Original Research Article

# Mesoscopic RLC Circuit and its Associated Occupation Number and Berry Phase

In this paper we consider the quantization of the time-dependent harmonic oscillator and its associated Berry phase using the invariant operator method, as well as the occupation number of the induced quasi-particle production. Furthermore, we point out that in the literature, there exist different methods for determining the solution to the Milne-Pinney equation, which leads to different results. By measuring the time-dependent occupation number and associated Berry phase, one can, in principle, determine which of these methods leads to physically realized results. As a concrete example, we consider the mesoscopic RLC circuit and derive the occupation number and associated Berry phase for each of these different methods. We find that, the solution to the Ermakov equations leads to a time-dependent occupation number and associated Berry phase, while the particular solution to the Milne-Pinney equation does not.

## I. INTRODUCTION

Modern electronic material techniques has allowed for the fabrication of small structures, called mesoscopic systems, with resolution that approaches the atomic scale, on the order of micro- and nanometer. As the devices and circuits are small enough that the inelastic coherence of the charge carriers approaches the Fermi wavelength, fluctuations about the average become important and hence the quantum effects of the device and circuit must be taken into account. A study of a mesoscopic system is usually done by examining an LC circuit [1], which is a non-dissipative circuit, and its more realistic counter part, the RLC circuit [2–5], which is a dissipative circuit. In this present paper, we are interested in the quantization of the mesoscopic RLC circuit without source. This system is modeled as a damped harmonic oscillator which is described by the Caldirola-Kanai Hamiltonian. To quantize the mesoscopic RLC circuit, we will make use of the quantum invariant method to solve the Schrödingier equation associated with this Hamiltonian. As is well-known, this system is cyclic in angular frequency $\Omega$. When the system is cyclic, there is a connection between the invariant, which is a constant of motion, and the generalized Berry or geometric phases. The exact solution to the quantum invariant method, however, depends on the solution to an auxiliary equation, known as the Milne-Pinney equation, which is a non-linear equation. Due to the non-linear nature of the Milne-Pinney equation, different methods exist for solving the equation. The methods are: (1) Solving for a particular solution of the Milne-Pinney equation [10]; (2) for a set of initial conditions, one can solve the Milne-Pinney equation numerically [13–15]; (3) the Ermakov equations, which give a relationship between the modulus of the damped harmonic equation for the coordinate and the solution to the Milne-Pinney equation [9, 16]. In general, methods (1) and (3) lead to different results, however, methods (2) and (3) lead to the same result. Hence, the mesoscopic RLC circuit gives a venue which can be used to distinguish between the different methods. Additionally, in the Appendix, we construct coherent and squeezed states for the quantized RLC circuit, as well as evaluate the quantum fluctuations of the charge and magnetic flux, which gives the uncertainty relation.

The paper is organized as follows. In Section II A, we derive the invariant operator and wave function for a time-dependent harmonic oscillator. In addition, we note that the Ermakov equations allow for the exact solution of the Milne-Pinney equation. In Section II B, we derive the occupation number of the induced quasi-particle that is induced due to the time-dependent nature of the system. In Section II C, we show that the Lewis phase can be decomposed into a generalized Berry (or geometric) phase and derive the Berry phase for the time-dependent harmonic oscillator. In Section III, we quantize the mesoscopic RLC circuit, described by the Caldirola-Kanai Hamiltonian, by use of the quantum invariant. Here, we consider the particular solution to the Milne-Pinney equation as well as the solution to the Ermakov equations. Most importantly, we show that the solution to the Ermakov equations leads to a time-dependent occupation number, as well as an associated Berry phase, while the particular solution to the Milne-Pinney equation leads to a time-independent occupation and no associated Berry phase. In Section IV, we conclude the paper with a short summary. Finally, even though these states are not germane to the Berry phase, in Appendix A, we derive the coherent states, expectation value of the coordinate and the uncertainty of the time-dependent harmonic oscillator for completeness. Here, we show that the coherent states are indeed squeezed states.

