Original Research Article

Green's Function (GF) For the Two Dimensional (2D) Time Dependent Inhomogeneous Wave Equation.

ABSTRACT

Interference effect that occurs when two or more waves overlap or intersect is a common phenomenon in physical wave mechanics. A carrier wave as applied in this study describes the resultant of the interference of a parasitic wave with a host wave. A carrier wave in this wise, is a corrupt wave function which certainly describes the activity and performance of most physical systems. In this work, presented in this paper, we used the Green's function technique to evaluate the behaviour of a 2D carrier wave as it propagates away from the origin in a pipe of a given radius. In this work, we showed quantitatively the method of determining the intrinsic characteristics of the constituents of a carrier wave which were initially not known. Evidently from this study the frequency and the band spectrum of the Green's function are greater than those of the general solution of the wave equation. It is revealed in this study that the retarded behaviour of the carrier wave described by the Green's function at some point away from the origin is much greater than the general wave solution of the carrier wave at the origin. The Green' function is spherically symmetric about the source, and falls off smoothly with increasing distance from the source. The anomalous behaviour exhibited by the carrier wave at some point during the damping, is due to the resistance pose by the carrier wave in an attempt to annul the destructive tendency of the interfering wave. Evidently it is shown in this work that when a carrier wave is undergoing attenuation, it does not consistently come to rest; rather it shows some resistance at some point in time during the damping process, before it finally comes to rest.

Keywords: Parasitic wave, Carrier wave, Host wave, Greens Function, Time dependent inhomogeneous wave

1. INTRODUCTION.

Interference effect that occurs when two or more waves overlap or intersect is a common phenomenon in physical wave mechanics. When waves interfere with each other, the amplitude of the resulting wave depends on the frequencies, relative phases and amplitudes of the interfering waves. The resultant amplitude can have any value between the differences and sum of the individual waves [1]. If the resultant amplitude comes out smaller than the larger of the amplitude of the interfering waves, we say the superposition is destructive; if the resultant amplitude comes out larger than both we say the superposition is constructive.

When a wave equation ψ and its partial derivatives never occur in any form other than that of the first degree, then the wave equation is said to be linear. Consequently, if ψ_1 and ψ_1 are any two solutions of the wave equation ψ , then $a_1\psi_1 + a_2\psi_2$ is also a solution, a_1 and a_2 being two arbitrary constants [2,3]. This is an illustration of the principle of superposition, which states that, when all the relevant equations are linear we may superpose any number of individual solutions to form new functions which are themselves also solutions.

There is a great need in differential equations to define objects that arise as limits of functions and behave like functions under integration but are not, properly speaking, functions themselves. These objects are sometimes called generalized functions or distributions. The most basic one of these is the so-called delta δ -function.

A distribution is a continuous linear functional on the set of infinitely differentiable functions with bounded support; this space of functions is denoted by D. We can write $d[\phi]:\ D \rightarrow \Re$ to represent such a map: for any input function ϕ, $d[\phi]$ gives us a number [4, 5].

Green's functions depend both on a linear operator and boundary conditions. As a result, if the problem domain changes, a different Green's function must be found. A useful trick here is to use symmetry to construct a Green's function on a semi-infinite (half line) domain from a Green's function on the entire domain. This idea is often called the method of images [6].

If a wave is to travel through a medium such as water, air, steel, or a stretched string, it must cause the particles of that medium to oscillate as it passes [7]. For that to happen, the medium must possess both mass (so that there can be kinetic energy) and elasticity (so that there can be potential energy). Thus, the medium's mass and elasticity property determines how fast the wave can travel in the medium.

The principle of superposition of wave states that if any medium is disturbed simultaneously by a number of disturbances, then the instantaneous displacement will be given by the vector sum of the disturbance which would have been produced by the individual waves separately. Superposition helps in the handling of complicated wave motions. It is applicable to electromagnetic waves and elastic waves in a deformed medium provided Hooke's law is obeyed [8].

A parasitic wave as the name implies, has the ability of destroying and transforming the intrinsic constituents of the host wave to its form after a sufficiently long time. It contains an inbuilt raising multiplier λ which is capable of increasing the intrinsic parameters of the parasitic wave to become equal to those of the 'host wave'. Ultimately, once this equilibrium is achieved, then all the active components of the 'host wave' would have been completely eroded and the constituted carrier wave ceases to exist [9].

Any source function $\psi(r)$ can be represented as a weighted sum of point sources. It follows from superposability that the potential generated by the source $\psi(r)$ can be written as the weighted sum of point source driven potentials i.e. Green's functions. It is evident that one very general way to solve inhomogeneous partial differential equations (PDEs) is to build a Green's function and write the solution as an integral equation [10,11]. Remarkably, a Green's function can be used for problems with inhomogeneous boundary conditions even though the Green's function itself satisfies homogeneous boundary conditions. This seems improbable at first since any combination or superposition of Green's functions would always still satisfy a homogeneous boundary condition [12]. The way in which inhomogeneous boundary conditions enter relies on the so-called "Green's formula", which depends both on the linear operator in question as well as the type of boundary condition (i.e. Dirichlet, Neumann, or a combination).

The organization of this paper is as follows. In section 1, we discuss the nature of wave and interference. In section 2, we show the mathematical theory of superposition of two incoherent waves using Green's function technique. The results emanating from this study is shown in section 3. The discussion of the results of our study is presented in section 4. Conclusion of this work is discussed in section 5. The paper is finally brought to an end by a few lists of references and appendix.

1.1 Research Methodology.

In this work, a carrier wave with an inbuilt raising multiplier is allowed to propagate in a narrow pipe containing air. The attenuation mechanism of the carrier wave is thus studied by means of the Green's function technique.

2. MATHEMATICAL THEORY.

2.1 General Wave Equation.

Generally, the wave equation (WE) can be described by two basic equations given below.

$$\nabla^2 \phi - \epsilon\mu \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\in} \tag{2.1}$$

$$\nabla^2 A - \in \mu \frac{\partial^2 A}{\partial t^2} = -\mu J \tag{2.2}$$

where ∇ is called the del operator, it is a three dimensional (3D) Laplacian operator in Cartesian coordinate system, the scalar potential is given by ϕ, the vector potential is given by A, the charge density is ρ, the permittivity is ϵ, while the permeability is μ, and the current density is J, the permittivity and the permeability of air is $\in$ and μ respectively. It is very obvious that both wave equations have the same basic structure; hence in a free space we can write a single wave that would connect the two equations as follows.

$$\left(\nabla^2 - \in \mu \frac{\partial^2}{\partial t^2}\right) \varphi(x,t) = -f(x,t;t) \tag{2.3}$$

Where $f(x, y; t)$ is a known source distribution having space – time functions. Since we are dealing with dynamic variable coordinates, the source function is normally represented by the delta function. Solving equation (2.3) using Green's Function $G(x,t \mid x',t')$ [13, 14], we obtains

$$\left(\nabla^2 - \in \mu \frac{\partial^2}{\partial t^2}\right) G(x;t \mid x';t') = -\delta(x-x')\,\delta(t-t') \tag{2.4}$$

Hence, (2.4) is the Green function for one dimensional (1D) space.
Recasting equation (2.4) to be 2-Dimensional in character, the variation in the Laplacian will also lead to a variation in the Green's Function. Accordingly, we get .

