we'll consider the constructions that will be important for us in the future.
In the Euclidean geometry, the Euclidean plane can be covered with the simplest polyhedra - squares, equilateral triangles or pentagons, figure 1.

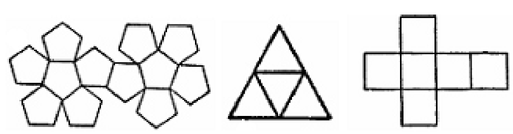

Fig. 1. Simplest figures for coverage of the Euclidean plane

It is interesting to note that for any surface covered with maps, the characteristic of Euler- Poincare is the following

$\chi = V - E + F$,

where *V* - vertices of the polygon, *E* - the number of edges, *F* - the number of polygonal areas or faces.

3. PROJECTIVE GEOMETRY AND HILBERT SCHEME

For the further it will be convenient to use the fact that projective geometry includes affine geometry and Euclidean geometry, [6]:

Projective geometry $\supset$ Affine geometry $\supset$ Euclidean geometry.

In the future we will deal with n-dimensional projective space [7]. n-dimensional projective space over the field *k*, P_k^n - is set of classes of equivalent collections $(a_0, a_1, \ldots, a_n)$ with respect to the equivalence

$$(a_0, a_1, \ldots, a_n) \sim (\lambda a_0, \lambda a_1, \ldots, \lambda a_n),$$
$$\lambda \in k, \lambda \neq 0.$$

If *f* - homogeneous polynomial of degree *d*, then

$$f(\lambda a_0, \lambda a_1, \ldots, \lambda a_n) = \lambda^d f(a_0, a_1, \ldots, a_n).$$

We have a set of zeros

$$Z(f)=\{P\in P^n \mid f(P)=0\}$$

in P^n of homogeneous polynomial *f*. *Y* of P^n is a projective algebraic set, if *Y = Z(T)* for the set *T* of homogeneous elements of the polynomial ring. Since the union and intersection of such algebraic sets defines the Zariski topology, then we can talk about the projective algebraic variety as of irreducible closed (in the Zariski topology) subset of the projective space P^n.

It is known that the schemes are an extension of the concept of manifolds [7]. They are determined by a topological space *X* and by a sheaf of rings over it, O_X (to each open set are mapped functions from which are built the rings of functions). In this case *X*, together with the open space covering, $(X_i, O_X \mid X_i)$ is isomorphic to the affine scheme $Spec\,\Gamma(X_i, O_X)$ of the ring of sections O_X over X_i. One of the methods for generating of new schemes is the transition to the quotient space by the equivalence relation over scheme, the special case of which is the orbifold X/Z_n (Z_n – is the cyclic group of order *n*). In this case, we have a flat family of closed subschemes in P_k^n [7], which is parameterized by the Hilbert scheme. It means that the set of rational k-points of Hilbert scheme is in one-to-one correspondence with the set of closed submanifolds in P_k^n. Thus, orbifold is a generalization of the concept of an algebraic variety.

4. COMPACTIFICATION OF HILBERT SCHEME

It is known that orbifolds are a special cases of a kind of an algebraic manifold, called toric variety, [8]. Since the scheme $\boldsymbol{Hilb}(\boldsymbol{X}/\boldsymbol{S})$, as a direct sum of subschemes $\boldsymbol{Hilb}^{\boldsymbol{P}}(\boldsymbol{X}/\boldsymbol{S})$ for all $\boldsymbol{P}\in \boldsymbol{Q}(\boldsymbol{z})$ with rational coefficients, is not compact, it can be "compactified" by gluing different maps of algebraic varieties [9]. As an example, it is convenient to consider the projective space as a result of gluing of three maps, or as a result of compactification of the torus when gluing zero and "infinity" (orbits of the torus action), that is represented in figure 2. Gluing functions (functions of coordinates change) are monomials of Laurent.

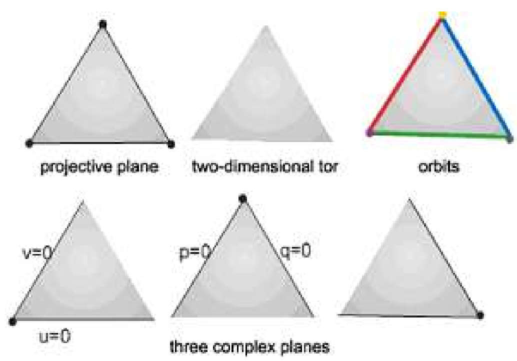

Fig. 2. Projective plane as the gluing of three complex planes [10]