## II. TIME-DEPENDENT HARMONIC OSCILLATOR

### A. Quantization

Before we quantize the mesoscopic RLC circuit, we will first quantize the time-dependent harmonic oscillator, that has both a time-dependent mass and frequency, using the invariant operator method. This will allow us to obtain the wave functional and discuss the generalized Berry phase associated with the oscillator. Throughout the text, we will set $\hbar = 1$.

To consider the quantization of a time-dependent harmonic oscillator, we will consider a general time-dependent harmonic oscillator equation[1]

$$H = \frac{1}{2M(t)}p^2 + \frac{1}{2}M(t)\omega^2(t)q^2 \tag{1}$$

where $p$ is the conjugate momentum to the coordinate $q$, $M(t)$ is a time-dependent mass, and $\omega(t)$ is a time-dependent frequency. In quantizing the time-dependent harmonic oscillator, we will work in the Heisenberg picture[2]. We can diagonalize the Hamiltonian at all moments of time by defining the new operators

$$a = \frac{1}{\sqrt{2M\omega}}\left(p - iM\omega q\right) \text{ and } a^\dagger = \frac{1}{\sqrt{2M\omega}}\left(p + iM\omega q\right), \tag{2}$$

which satisfy the commutation relation $[a, a^\dagger] = 1$, as well as the Heisenberg equation

$$\frac{da}{dt} = -\frac{1}{2M\omega}\frac{d}{dt}(M\omega)a^\dagger + i\left[H(a, a^\dagger), a\right]. \tag{3}$$

Let's first make some observations about (3). In (3), the second term is the usual time-evolution of an operator in the Heisenberg picture, while the first term describes the moment to moment redefinition of the notion of what the operator, and hence the quasi-particle, is for every moment $t$. The ground state associated with these operators is defined by $a|0\rangle_a = 0$ and leads to the normalized wavefunctional

$$\langle q|0\rangle_a = \varphi(q) = \left(\frac{M\omega}{\pi}\right)^{1/4} e^{-M\omega q^2/2} \tag{4}$$

which is just the harmonic oscillator ground state wavefunctional.

Alternatively, one can use the invariant operator method to study the time-dependence of the quantum system [6, 7]. In this method, one defines a Hermitian invariant operator that satisfies the operator equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} - i[I, H] = 0,$$

which has real, time-independent, eigenvalues. For our purposes, the invariant operator may be decomposed in terms of two linear invariants given as

$$c = \frac{1}{\sqrt{2}}\left[\rho p - M\frac{d\rho}{dt}q - i\frac{q}{\rho}\right] \text{ and } c^\dagger = \frac{1}{\sqrt{2}}\left[\rho p - M\frac{d\rho}{dt}q + i\frac{q}{\rho}\right],$$

which satisfy the commutation relation $[c, c^\dagger] = 1$, where $\rho$ is the real solution to the auxiliary equation, known as the Milne-Pinney equation,

$$\frac{d^2\rho}{dt^2} + \sigma\frac{d\rho}{dt} + \omega^2(t)\rho = \frac{1}{M^2\rho^3}, \tag{5}$$

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[1] This is not the most general time-dependent harmonic oscillator equation, since the most general equation involves terms that involve products of the conjugate momentum and the coordintate.

[2] We could equally well work in the Interaction picture. Here the creation operator defined in (2) is related to the creation operators in the Interaction picture in the usual manner

$$a(t) = \tilde{a}(t)\exp\left[-i\int_{t_0}^{t}\omega(t')dt'\right].$$

where

$$\sigma = \frac{d}{dt}\ln(M) = \frac{1}{M}\frac{dM}{dt}.$$

In terms of the operators $c$ and $c^\dagger$, the Hermitian quadratic invariant operator is then given by $I(t) = \left(c^\dagger c + \frac{1}{2}\right)$. The ground state associated with these operators is defined by $c|0\rangle_c = 0$ and leads to the normalized wavefunctional