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \in \mu \frac{\partial^2}{\partial t^2}\right) G(x,y;t \mid x',y';t') = -\delta(x-x')\,\delta(y-y')\,\delta(t-t') \tag{2.5}$$

Solving equation (2.7) using the Green's Function of the Helmholtz equation, we get

$$G(x,y;t \mid x'.y';t') = \frac{1}{(2\pi)^4} \int d^3k \int d\omega\, g(k,\omega)\, e^{i(k_x - k'_x\lambda)(x-x')} e^{j(k_y - k'_y\lambda)(y-y')} \times e^{-2i\,[(\omega_x - \omega'_x\lambda)(t-t') - E(t)]} \tag{2.6}$$

Putting equation (2.6) into equation (2.5) and equating the result into equation (2.8) and if the wave number conserves parity or reciprocricity, then $k_y = k_x$ and $j - i$ hence

$$G(x,y;t \mid x'.y';t') = \frac{1}{(2\pi)^4} \int d^3k \int d\omega \frac{e^{2i\left[(k-k'\lambda)\left|x-x'\right| - \left((\omega-\omega'\lambda)(t-t') - E(t)\right)\right]}}{2(k-k'\lambda)^2 - 4\in\mu\left((\omega-\omega'\lambda) - z(t)\right)^2} \tag{2.7}$$

2.2 Evaluation of the Retarded Distance and the Retarded Time of the Green's Equation.

Factorizing the denominator of equation (2.7) and taking the value of the exponential power $\neq o$ then

$$x' = x - \frac{(\omega-\omega'\lambda)(t-t') - E(t)}{\sqrt{2\in\mu}\,\left((\omega-\omega'\lambda) - z(t)\right)} = 0 \tag{2.8}$$

$$t' = t - \frac{\left(\sqrt{2\in\mu}\,\left((\omega-\omega'\lambda) - z(t)\right)\right)\left|x-x'\right| + E(t)}{(\omega-\omega'\lambda)} \tag{2.9}$$

This means that the causal behaviour associated with a wave distribution, that is, the effect observed at the point x and time t is due to a disturbance which originated at an earlier or retarded time t'. The reader should note that $\sqrt{\in \mu}\,|x - x'|$ is a time component. Hence equation (2.7) becomes

$$G(x, y, t \,|\, x'.y', t') = \frac{1}{2(2\pi)^4} \int d^3k \int d\omega\, e^{2i\left[(k - k'\lambda)|x - x'| - \left((\omega - \omega'\lambda)(t - t') - E(t)\right)\right]} \times$$

$$\frac{1}{\left((k - k'\lambda) + \sqrt{2 \in \mu}\,\left((\omega - \omega'\lambda) - z(t)\right)\right)\left((k - k'\lambda) - \sqrt{2 \in \mu}\,\left((\omega - \omega'\lambda) - z(t)\right)\right)} \qquad (2.10)$$

2.3 Evaluation of the Green's Function using Contour Integration Method.

Solving equation (2.9) by contour integration and determine the validity of the Green's function $G(x, y; t \,|\, x'.y'; t')$ by observing the poles of the equation, we have

$$f(z_1) \equiv f(k - k'\lambda) = -\sqrt{2 \in \mu}\,\left((\omega - \omega'\lambda) - z(t)\right) \qquad (2.11)$$

$$f(z_2) \equiv f(k - k'\lambda) = +\sqrt{2 \in \mu}\,\left((\omega - \omega'\lambda) - z(t)\right) \qquad (2.12)$$

Thus Setting $\theta = 2\sqrt{2\in\mu}\,\left((\omega - \omega'\lambda) - z(t)\right)|x - x'|$; $\beta = 2\left((\omega - \omega'\lambda)(t - t') - E(t)\right)$

the sum residue of $f(z_1)$ and $f(z_2)$ at the poles is

$$f(z_1) + f(z_2) = \frac{i\, e^{-2i\left((\omega - \omega'\lambda)(t - t') - E(t)\right)} \sin\left(\sqrt{8 \in \mu}\,\left((\omega - \omega'\lambda) - z(t)\right)|x - x'|\right)}{\sqrt{2 \in \mu}\,\left((\omega - \omega'\lambda) - z(t)\right)} \qquad (2.13)$$

Hence by Cauchy's Residue theorem the integral (2.7) becomes

$$G(x, y; t \,|\, x'.y'; t') = -\frac{e^{-2i\left((\omega - \omega'\lambda)(t - t') - E(t)\right)} \sin\left(\sqrt{8 \in \mu}\,\left((\omega - \omega'\lambda) - z(t)\right)|x - x'|\right)}{2(2\pi)^3 \sqrt{2 \in \mu}\,\left((\omega - \omega'\lambda) - z(t)\right)} \qquad (2.14)$$

Reducing the numerator of equation (2.14) and taking the absolute value of the resulting equation, we get after simplification

$$G(x, y; t \,|\, x'.y'; t') = -\frac{\sin\left(\sqrt{8 \in \mu}\,\left((\omega - \omega'\lambda) - z(t)\right)|x - x'|\right)}{(2\pi)^3 \sqrt{8 \in \mu}\,\left((\omega - \omega'\lambda) - z(t)\right)} \qquad (2.15)$$

Thus the dimension of the Green's function is metres m. Note that the point source driven potential (2.15) is perfectly sensible. It is spherically symmetric about the source, and falls off smoothly with increasing distance from the source.

2.4 General Solution of the Wave Equation and the Carrier Wave CW which is the Source Function.

It follows that the potential generated by $\Psi(r, t)$ can be written as the weighted sum of point impulse driven potentials. Hence generally, the solution to the wave equation (2.3) is

$$\Psi(r, t) = \int |\psi(x, y; t)|\, G(r, t \,|\, r', t')\, dr' dt' \qquad (2.16)$$

$$\Psi(x, y; t) = \int |\psi(x, y; t)|\, G(x, y, t \,|\, x', y', t')\, dx'\, dy'\, dt' \qquad (2.17)$$

If such a representation exists, the kernel of this integral operator $G(x, y; t \,|\, x', y'; t')$ is called the Green's function. Hence we think of $\Psi(x, y; t)$ as the response at x and y to the influence given by a source function $\psi(x, y; t)$. For example, if the problem involved elasticity, $\Psi(x, y; t)$ might be the displacement caused by an external force $f(x, y; t)$. If this were an equation describing heat flow, $\Psi(x, y; t)$ might be the temperature arising from a heat source described by $f(x, y; t)$. The integral can be thought of as the sum over influences created by sources at each value of x' and y'. For this reason, G is sometimes called the influence function [15]. Thus in this study we assume that the carrier wave which is the source distribution is given by the equation

$$\psi(x,y;t) = \left\{(a^2 - b^2\lambda^2) - 2\left(a - b\lambda\right)^2 \cos\left((\omega - \omega'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right)\right\}^{\frac{1}{2}} \cos\left(\vec{k}_C \cdot \vec{r} - (\omega - \omega'\lambda)t - E(t)\right)$$
(2.18)

$$E(t) = \tan^{-1}\left(\frac{a\sin\varepsilon + b\lambda\sin\left(\varepsilon'\lambda - (\omega - \omega'\lambda)t\right)}{a\cos\varepsilon + b\lambda\cos\left(\varepsilon'\lambda - (\omega - \omega'\lambda)t\right)}\right) \qquad (2.19)$$

From the geometry of the resultant of the two interfering waves (please see appendix), the carrier wave CW is two dimensional 2D in character since it is a transverse wave, the position vector of the particle in motion is represented as $\vec{r} = r(\cos\theta\, i + \sin\theta\, j)$ and hence the motion is constant with respect to the z- axis, the combined wave number or the spatial frequency of the carrier wave is $\vec{k}_C = (k - k'\lambda)\, i + (k - k'\lambda)\, j$. Then, $\vec{k}_C \cdot \vec{r} = r\ (k - k'\lambda)\ \left(\cos\theta + \sin\theta\right)$ is the coordinate of two dimensional (2D) position vectors and $\theta = \pi - (\varepsilon - \varepsilon'\lambda)$, the total phase angle of the CW is represented by $E(t)$. A complete detail of the derivation of the carrier wave (2.19) is shown in a previous paper [16].