Laurent polynomial is determined by the set of lattice points $M\subset Z^2$, $\operatorname{supp} f=\{a \mid \lambda_a\neq 0\}\subset Z^2$. With these points is constructed the cone $pos(M)=\{\lambda_1 y_1+\ldots+\lambda_k y_k : \lambda_i\geq 0, y_i\in M\}$. To each map corresponds its own cone σ, and the glue a few maps gives the toric variety. At the same time the cones $\sigma\in\Sigma$ are glued to the fan, Σ, according to certain rules [9]. According to Batyrev's technique [11], a toric variety is represented as a polyhedron Δ, which is determined by the set of convex in R^d cones σ, $\sigma=R_{\geq 0}\overrightarrow{n_1}+\ldots+R_{\geq 0}\overrightarrow{n_r}$ for some linearly independent vectors $\vec{n}_1,\cdots,\vec{n}_r\in Z^d$ satisfying the following conditions: 1) any of two cones intersect along a common face 2) for any cone belonging to polyhedron Δ, all its faces also belong to Δ.To each reflexive polyhedron there corresponds a dual polyhedron ∇. According to the Theorem 4.2.2 of [11], there exist at least one toroidal desingularization of any projective toric variety which corresponds to any maximal projective triangulation.

5. BLOWING UP OF SINGULARITIES OF TORIC VARIETY

An important structure that carries information about the algebraic variety is the ring of regular functions, $R = C[z_1,\dots,z_n] = C[z]$, for multivariable $z = (z_1,\dots,z_n)$ and $a = (a_1,\dots,a_n) \in Z^n$, $z^a = z_1^{a_1} \cdot \dots \cdot z_n^{a_n}$. This ring of regular functions allows the construction of an algebraic variety *X* as a scheme $X = Spec\ R$. Since the toric variety, studied in the paper, has singularities, to remove them is used the procedure of blowing up of singularities associated with the defragmentation of fan Σ. An example of such a blow-up procedure is Nakamura's algorithm [12] demonstrated for blowing up of orbifold singularity C^3/Z_3. McKay quiver tessellated by tripods for the model $\frac{1}{3}(1,1,1)$ is illustrated in figure 3

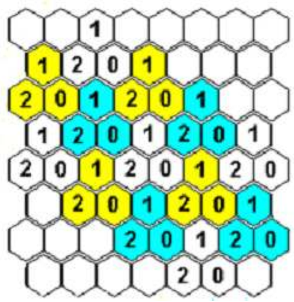

Fig. 3. McKay quiver for $\frac{1}{3}(1,1,1)$ model

The other model that demonstrates the blowing up of orbifold C^3/Z_n singularity is $\frac{1}{13}(1,2,10)$. McKay quiver tesselated by tripods for this model is presented in figure 4

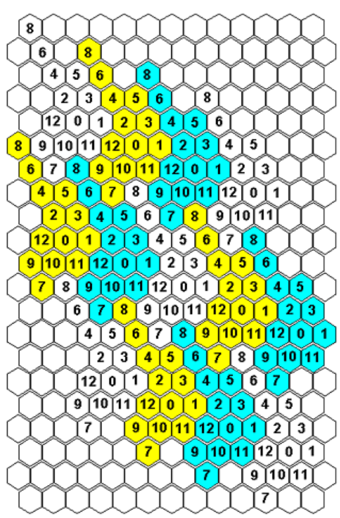

Fig. 4. McKay quiver for $\frac{1}{13}(1,2,10)$ model

The corresponding monomial representation of this quiver is presented in our article [12].

The concept of a structure sheaf O_{X_Σ} is introduced to distinguish compact manifolds X_Σ. This concept associates the ring of regular functions, $O_{X_\Sigma}(U) = R_U$, to each open set. The structure sheaves or sheaves of rings are introduced to differ X_Σ. Structure sheaf $O_{X_\Sigma}(U)$ is the sheaf of O_{X_Σ} modules. For a sheaf *F* on a manifold X_Σ, $f \in F(U)$ is a section of sheaf *F* over *U* and the sections of sheaf *F* over X_Σ are global sections. After gluing the disjoint cones in the fan, set of global sections is empty, ie, there are no constant functions. It is useful for further physical interpretations. Thus, the local model of an algebraic variety over a field *k* is subset of algebraic variety defined by a system of algebraic equations or ringed space with a structure sheaf of rational functions together with Zariski topology. The modern version of this definition is the variety defined by a scheme over a field *k*.

6. DIFFERENTIAL FORMS AND THE EULER CHARACTERISTIC ON THE MANIFOLD

Let's consider the ringed space (*X, O*), equipped with a sheaf of holomorphic functions. Since the functions are tensor fields of rank 0, and the vector fields are tensors of rank 1, it will be natural to use tensor fields as the common types of functions. Among tensor fields differential forms are widely used in applications [13]

$$\omega = \sum_{i_1,\dots,i_k} a_{i_1\dots i_k}(x) dx^{i_1} \wedge \dots \wedge dx^{i_k} .$$

These forms can be closed, $d\omega = 0$, and exact, $\omega = d\omega'$, for some form ω'. Factor group of closed forms over the subgroup of exact forms determines de Rham cohomology group $H^k(M, K)$, $K = R, C$ for real, *R* or complex, *C* fields.