$$\langle q|0\rangle_c = \psi_0(q,t) = e^{i\alpha_0}\left(\frac{1}{\pi\rho^2}\right)^{1/4} \exp\left[\frac{iM}{2}\left(\frac{1}{\rho}\frac{d\rho}{dt} + \frac{i}{M\rho^2}\right)q^2\right], \tag{6}$$

where $\alpha_0$, known as the ground state Lewis phase, is defined by

$$\frac{d\alpha_0}{dt} = {}_c\langle 0\,|i\partial_t - H|\,0\rangle_c, \tag{7}$$

Using (6) and (7), one can then easily show that the phase is given by

$$\alpha_0 = -\frac{1}{2}\int \frac{dt'}{M(t')\rho^2(t')}.$$

Using (6), the exact solution of the Schrödinger equation for any state is given by

$$\psi_n(q,t) = e^{i\alpha_n}\left(\frac{1}{\pi 4^n (n!)^2\rho^2}\right)^{1/4} \exp\left[\frac{iM}{2}\left(\frac{\dot\rho}{\rho} + \frac{i}{M\rho^2}\right)q^2\right] H_n\left[\frac{q}{\rho}\right] \tag{8}$$

where $H_n(x)$ are the Hermite polynomials and

$$\alpha_n = \int_0^t dt'\,{}_c\langle n\,|i\partial_{t'} - H|\,n\rangle_c = -\left(n + \frac{1}{2}\right)\int_0^t \frac{dt'}{M(t')\rho^2(t')}, \tag{9}$$

which is known as the Lewis phase.

Here, we note that the invariant operator ground state, $|0\rangle_c$, is distinct from the harmonic oscillator ground state, $|0\rangle_a$, in that the operators $a$ and $c$ are related through a Bogolyubov transformation [8]

$$a = \mu(t)c + \nu(t)c^\dagger, \tag{10}$$

where

$$\mu(t) = \frac{1}{2\sqrt{M\omega}}\left(\rho M\omega + \frac{1}{\rho} + iM\frac{d\rho}{dt}\right), \text{ and } \nu(t) = -\frac{1}{2\sqrt{M\omega}}\left(\rho M\omega - \frac{1}{\rho} + iM\frac{d\rho}{dt}\right) \tag{11}$$

are the Bogolyubov coefficients. That is, the transformation (10) is between two different Fock space basis at equal times, not between the same basis at different times. Here we note that at the initial time $t = t_0$, $\mu(t = t_0) = 1$ and $\nu(t = t_0) = 0$ and the Bogolyubov transformation satisfies

$$|\mu(t)|^2 - |\nu(t)|^2 = 1 \tag{12}$$

for all time $t$. Thus, from (10) and (12), we can also see that at the initial time, $a = c$ and $\psi_0(q, t = t_0) = \varphi(q)$ so that the operators for the two methods are equivalent and the ground states are equivalent at the initial time and thus there is no mixing at the initial time.

From (8) and (9), we can see that in order to quantize the time-dependent harmonic oscillator, one must solve for the auxiliary equation (5). The non-linear nature of (5), on the other hand, suggests that it must be solved numerically. Alternatively, one can choose to consider a particular solution of the Milne-Pinney equation [10]. However, using the Ermakov equations [9], the relationship between the time-dependent amplitude $q$ and $\rho$ is given by $q = \rho e^{-i\gamma}$, where $\gamma$ satisfies the differential equation

$$\frac{d\gamma}{dt} = \frac{\mathrm{A}}{M\rho^2}$$

From (1), the time-dependent amplitude of the coordinate satisfies the equation of motion

$$\frac{d^2q}{dt^2} + \sigma\frac{dq}{dt} + \omega^2 q = 0. \tag{13}$$

Hence, one can instead solve the linear equation (13) and use the fact that $\rho = |q|$.