If $\omega = \omega'$, then the average angular frequency say $(\omega + \omega'\lambda)/2$ will be much more greater than the modulation angular frequency say $(\omega - \omega'\lambda)/2$ and once this is achieved then we will have a slowly varying carrier wave with a rapidly oscillating phase. Driving forces in anti-phase $(\varepsilon - \varepsilon' = \pm\pi)$ provide full destructive superposition and the minimum possible amplitude; driving forces in phase $(\varepsilon = \varepsilon')$ provides full constructive superposition and maximum possible amplitude.

2.5 The Calculus of the Total Phase Angle E of the Carrier Wave Function.

Let us now determine the variation of the total phase angle with respect to time t. Thus from (2.19),

$$\frac{dE}{dt} = \left(1 + \left(\frac{a\sin\varepsilon - b\lambda\sin\left((\omega - \omega'\lambda)t - \varepsilon'\lambda\right)}{a\cos\varepsilon - b\lambda\cos\left((\omega - \omega'\lambda)t - \varepsilon'\lambda\right)}\right)^2\right)^{-1} \times \frac{d}{dt}\left(\frac{a\sin\varepsilon - b\lambda\sin\left((\omega - \omega'\lambda)t - \varepsilon'\lambda\right)}{a\cos\varepsilon - b\lambda\cos\left((\omega - \omega'\lambda)t - \varepsilon'\lambda\right)}\right) \qquad (2.20)$$

$$\frac{dE}{dt} = \left\{\frac{\left(a\cos\varepsilon - b\lambda\cos\left((\omega - \omega'\lambda)t - \varepsilon'\lambda\right)\right)^2}{\left(a\cos\varepsilon - b\lambda\cos\left((\omega - \omega'\lambda)t - \varepsilon'\lambda\right)\right)^2 + \left(a\sin\varepsilon - b\lambda\sin\left((\omega - \omega'\lambda)t - \varepsilon'\lambda\right)\right)^2}\right\} \times$$

$$\frac{d}{dt}\left(\frac{a\sin\varepsilon - b\lambda\sin\left((\omega - \omega'\lambda)t - \varepsilon'\lambda\right)}{a\cos\varepsilon - b\lambda\cos\left((\omega - \omega'\lambda)t - \varepsilon'\lambda\right)}\right) \qquad (2.21)$$

After a lengthy algebra (2.21) simplifies to

$$\frac{dE}{dt} = -Z \qquad (2.22)$$

where we have introduced a new variable defined by the symbol Z as the characteristic angular velocity of the carrier wave and is given by

$$Z = (\omega - \omega'\lambda)\left(\frac{b^2\lambda^2 - ab\lambda\cos\left((\varepsilon + \varepsilon'\lambda) - (\omega - \omega'\lambda)t\right)}{a^2 + b^2\lambda^2 - 2ab\lambda\cos\left((\varepsilon + \varepsilon'\lambda) - (\omega - \omega'\lambda)t\right)}\right) \qquad (2.23)$$

Hence, Z has the dimension of *rad./s.* In other to avoid unnecessary complications we can set

$$Q = \left\{(a^2 - b^2\lambda^2) - 2\left(a - b\lambda\right)^2 \cos\left((\omega - \omega'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right)\right\}^{\frac{1}{2}} \qquad (2.24)$$

$$\Psi(x,y;t) = -\int \frac{\sin\left(\sqrt{8\in\mu}\ \left((\omega - \omega'\lambda) - z(t)\right)\left|\ x - x'\ \right|\right)}{(2\pi)^3\sqrt{8\in\mu}\ \left((\omega - \omega'\lambda) - z(t)\right)}\, Q\ \cos\left(\vec{k}_C \cdot \vec{r} - (\omega - \omega'\lambda)t - E(t)\right) dx'\,dt' \qquad 2.25)$$

But according to (2.8) and (2.9); $dx' = dt' = 1$, as a result,

$$\Psi(x,y;t) = -\frac{\sin\left(\sqrt{8\in\mu}\left((\omega-\omega'\lambda)-z(t)\right)\left|x-x'\right|\right)}{(2\pi)^3\sqrt{8\in\mu}\left((\omega-\omega'\lambda)-z(t)\right)}Q \quad \cos\left(\vec{k}_C.\vec{r}-(\omega-\omega'\lambda)t-E(t)\right) \qquad (2.26)$$

Now in equation (2.26) we can simply replace $x \to \left|x-x'\right|$ which is just the distance covered by the carrier wave in metres m as it propagates in a pipe of radius $r = 0.03$ metres m.

$$\Psi(x,y;t) = -\frac{\sin\left(\sqrt{8\in\mu}\left((\omega-\omega'\lambda)-z(t)\right)x\right)}{(2\pi)^3\sqrt{8\in\mu}\left((\omega-\omega'\lambda)-z(t)\right)} \times$$

$$\left\{(a^2-b^2\lambda^2)-2\left(a-b\lambda\right)^2\cos\left((\omega-\omega'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\right\}^{\frac{1}{2}} \cos\left(\vec{k}_C.\vec{r}-(\omega-\omega'\lambda)t-E(t)\right) \qquad (2.27)$$

The reader should not ignore or forget that the motion under study is still a 2D one. The fact that we have constrained it to $x-$ axis does not mean that the $y-$axis is not implied. The factor 2 which appear in (2.7) is a reflection that the motion is still 2D. Note that it is the absolute values of the carrier wave $\psi(x,y;t)$ that we used in our computation.

2.6 Determination of the Host Wave Parameters (a, ω, ε and k) contained in the Carrier Wave.

Let us now discuss the possibility of obtaining the parameters of the host wave which were initially not known from the carrier wave equation. This is a very crucial stage of the study since there was no initial knowledge of the values of the host wave and the parasitic wave contained in the carrier wave. However, the carrier wave given by (2.18) can only have a maximum value provided the spatial oscillating phase is equal to one. As a result, the non-stationary amplitude A and the oscillating phase angle ϕ becomes after disengaging them as

$$A = \left\{\left(a^2-b^2\lambda^2\right)-2\left(a-b\lambda\right)^2\cos\left(\left(\omega-\omega'\lambda\right)t-\left(\varepsilon-\varepsilon'\lambda\right)\right)\right\}^{\frac{1}{2}} \qquad (2.28)$$

$$\phi = \cos\left((k-k'\lambda)\; r\; (\cos\theta+\sin\theta)-(\omega-\omega'\lambda)t-E(t)\right) \qquad (2.29)$$

Using the boundary conditions that at time $t = 0$, $\lambda = 0$ and $A = a$, then

$$A = \left\{a^2-2a^2\cos\left(-\varepsilon\right)\right\}^{\frac{1}{2}} = a\left\{1-2\cos\left(\varepsilon\right)\right\}^{\frac{1}{2}} \qquad (2.30)$$

$$\left\{1-2\cos\left(\varepsilon\right)\right\}^{1/2} = 1 \quad \Rightarrow \quad \varepsilon = \cos^{-1}(0) = 90^0 \;(1.5708\, rad.) \qquad (2.31)$$