It is interesting to note that Euler characteristic of a manifold *M*, $\chi(M)$, is determined by the differential form

$$\eta = \frac{1}{N!(4\pi)^N} \varepsilon_{i_1\dots i_{2N}} \cdot F^{i_1 i_2} \wedge \dots \wedge F^{i_{2N-1} i_{2N}}$$

or the Euler class in de Rham cohomology group $H^{2N}(M, R)$. There F^{ij} - the field strength of the Yang - Mills and $\varepsilon_{i_1\dots i_{2N}}$ is antisymmetric tensor. Wherein

$$\chi(M) = \int_M \eta .$$

Similarly, it is possible to enter Dolbeault cohomology group in the complex space through *p, q* - forms, [14]

$$A^{p,q}(M) = \{\varphi \in A^n(M) : \varphi(z) \in \wedge^p T_z^{*'}(M) \otimes \wedge^q T_z^{*''}(M)\}$$

$$for\ all\ z \in M$$

for the decomposition of the cotangent space at any point *z*

$$\wedge^n T^*_{C,z}(M) = \oplus_{p+q=n} \left(\wedge^p T_z^{*'}(M) \otimes \wedge^q T_z^{*''}(M)\right).$$

Factor of d-exact forms of type (*p, q*), $Z_{\bar\partial}^{p,q}(M)$ over exact forms $\bar\partial\left(A^{p,q}(M)\right) \subset$ $Z_{\bar\partial}^{p,q+1}(M)$ determines Dolbeault cohomology group

$$H_{\bar\partial}^{p,q}(M) = Z_{\bar\partial}^{p,q}(M) / \bar\partial\left(A^{p,q-1}(M)\right).$$

Relation between cohomology groups of de Rham and Dolbeault is realized in the form of the Hodge decomposition $H_D^n = \oplus_{p+q=n} H^{p,q}$. This implies the relationship between the dimensions of the de Rham cohomology groups - Betti numbers, b_n, and dimensions of the Dolbeault cohomology group - Hodge numbers, $h^{p,q}$ [15]

$$b_n = \sum_{p+q=n} h^{p,q} .$$

In this case the Euler characteristic is given by the expression

$$\chi = \sum_n (-1)^n b_n = \sum_{p,q} (-1)^{(p+q)} h^{p,q}.$$

It is also important to stress the existence of an alternative formula for Euler characteristic,

$$\frac{1}{2}\chi(Z_f) = h^{1,1}(Z_f) - h^{2,1}(Z_f),$$

where the Hodge numbers of toric variety, Z_f are defined by Laurent polynomial. These Laurent polynomials defines Newton polyhedron of such toric variety [11].

7. HILBERT SCHEME OF $\frac{1}{3}(1,1,1)$ MODEL AND THE NUMBER OF GENERATIONS OF PARTICLES IN STANDARD MODEL

The article of contemporary theorists in the field of high energy physics [16] make it possible to interpret the Hodge numbers in terms of particle multiplets

$$h_{11} = rank\ G_2^{(0)}(k) + rank\ H + n_T(k) + 2$$

$$h_{21} = 272 + \dim G_2^{(0)}(k) + \dim H - 29 n_T(k) - a_H - b_H k\ ,$$

where a_H and b_H encode the number of *H*-charged fields, n_T - tensor multiplets and gauge groups *H* and $G_2^{(0)}(k) = E_8, E_7, E_6, SO(8)$ for $k = 6,4,3,2$ and $G_2^{(0)}(k) = SU(1)$ for $k = 1,0$ of $E_8 \times E_8$ heterotic string. Hence the obvious connection of multiplet content of the particles with the Euler characteristic is presented by formula [15]:

$$N_{gen} = |\chi(K)/2|,$$

ie, the number of generations of particles in nature is determined by the Euler characteristic.

It will be important to calculate the Hilbert scheme for the considered model $\frac{1}{3}(1,1,1)$, since it contains important information about the number of generations of quarks and leptons in the Standard Model (SM). Hilbert scheme is a space related to representation theory and mathematical physics [17]. This fact was presented in the study of the instanton moduli space associated with Hilbert schemes through the moduli space of sheaves. In addition, the Hilbert scheme is a special case of the moduli space, as shown in [17]. The spaces of modules in high-energy physics are associated with the multiplet content of matter fields [18], what is encoded in the Hilbert schemes.