### B. Occupation Number

An interesting quantity to consider is the number of quasi-particles that are induced as a function of time. The occupation number of quasi-particles created during the time of oscillation then amounts to determining the number of $\hat{a}$ particles in the ground state[3] $|0\rangle_c$:

$$N = {}_c\langle 0|\hat{a}^\dagger \hat{a}|0\rangle_c = |\nu(t)|^2. \tag{14}$$

Using (11), it is then easy to show that the spectrum of instantaneous excitations from (14), is given by

$$N(\omega, t) = \frac{M\omega\rho^2}{4}\left[\left(1 - \frac{1}{M\omega\rho^2}\right)^2 + \left(\frac{1}{\omega\rho}\frac{d\rho}{dt}\right)^2\right]. \tag{15}$$

### C. Berry Phase

From the structure of the Lewis phase (9), we can see that the Lewis phase actually consists of two parts

$$\begin{aligned}\frac{d\alpha_n}{dt} &= i\,{}_c\langle n\,|\partial_t|\,n\rangle_c - {}_c\langle n\,|H|\,n\rangle_c \\ &\equiv \frac{d\alpha_B}{dt} + \frac{d\alpha_D}{dt}\end{aligned} \tag{16}$$

where the first term in (16) is the well-known generalized Berry phase in the adiabatic limit and the second term is the dynamic phase of a time-dependent system. In general, the Berry phase is a real quantity, which leads to physically measurable results. Using (8), we can determine the Berry phase to be

$$\frac{d\alpha_B}{dt} = i\,{}_c\langle n\,|\partial_t|\,n\rangle_c = -\left(n + \frac{1}{2}\right)\left(\frac{1}{M\rho^2} - M\rho^2\omega^2 - M\dot{\rho}^2\right).$$

Now, assuming that the invariant $I(t)$ is $T$-periodic and that its eigenvalues are nondegenerate, then the eigenstates of the dynamical invariant satisfy $\phi(q, T) = \phi(q, 0)$. Thus, the Berry phase for an arbitrary state becomes

$$\alpha_B = -\left(n + \frac{1}{2}\right)\int_0^T dt\left(\frac{1}{M\rho^2} - M\rho^2\omega^2 - M\dot{\rho}^2\right). \tag{17}$$

## III. MESOSCOPIC RLC CIRCUIT

Let us now consider the quantization of the mesoscopic RLC circuit using the invariant operator method. We will quantize the mesoscopic RLC circuit using two different choices for the solution to the auxiliary equation: The first solution will be determined by solving the Ermakov equations and the second solution will be to choose a particular solution to the Milne-Pinney equation. We can then compare the two solutions and point out differences in each in terms of the instantaneous occupation number and the Berry phase.

The classical Hamiltonian associated with the RLC circuit, which is known as the Caldirola-Kanai Hamiltonian [11, 12], is given by

$$H(t) = e^{-Rt/L}\frac{\Phi^2}{2L} + \frac{1}{2}e^{Rt/L}L\omega^2 q^2, \tag{18}$$

where $q$ is the charge, $\Phi$ is the magnetic flux (which is the conjugate momentum to the charge, $p = \Phi = L\frac{dq}{dt}$), $L$ is the inductance, $R$ is the resistance, and $\omega^2 = \frac{1}{LC}$ is the frequency ($C$ is the capacitance). From (18), we can see that the structure is that of a time-dependent harmonic oscillator with a time dependent mass, $M(t) = Le^{Rt/L}$, and constant frequency. From (13) and (18), we can determine the equation of motion for the charge to be

$$q(t) = Ae^{-Rt/2L}\sin(\Omega t + \theta),$$

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[3] Equivalently one can consider the number of $\hat{c}_k$ particles in the ground state $|0\rangle_a$.

where $A$ and $\theta$ are constants to be determined by initial conditions and $\Omega^2 = \omega^2 - \left(\frac{R}{2L}\right)^2$.