Any slight variation in the combined amplitude A of the carrier wave due to displacement with time $t = t+\delta t$ would invariably produce a negligible effect in the amplitude a of the host wave and under this situation $\lambda \approx 0$. Hence we can write

$$\lim_{\delta t \to 0}\left\{A+\frac{\delta A}{\delta t}\right\} = a \qquad (2.32)$$

$$\lim_{\delta t \to 0}\left\{\left(a^2-2a^2\cos\left(\omega(t+\delta t)-\varepsilon\right)\right)^{1/2} + \frac{n\,a^2\sin\left(\omega(t+\delta t)-\varepsilon\right)}{\left(a^2-2a^2\cos\left(\omega(t+\delta t)-\varepsilon\right)\right)^{1/2}}\right\} = a \qquad (2.33)$$

$$\left\{\left(a^2-2a^2\cos\left(\omega t-\varepsilon\right)\right)^{1/2} + \frac{\omega a^2\sin\left(\omega t-\varepsilon\right)}{\left(a^2-2a^2\cos\left(\omega t-\varepsilon\right)\right)^{1/2}}\right\} = a \qquad (2.34)$$

$$\left(a^2-2a^2\cos\left(\omega t-\varepsilon\right)\right)+\omega a^2\sin\left(\omega t-\varepsilon\right) = a\left(a^2-2a^2\cos\left(\omega t-\varepsilon\right)\right)^{1/2} \qquad (2.35)$$

$$1-2\cos\left(\omega t-\varepsilon\right)+\omega\sin\left(\omega t-\varepsilon\right) = \left(1-2\cos\left(\omega t-\varepsilon\right)\right)^{1/2} \qquad (2.36)$$

At this point of our work, it may not be easy to produce a solution to the problem; this is due to the mixed sinusoidal wave functions. However, to get out of this complication we have implemented a special approximation technique to minimize the right hand side of (2.36). This approximation states that

$$\left(1+\xi\, f(\phi)\right)^{\pm\, n} = \frac{d}{d\phi}\left(1+n\,\xi\, f(\phi)+\frac{n(n-1)}{2!}\left(\xi\, f(\phi)\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\xi\, f(\phi)\right)^3+\ldots\right) \tag{2.37}$$

The general background of this approximation is the differentiation of the resulting binomial expansion of a given variable function. This approximation has the advantage of converging functions easily and also it produces minimum applicable value of result. Consequently, (2.36) becomes

$$1-2\cos\left(\omega t-\varepsilon\right)+\omega\sin\left(\omega t-\varepsilon\right)=\omega\sin\left(\omega t-\varepsilon\right) \tag{2.38}$$

$$\omega t-\varepsilon=\cos^{-1}(0.5)=60^{0}=1.0472\,rad. \quad \Rightarrow \omega t=2.6182\,rad. \quad \Rightarrow \omega=2.6182\,rad./s \tag{2.39}$$

From (2.33), by using the boundary conditions that for stationary state when $\delta t=0$, $\lambda\approx 0$, $\theta=\pi-(\varepsilon-\varepsilon'\lambda)=\pi-\varepsilon=3.142-1.5708=1.5712\,rad$, $E=\varepsilon=1.5708\,rad$, then we have that

$$\lim_{\delta t\to 0}\cos\left\{(k-k'\lambda)\,r\cos\theta+(k-k'\lambda)\,r\sin\theta-(\omega-\omega\lambda)\,(t+t\,\delta t)-E\right\}=1 \tag{2.40}$$

$$\left(k\,r\,(\cos\theta+r\sin\theta)-\omega t-\varepsilon\right)=0 \qquad (\text{ since, } \cos^{-1}1=0\ ) \tag{2.41}$$

$$\left(k\,r\,(0.9996)-2.6182-1.5708\right)=0 \quad\Rightarrow\ k\,r=4.1907\,rad \quad\Rightarrow\ k=4.1907\,rad\,/\,m \tag{2.42}$$

The change in the resultant amplitude A of the carrier wave is proportional to the frequency of oscillation of the spatial oscillating phase ϕ multiplied by the product of the variation with time t of the inverse of the oscillating phase with respect to the radial distance r, and the variation with respect to the wave number $(k-k'\lambda)$. This condition would make us write (2.28) and (2.29) separately as

$$\frac{dA}{dt}=\frac{(\omega-\omega'\lambda)(a-b\lambda)^2\sin\left((\omega-\omega'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)}{\left((a^2-b^2\lambda^2)-2(a-b\lambda)^2\cos\left((\omega-\omega'\lambda)t-(\varepsilon-\varepsilon'\lambda)\right)\right)^{1/2}} \tag{2.43}$$

$$\frac{d\phi}{dr}=-(k-k'\lambda)(\cos\theta+\sin\theta)\sin\left((k-k'\lambda)r\,(\cos\theta+\sin\theta)-(\omega-\omega'\lambda)t-E\right) \tag{2.44}$$

$$\frac{d\phi}{dt}=\left((\omega-\omega'\lambda)+Z\right)\sin\left((k-k'\lambda)r\,(\cos\theta+\sin\theta)-(\omega-\omega'\lambda)t-E\right) \tag{2.45}$$

$$\frac{d\phi}{d\,(k-k'\lambda)}=\left(-\,r\,(\cos\theta+\sin\theta)-E\right)\sin\left((k-k'\lambda)r\,(\cos\theta+\sin\theta)-(\omega-\omega'\lambda)t-E\right) \tag{2.46}$$

$$\frac{dA}{dt}=\left(\frac{1}{2\pi}\frac{\partial\phi}{\partial t}\right)\left(\frac{1}{r}\frac{\partial r}{\partial\phi}\right)\left(\frac{\partial\phi}{\partial(k-k'\lambda)}\right)=f\,l \tag{2.47}$$

$$A=f\,l\,t \tag{2.48}$$

That is the time rate of change of the resultant amplitude is equal to the frequency f of the spatial oscillating phase multiplied by the length l of the arc covered by the oscillating phase. Under this circumstance, we refer to A as the instantaneous amplitude of oscillation. The first term in the parenthesis of (2.47) is the frequency dependent term, while the combination of the rest two terms in the parenthesis represents the angular length or simply the length of an arc covered by the spatial oscillating phase. Note that the second term in the right hand side of (2.47) is the inverse of (2.44).

With the usual implementation of the boundary conditions that at

$t=0$, $\lambda=0$, $\theta=\pi-(\varepsilon-\varepsilon'\lambda)=\pi-\varepsilon=3.142-1.5708=1.5712\,rad$, $E=\varepsilon=1.5708\,rad$, $dA/dt=a$

we obtain the expression for the amplitude as

$$a=-\left(\frac{1}{2\pi}\right)\left(\frac{\left((\cos\theta+\sin\theta)-\varepsilon\right)}{k\sin\varepsilon(\cos\theta+\sin\theta)}\right)=0.0217m \tag{2.49}$$

Note that $\cos(-\varepsilon) = \cos\varepsilon$ (even and symmetric function) and $\sin(-\varepsilon) = -\sin\varepsilon$ (odd and screw symmetric function). Thus generally we have established that the basic constituent's parameters of the host wave are

$$a = 0.0217m\,,\ \omega = 2.6182 rad/s\,,\ \varepsilon = 1.5708 rad\,, \text{ and } k = 4.1907 rad/m \qquad (2.50)$$

2.7 Determination of the Parasitic Wave Parameters (b, ω', ε' and k') Contained in the Carrier Wave.

Let us now determine the basic parameters of the parasitic wave which were initially not known before the interference from the derived values of the resident 'host wave' using the below method. The gradual depletion in the physical parameters of the system under study would mean that after a sufficiently long period of time all the active constituents of the resident host wave would have been completely attenuated by the destructive influence of the parasitic wave. On the basis of these arguments, we can now write as follows.

$$\left.\begin{aligned} a - b\lambda = 0 &\Rightarrow 0.0217 = b\lambda \\ \omega - \omega'\lambda = 0 &\Rightarrow 2.6182 = \omega'\lambda \\ \varepsilon - \varepsilon'\lambda = 0 &\Rightarrow 1.5708 = \varepsilon'\lambda \\ k - k'\lambda = 0 &\Rightarrow 4.1907 = k'\lambda \end{aligned}\right\} \qquad (2.51)$$