The applicaton of The Nakamura's algorithm for computation of the Hilbert scheme for the D-brane model $\frac{1}{3}(1,1,1)$ gives us the cones of the fan

$$P = (3,0,0) \qquad Q = (1,1,1) \qquad R = (0,0,3)$$
$$P = (3,0,0) \qquad Q = (0,3,0) \qquad R = (1,1,1)$$
$$P = (1,1,1) \qquad Q = (0,3,0) \qquad R = (0,0,3)$$

The Hilbert scheme as the unifiication of fans is illustrated in figure 5.

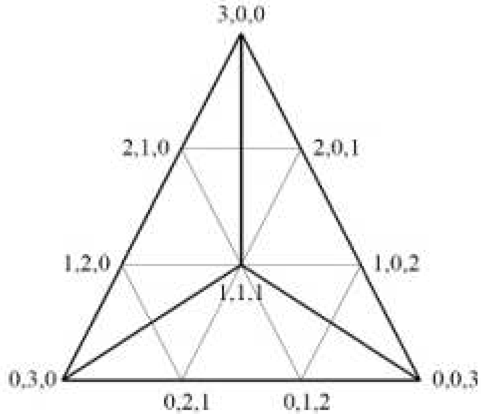

Fig. 5. Hilbert scheme of $\frac{1}{3}(1,1,1)$ model

As we considered the blowing up of orbifold C^3/Z_3, where Z_3 – subgroup of SU(3) [19], and group SU(3) classifies three possible quark states that realizes the fundamental representation of group of dimension three in the SM [20], then we can insist that Hilbert scheme for the model $\frac{1}{3}(1,1,1)$ gives the number of generations of SM. This number of generations in SM is equal to three that agrees with the experimental data.

The other example is Hilbert scheme for the model $\frac{1}{13}(1,2,10)$, presented in figure 6.

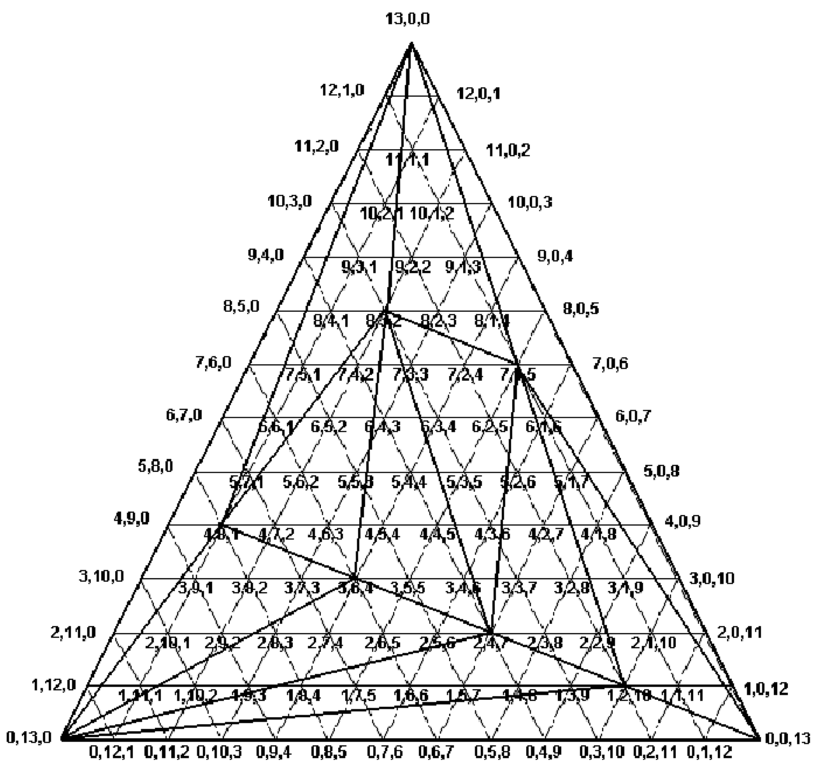

Fig. 6. Hilbert scheme of $\frac{1}{13}(1,2,10)$ model

8. CONCLUSION

We have considered orbifolds in terms of Hilbert scheme within the framework of toric geometry, which is the subsection of projective geometry. It is shown that the blowing up of orbifold singularities is associated with grinding or gluing of several cones in fan, as demonstrated by two examples of orbifold C^3/Z_n. The interpretation of the Euler characteristic in terms of Hodge numbers expressed in two different formulas for reflexive polyhedron on the one hand and for the matter content on the other hand is of importance for the physical interpretation of the mathematical constructions of this paper, as multiplet content of particles gives the number of generations of quarks and leptons. This theoretical result is confirmed by the specific example of the construction of the Hilbert scheme for two models $\frac{1}{3}(1,1,1)$ and $\frac{1}{13}(1,2,10)$. Thus, to sum up our research, we can prove that the construction of the Hilbert scheme in accordance with Nakamura's algorithm is identical to the blow-up of singularities of orbifold. The blowing up of singularities makes it possible to calculate topological invariant of manifold, which is associated with the number of particle generations in physics.

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