We can now determine the solution to the auxiliary equation $\rho$. From the Ermakov equations, the auxiliary equation is then given by

$$\rho_E = |q| = Ae^{-Rt/2L}\sin(\Omega t + \theta) = A\sqrt{\frac{L}{M}}\sin(\Omega t + \theta). \tag{19}$$

From [5], a particular solution to the Milne-Pinney equation is given by

$$\rho_P = \sqrt{\frac{1}{M\Omega}}. \tag{20}$$

From (19) and (20), one can then determine the wave functional (8), coherent states, and the uncertainty product for the coherent states (A2) for each of the two cases. However, as stated above, we are mostly interested in the occupation number and the Berry phase.

### A. Occupation Number

Using (19) and (15), we can determine the occupation number associated with the solution to the Ermakov equations to be

$$N_E(\omega, t) = \frac{1}{4}A^2 L\Omega\sin^2(\Omega t + \theta)\left[\frac{(R - 2L\Omega\cot(\Omega t + \theta))^2}{4L^2\Omega^2} + \left(\frac{1}{A^2 L\Omega\sin(\Omega t + \theta)} - 1\right)^2\right], \tag{21}$$

which is an oscillatory function of time. Hence, the number of quasi-particles present varies over time, but repeats itself from period to period.

Using (20) and (15), we can determine the occupation number associated with the particular solution to the Milne-Pinney equation to be

$$N_P(\omega, t) = \frac{R^2}{16L^2\Omega^2}, \tag{22}$$

which is constant in time. That is, regardless of the time, the number of induced quasi-particles stays the same regardless of the time.

Therefore, if we measure the occupation number of the induced quasi-particles that are produced over one-period of oscillation for the RLC circuit, we can then provide a direct insight into the time-dependence of the harmonic oscillator and the method to use to determine the solution for the auxiliary equation.

### B. Berry Phase

From (17), one may also determine the Berry phase for the mesoscopic RLC circuit for each of the two cases. Using (19), we can determine the Berry phase associated with the solution to the Ermakov equations to be

$$\alpha_{B,E} = \left(n + \frac{1}{2}\right)\frac{2A^2 L\pi\omega^2}{\Omega}. \tag{23}$$

Hence, there is an associated Berry phase with the solution to the Ermakov equations. Moreover, notice that in the limit the resistance goes to zero, hence the system reduces to that of a mesoscopic LC circuit, (23) reduces to

$$\begin{aligned}\alpha_{B,E} &= \left(n + \frac{1}{2}\right)2\pi\omega LA^2 \\ &= \left(n + \frac{1}{2}\right)2\pi A^2\sqrt{\frac{L}{C}};\end{aligned}$$

that is, even the mesoscopic LC circuit has an associated Berry phase.

Using (20), we can then determine the Berry phase associated with the particular solution to the Milne-Pinney equation to be

$$\alpha_{B,P} = 0, \tag{24}$$

hence there is no associated Berry phase since the system is not periodic.

Thus, just like in the case of the occupation number, we can in principle use the Berry phase to establish the auxiliary equation for the time-dependent system; that is, by measuring a Berry phase one can establish if one must use the Ermakov equations to solve for the solution to the auxiliary equation or use the particular solution to the Milne-Pinney equation or some other method.

## IV. CONCLUSION

In this paper, we investigated the quantization of a general time-dependent harmonic oscillator using the instantaneous diagonalization and invariant operator methods and showed that these two methods are related by a Bogolyubov type transformation. We also constructed the occupation number for the induced quasi-particles that are produced at any moment in time. Finally, we showed that the Lewis phase may be decomposed into a dynamic phase and the well-known Berry phase in the adiabatic limit. Furthermore, if the invariant operator is $T$-periodic, the Berry phase is the integral over the period of a single oscillation, (9). In order to determine the exact solution of the Schrödinger equation and the Berry phase, one must first determine the solution of the auxiliary equation, which, in general, is a non-linear differential equation. In the literature, there are different methods for handling such a non-linear differential equation. One such method is to consider a particular solution of the Milne-Pinney equation. Another is, for a set of initial conditions, to solve the auxiliary equation numerically. Finally, using the Ermakov equations, one can determine the solution by solving a much simpler equation for the time-dependent coordinate and then use the fact that the auxiliary solution is related to the time-dependent coordinate by taking the modulus of the time-dependent coordinate. By measuring the instantaneous occupation number and associated Berry phase will, in principle, give a method for determining the appropriate method for solving the auxiliary equation.