Upon dividing the sets of relations in (2.75) with one another with the view to eliminate λ we get

$$\left.\begin{aligned} 0.008288\,\omega' &= b \\ 0.013820\,\varepsilon' &= b \\ 0.005178k' &= b \\ 1.6668\,\varepsilon' &= \omega' \\ 0.6248\,k' &= \omega' \\ 0.3748\,k' &= \varepsilon' \end{aligned}\right\} \qquad 2.52)$$

However, there are several possible values that each parameter would take according to (2.49). But for a gradual decay process, that is for a slow depletion in the constituent of the host parameters we choose the least values of the parasitic parameters. Thus a more realistic and applicable relation is when $0.008288\,\omega' = 0.005178\,k'$. Based on simple ratio we eventually arrive at the following results.

$$\omega' = 0.00518 rad/s\,,\quad k' = 0.00829 rad/m\,,\quad \varepsilon' = 0.00311 rad\,,\quad b = 0.0000429m \qquad (2.53)$$

Any of these values of the constituents of the parasitic wave shall produce a corresponding approximate value of lambda $\lambda = 505$ upon substituting them into (2.51). Hence the interval of the multiplier is $0 \le \lambda \le 505$. Now, so far, we have systematically determined the basic constituent's parameters of both the host wave and those of the parasitic wave both contained in the carrier wave.

2.8 Determination of the Attenuation Constant (η).

Attenuation is a decay process. It brings about a gradual reduction and weakening in the initial strength of the basic parameters of a given physical system. In this study, the parameters are the amplitude (a), phase angle (ε), angular frequency (ω) and the spatial frequency (k). The dimension of the attenuation constant (η) is determined by the system under study. However, in this work, the attenuation constant is the relative rate of fractional change (FC) in the basic parameters of the carrier wave. There are 4 (four) attenuating parameters present in the carrier wave. Now, if a, ω, ε, k represent the initial basic parameters of the host wave that is present in the carrier wave and $a - b\lambda$, $\omega - \omega'\lambda$, $\varepsilon - \varepsilon'\lambda$, $k - k'\lambda$ represent the basic parameters of the host wave that survives after a given time. Then, the FC is

$$\sigma = \frac{1}{4} \times \left(\left(\frac{a - b\lambda}{a} \right) + \left(\frac{\varepsilon - \varepsilon'\lambda}{\varepsilon} \right) + \left(\frac{\omega - \omega'\lambda}{\omega} \right) + \left(\frac{k - k'\lambda}{k} \right) \right) \qquad (2.54)$$

$$\eta = FC/_{\lambda=i} - FC/_{\lambda=I+1} = \sigma_I - \sigma_{I+1} \qquad (2.55)$$

The dimension is *per second* (s^{-1}). Thus (2.55) gives $\eta = 0.001978\,s^{-1}$ for all values of the raising multiplier $\lambda\,(i = 0, 1, 2, \ldots, 505)$. The reader should note that we have adopted a slowly varying regular interval for the raising multiplier since this would help to delineate clearly the physical parameter space accessible to our model.

2.9 Determination of the Decay or Attenuation Time (t).

We used the information provided in (2.55), to compute the various times taken for the carrier wave to attenuate to zero. The maximum time the carrier wave lasted as a function of the raising multiplier λ is also calculated from the attenuation equation. However, it is clear from the calculation that the different attenuating fractional changes contained in the carrier wave are approximately equal to one another. We can now apply the attenuation time equation given below.

$$\sigma = e^{-(2\eta t)/\lambda} \quad (2.56)$$

$$t = -\left(\frac{\lambda}{2\eta}\right)\ln\sigma \quad (2.57)$$

The equation is statistical and not a deterministic law. It gives the expected basic intrinsic parameters of the 'host wave' that survives after time t. Clearly, we used (2.57) to calculate the values of the decay time as a function of the raising multiplier λ (0, 1, 2, . . . , 505).

Table 2.1: **Shows the calculated values of the characteristics of the carrier wave** $\psi(x, y; t)$ **.**

S/N	Physical Quantity	Symbol	Value	Unit
1	Amplitude of the host wave	a	0.0217	m
2	Angular frequency of the host wave	ω	2.6182	rad/s
3	Phase angle of the host wave	ε	1.5708	$radian$
4	Spatial frequency of the host wave	k	4.1907	rad/m
5	Amplitude of the parasitic wave	b	0.0000429	m
6	Angular frequency of the parasitic wave	ω'	0.00518	rad/s
7	Phase angle of the parasitic wave	ε'	0.00311	$radian$
8	Spatial frequency of the parasitic wave	k'	0.00829	rad/m
9	Raising multiplier	λ	0, 1, 2, … ,505	--

Table 2.2: **Shows calculated values of some of the parameters and some constants required for the work**.

S/N	Physical Quantity	Symbol	Value	Unit
1	Attenuation constant	η	0.001978	s^{-1}
2	Radius of the pipe	r	0.03	m
3	Maximum attenuation time corresponding to the maximum multiplier	t	892180	s
4	Sum of the total time that the carrier wave lasted as a function of the multiplier	t	48429885	s
5	Sum of the total distance covered by the carrier wave as a function of the multiplier	x	4.67691×10^{15}	m
6	Permittivity of air	$\in$	8.85×10^{-12}	$C^2 N^{-1} m^{-2}$
7	Permeability of air	μ	1.2566×10^{-6}	H/m
8	The product of Permittivity and	$\in\mu$	$1.11209 \times 10^{-}$	s^2/m^2

Permeability of air 17

3. RESULTS AND DISCUSSION.

The relevant results obtained which is given by the equations (2.8), (2.14), (2.18), (2.22) and (2.26) respectively are shown graphically below.

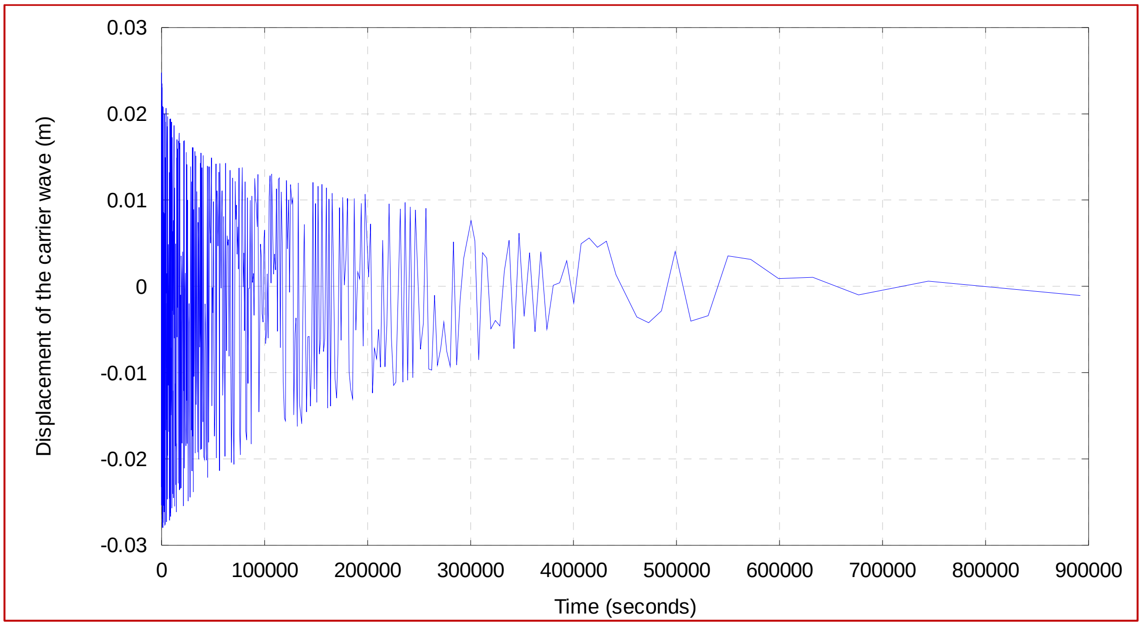

Fig. 3.1: **Shows the displacement of the carrier wave as function of time and multiplier. The figure represents equation (2.18).**