As a concrete example, we consider the mesoscopic RLC circuit, which is described by the Caldirola-Kanai Hamiltonian. The Caldirola-Kanai Hamiltonian has a time-dependent "mass" term and a constant angular frequency. First, we showed that the solution to the Ermakov equations leads to a time-dependent quasi-particle occupation number (21), while the particular solution to the Milne-Pinney solution leads to a time-independent quasi-particle occupation number (22). Next, we showed that the solution to the Ermakov equations leads to an associated Berry phase, (23), however, the particular solution to the particular solution to the Milne-Pinney solution leads no associated Berry phase, (24). Therefore, measurements of the occupation number and associated Berry phase, or no measurement, can, in principle, lead to a direct method for determining the appropriate method for determining the solution to the auxiliary equation.

### Appendix A: Coherent and Squeezed States

Even though the coherent and squeezed states are not necessary for our purposes of deriving the generalized Berry phase for the time-dependent harmonic oscillator, and hence the mesoscopic RLC circuit, they are germane to quantization of the time-dependent harmonic oscillator. Therefore here, we construct coherent states for the quantized time-dependent harmonic oscillator for completeness. Let us define the annihilation and creation operators of the time-dependent harmonic oscillator as

$$b = \sqrt{\frac{1}{2}}\left[\rho p - i\frac{q}{\rho}\right],\ b^{\dagger} = \sqrt{\frac{1}{2}}\left[\rho p + i\frac{q}{\rho}\right],$$

where $[b, b^{\dagger}] = 1$, so that the invariant operator may be written as $I' = \left(b^{\dagger}b + \frac{1}{2}\right)$. The coherent states associated with $I'$ are then

$$\varphi_{\beta}(\sigma, t) = e^{-|\beta|^2/2}\sum_{n}\frac{\beta^n}{\sqrt{n!}}e^{i\alpha_n(t)}\varphi_n(\sigma),$$

where $\beta$ is an arbitrary complex number. The coherent states for the time-dependent harmonic oscillator are given by

$$\phi_{\beta}(q, t) = \frac{1}{\sqrt{\rho}}\exp\left[\frac{iM\dot{\rho}}{2\rho}q^2\right]\varphi_{\beta}(\sigma, t).$$

These states must satisfy the eigenvalue equation

$$c\phi_{\beta}(q, t) = \alpha(t)\phi_{\beta}(q, t), \tag{A1}$$

where $c$ and $b$ are related by

$$c = \mathcal{U}^\dagger b \mathcal{U}.$$

Thus, the invariant operator takes the form $I = \left(c^\dagger c + \frac{1}{2}\right)$ as above.

We can now consider the expectation value of the coordinate $q$ in the state $\phi_\beta(q,t)$. Here, we find that the expectation value of the coordinate is given by

$$\langle q \rangle = \sqrt{2|\beta|^2}\rho \sin(\alpha_0 t + \delta)$$

where $\delta$ is the argument of $\beta$. The uncertainty product is given by,

$$(\Delta q)(\Delta p) = \frac{1}{2}\sqrt{1 + M^2\rho^2\dot{\rho}^2}. \tag{A2}$$

From (A1), (11) and (12), we can see the states $\phi_\beta(q,t)$ are in fact the well-known squeezed states. In terms of the Bogoliubov coefficients, the quantum fluctuations in $q$ and $p$ in the squeezed states may be written as

$$(\Delta q)^2 = \frac{1}{2M\omega}|\mu - \nu|^2, \ (\Delta p)^2 = \frac{M\omega}{2}|\mu + \nu|^2,$$

and hence the uncertainty product takes the form

$$(\Delta q)(\Delta p) = \frac{1}{2}|\mu - \nu||\mu + \nu|,$$

which is the same as in (A2).


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