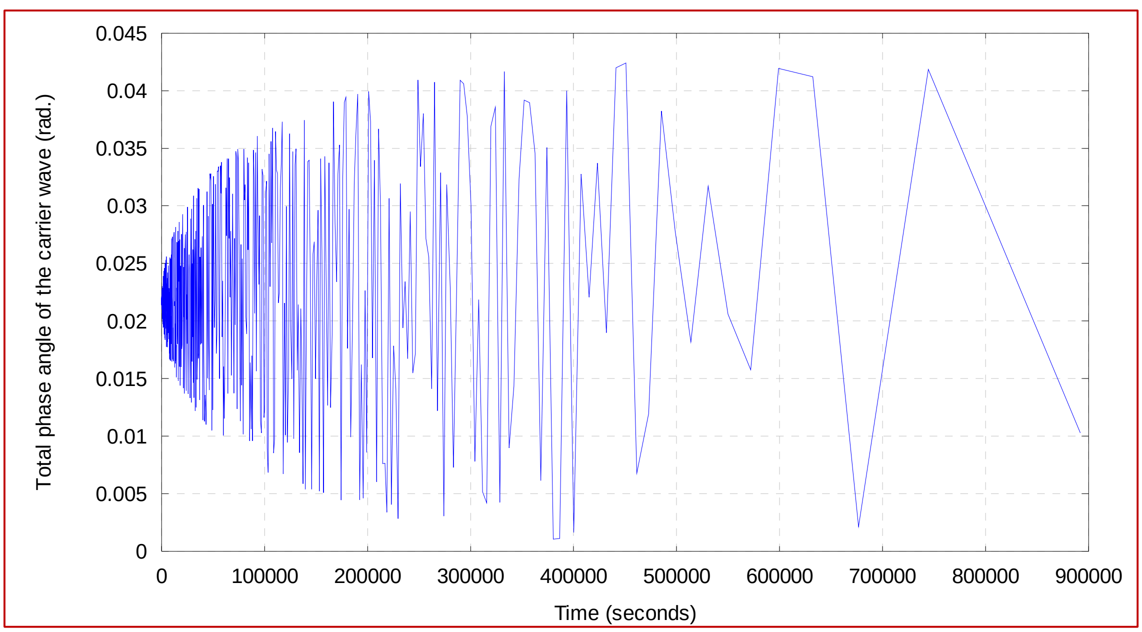

Fig. 3.2: **Shows the spectrum of the amplitude of the total phase angle of the carrier wave as function of time and multiplier. The figure represents equation (2.18).**

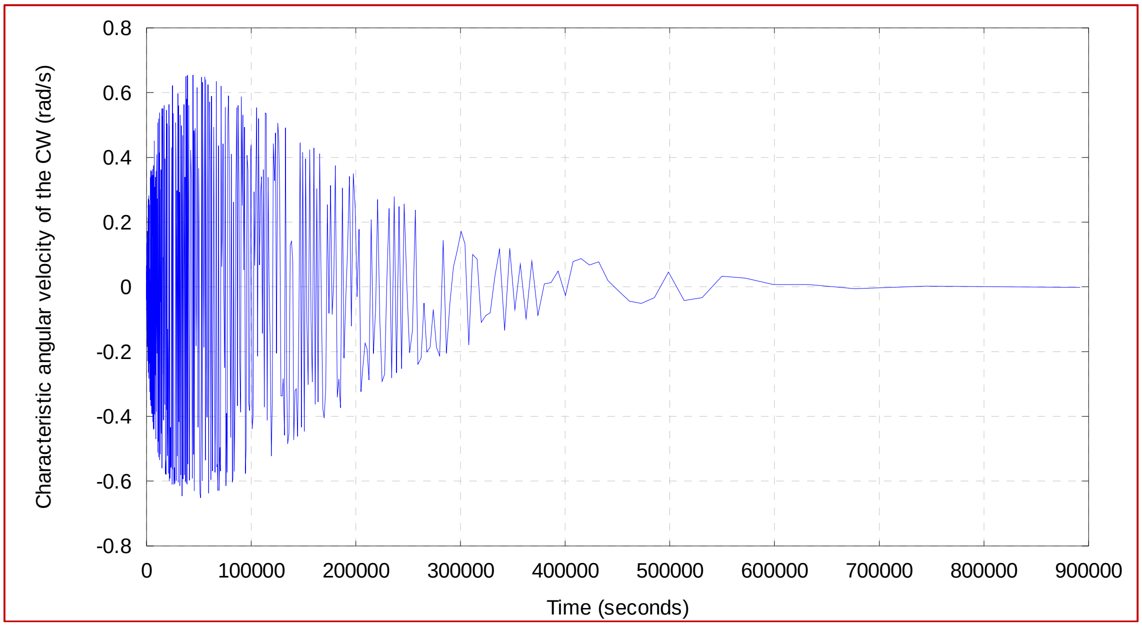

Fig. 3.3: **Fig. 3.2: Shows the spectrum of the amplitude of the characteristic angular velocity** Z **of the CW as function of time and multiplier. The figure represents equation (2.22).**

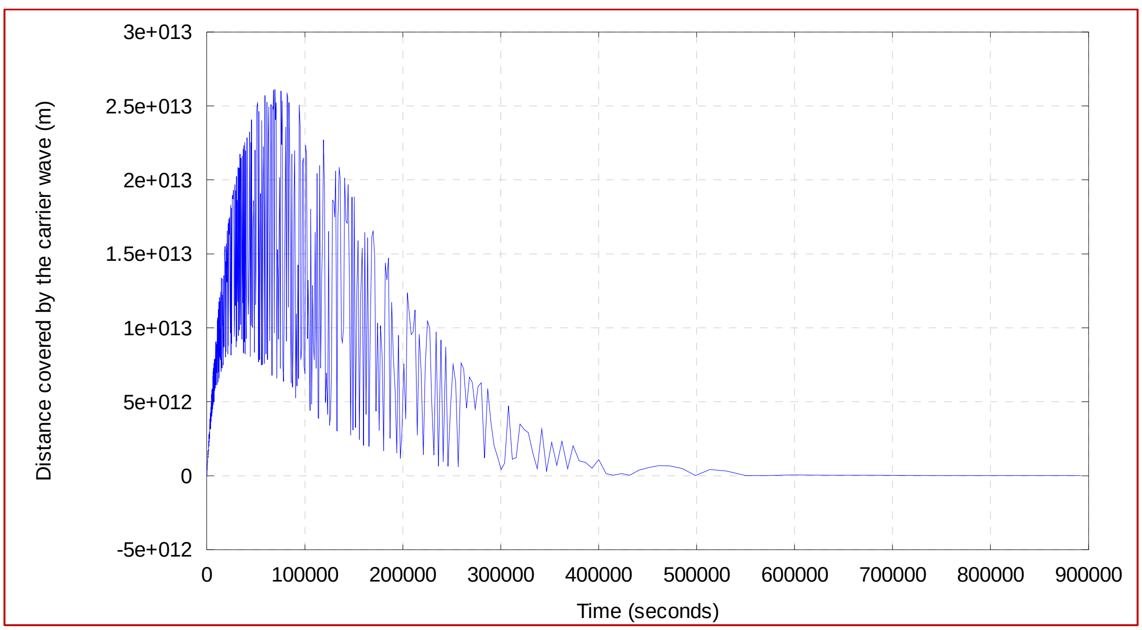

Fig. 3.4: **Shows the spectrum of the distance covered as a function of time and the raising multiplier. The figure represents equation (2.8).**

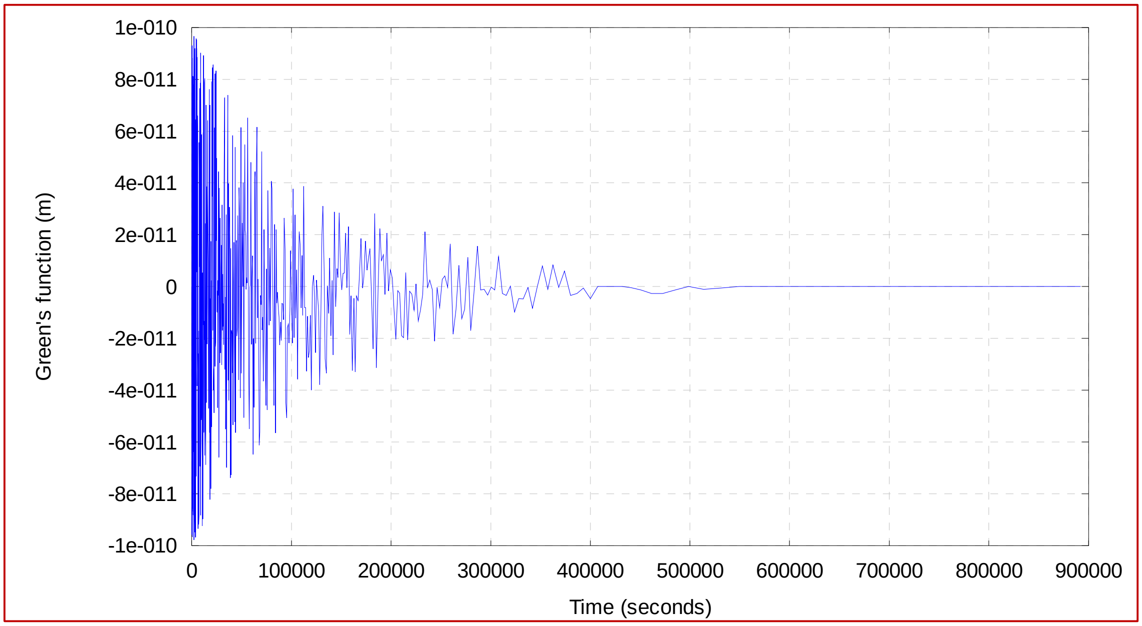

Fig 3.5: **Shows the Green's function representation of the 2D carrier wave as function of time and the multiplier. The figure represents equation (2.14).**

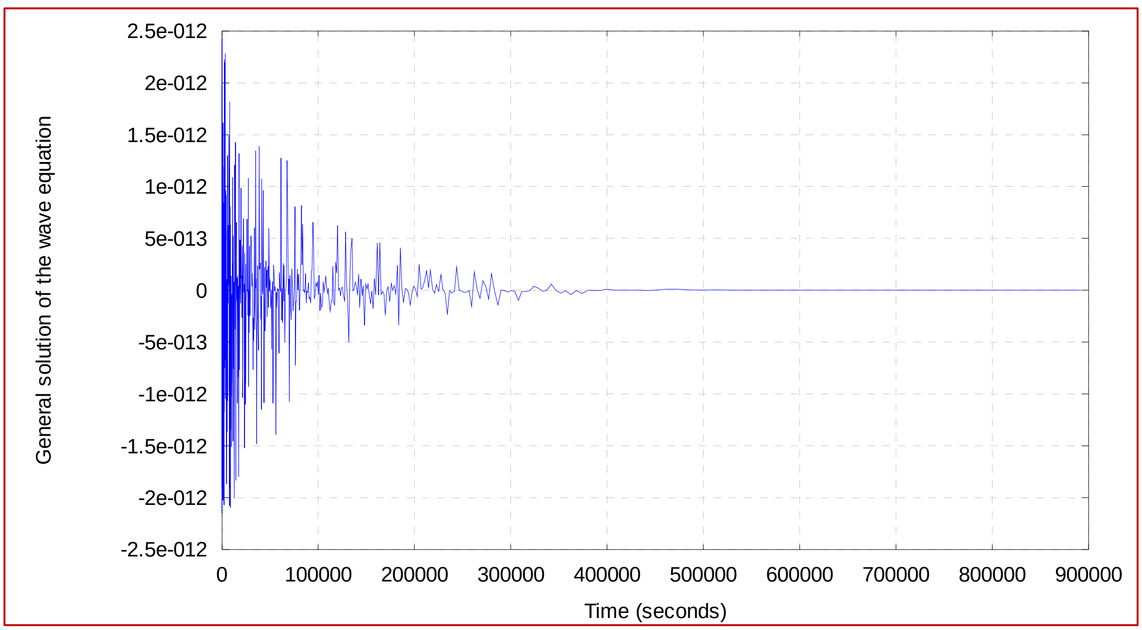

Fig. 3.6: **Shows the spectrum of the general wave solution as function of time and multiplier. The figure represents equation (2.26).**

The displacement of the carrier wave as a function of time and the multiplier is shown in Fig. 3.1. At the origin the frequency of the carrier is initially very high and with well-defined amplitude of about $+0.02477\,m$ and $-0.0279\,m$. When the value of the raising multiplier is 486 and the time is about 400000s the frequency of the CW decreases. However, the CW has a longer wavelength when the amplitude is decreasing. The carrier wave becomes monochromatic with no definite frequency after $600000\,s$. The carrier wave finally decays to zero after $892180\,s$. The simple explanation here is that the components of the host wave in the carrier wave could have been completely depleted by the effect of the interfering parasitic wave.

In fig. 3.2 the amplitude of the total phase angle of the CW first increases with high frequency and a narrow band. However, after a period of time say about 400000 s, the frequency becomes dispersed and the CW has a pronounced wavelength. The total phase angle is thus generally positive and basically it does not attenuate to zero. The positive nature of the total phase angle means that the constituents of the CW are highly repulsive, in other words, there is a general disagreement between the parameters of the host wave and those of the parasitic wave in the CW.

The spectrum of the amplitude of the characteristic angular velocity of the CW as shown in fig. 3.3 first increases to a maximum value of about $\pm$0.65011 rad/s and time 37526 s. The maximum value of the characteristic angular velocity corresponds to when the raising multiplier is 236. However, after this time the amplitude decreases to a minimum value of $\pm$0.00896 rad/s when the time is about 380133 s and the multiplier is 483. The attenuation of the characteristic angular velocity is exponential and it decays to zero after about 600000 s. The disband frequency shows that the parameters of the host wave in the CW equation are already experiencing a rapid decay process due the destructive tendency of the interfering parasitic wave.

Fig. 3.4 represents the spectrum of the total distance covered by the carrier wave as a function of the raising multiplier. The maximum distance covered by the CW as a function of the raising multiplier is about 2.60893 x10^{13} m. This maximum value corresponding to λ = 300 and time t = 68265 s. In the interval of the multiplier [100 – 446] and time [6055 – 241272] s the distance covered by the carrier wave is longer with high frequency. The distance of propagation by the CW first go to zero at t =407419s. The CW then propagates to some distance before it comes to zero again for a second time at t = 500000 s. Thereafter, it then covered a small distance before it finally attenuates to zero. The repeated relative zero distance – time behaviour of the CW equation shows that there is existence of some residual energy in the host wave that tends to resist an end to the propagation of the CW due to the destructive influence of the parasitic wave. From the calculation the sum of the total distance covered by the carrier wave as a function of the multiplier is 4.67691 x 10^{15} m while the sum of the total time that the carrier wave lasted to travel the distance as a function of the multiplier is 48429885 s.

It is obvious from fig 3.5 that the Green's function first show initial increase in the displacement from the equilibrium position. The spectrum of the Green's function also show a very high frequency before it starts to decrease exponentially. The exponential decrease in the amplitude finally becomes a plane wave at time t = 450873 s with an amplitude of about -1.29941 x 10^{-12} m. The wave finally comes to rest at time 892180 s (247 hours) and this corresponds to a critical value of the multiplier which is 505.

It is clear from fig. 3.6 that the general wave equation first show initial increase in the displacement from equilibrium position. The spectrum of the general wave equation also show a very high frequency before it starts to decrease. The decrease is exponential and it finally becomes a plane wave at time t = 391988 s with final amplitude of about 6.03426 x 10^{-14} m. Beyond this time the general wave equation becomes a plane wave and it is no longer sinusoidal before it finally comes to rest. It is clear from fig. 3.5 and fig. 3.6 that the decay time and the relative amplitude of the Greens function representation of the carrier wave are greater than those of the general solution of the wave equation. From the figure while there is still pronounced amplitude in the Green's function beyond 300000 s but in the general wave solution the amplitude is already almost zero. Thus the attenuation time of the Green's function representation of the carrier wave lags the decay time for the general wave solution by 100000 s. That is, while the general wave solution finally goes to zero at 400000 s the Green's function finally goes to zero at 500000 s. This shows that the retarded behaviour of the carrier wave described by the Green's function at some point away from the origin is much greater than the general wave solution of the carrier wave at the origin.

4. CONCLUSION.

This study shows that the process of attenuation in most physically active system does not obviously begin immediately when they encounter an oppositely interfering system. The general wave equation that defines the activity and performance of a given wave away from the origin is guided by some internal inbuilt factor which enables it to resist any external interfering influence that is destructive in nature. Consequently, the anomalous behaviour exhibited by the carrier wave during the decay process, is due to the resistance pose by the intrinsic parameters of the host wave in the constituted carrier wave in an attempt to annul the destructive tendency of the parasitic wave. It is evident from this work that when a carrier wave is undergoing attenuation, it does not steadily or consistently come to rest; rather it shows some resistance at some point in time during the decay process, before it finally comes to rest. The attenuation time of the Green's function representation of the carrier wave equation lags the attenuation time for the general wave solution. This shows that the retarded behaviour of the carrier wave described by the Green's function at some point away from the origin is much greater than the general wave solution of the carrier wave at the origin. The Green' function is spherically symmetric about the source, and falls off smoothly with increasing distance from the source.

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APPENDIX: The vector representation shown below is the resultant of the superposition of the parasitic wave on the host wave. The amplitudes of the waves y_1 , y_2 and the resultant wave y are not constant with time but they oscillate at a given frequency.

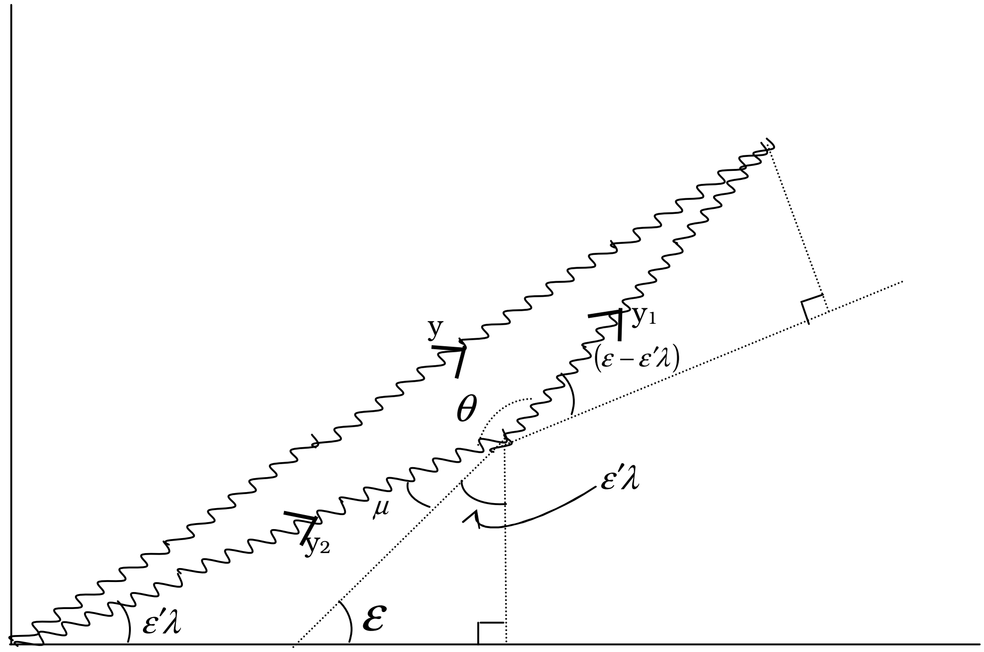

Fig. A 1: Represents the host wave y_1 and the parasitic wave y_2 after the interference. The superposition of both waves y_1 and y_2 is represented by the carrier wave displacement y . It is clear from the geometry of the figure: $\mu + \varepsilon'\lambda + 180^0 - \varepsilon = 180^0$; $\mu = \varepsilon - \varepsilon'\lambda$; $\theta = 180^0 - (\varepsilon - \varepsilon'\lambda)$; and $\theta = \pi - (\varepsilon - \varepsilon'\lambda)$. Note that θ is the variable angle between the waves y_1 and y_2 .

Let us consider two incoherent waves defined by the non - stationary displacement vectors

$$y_1 = a \cos (\vec{k}.\vec{r} - nt - \varepsilon)$$

(A. 1)

$$y_2 = b\lambda \cos (\vec{k}'\lambda.\vec{r} - n'\lambda t - \varepsilon'\lambda)$$

(A. 2)

where all the symbols retain their usual meanings. In this study, (A1) is regarded as the host wave. While (A. 2) represents a parasitic wave with an inbuilt multiplicative factor or raising multiplier $\lambda (= 0, 1, 2, \cdots, \lambda_{\max})$. The inbuilt raising multiplier is dimensionless and as the name implies, it is capable of gradually raising the basic intrinsic parameters of the parasitic wave. Now let us add the two waves given by (A. 1) and (A. 2) using vector summation rule. Consequently, after a lengthy algebra we shall get that the resultant equation y is given as

$$y = \sqrt{\left(a^2 - b^2\lambda^2\right) - 2\left(a - b\lambda\right)^2 \cos\left((n - n'\lambda)t - (\varepsilon - \varepsilon'\lambda)\right)} \times \cos\left(k_c.\vec{r} - (n - n'\lambda)t - E(t)\right)$$

(A. 3)

where

$$E(t) = \tan^{-1}\left(\frac{a \sin \varepsilon - b\lambda \sin((n - n'\lambda)t - \varepsilon'\lambda)}{a \cos \varepsilon - b\lambda \cos((n - n'\lambda)t - \varepsilon'\lambda)}\right)$$

(A. 4)

Equation (A. 3) is now the required carrier wave CW equation necessary for our study. It is clear from (A 3) that when the raising multiplier increases the intrinsic parameters of the parasitic wave to become equal to those of the host wave then the carrier wave ceases to exist after a specified time.