

Original Research Article

Generalized Uncertainty Relations in Quantum Mechanics and the Principles of Completeness in Physics

Abstract. Focusing on the initial development of quantum mechanics, we will give a brief historical synopsis of the theory foundations based on Fourier's framework and about the philosophical conclusions inspired by that same mathematical formalism. Next, we shall present an alternative local way of describing the undulatory aspects of quantum entities using gaussian Morlet wavelets. As we shall see, this change implies different philosophical conclusions about quantum reality and even more, about the contemporary accepted differences between quantum and macroscopic realms. From these we shall see the formal and heuristic power of wavelet local analysis for the physical description of Nature.

Keywords: Orthodox quantum mechanics, nonlinear quantum physics, nonlocal Fourier analysis, local analysis by wavelets, Fourier ontology, Heisenberg uncertainty relations, general uncertainty relations, Principles of Completeness.

1. Introduction

It is always a motive of wonder and surprise when a mathematical formulation adequately describes physical reality and inspires human thought about Nature's inner processes. Perhaps, even more important, when it deeply influences the relationship between man as a sentient creature and the world he tries to understand.

In this paper, we shall give a brief historical synopsis of the development of quantum mechanics based on Fourier's ontology¹, focusing on the philosophical conclusions we usually hear whenever the theory is described or taught by the orthodox stand. Afterward, we will present an alternative local way of describing the undulatory aspects of quantum entities using gaussian Morlet wavelets. As we shall see, this will imply different philosophical conclusions about the quantum reality and even about contemporary accepted differences between atomic and macroscopic phenomena.

2. The Fourier formalism at the beginning of quantum mechanics

In 1927, in a lecture held at the Volta Conference in Como², Italy, Niels Bohr argued that the phenomenon of physical quantization put in evidence universal limitations on our knowledge about Nature. This was the "epistemological lesson taken from quantum mechanics", as he so eloquently put it. His narrative began by stating that the quantized nature of atomic systems was a consequence of the quantum of action given by Planck's constant h , that has a very small value, but which is, nevertheless, non-null. Due to the fact, that such a value is different from zero it became possible to quantify energy, as Planck concluded earlier in 1900, to consider light behaving like corpuscles, as Einstein suggested in 1905, and finally to universally associate each particle with its own matter wave, as de Broglie conjectured in his Ph.D. thesis in 1924.

By 1927, at the time Bohr was fully developing his own philosophical view about Nature, the Planck-Einstein relation between energy and frequency, $E = h\nu$, was well known and de Broglie had already introduced his famous relation $p = h/\lambda$ between linear

momentum and matter wavelengths, for all corpuscles. It was then quite remarkable that Niels Bohr, from the initial work of Heisenberg could develop, in the summer of 1927, a mathematical framework describing in a relational way both the corpuscular and undulatory aspects of quantum entities. Indeed, based on Fourier nonlocal and nontemporal analysis, Bohr derived in a very beautiful way the same uncertainty relations Heisenberg told him about some months earlier, in March of 1927.

The basic idea of Niels Bohr was to associate the spatial k and temporal ω frequencies, present in the most fundamental phenomenological formulas of quantum physics, *viz.* de Broglie's

$$p = \hbar k \quad (1)$$

and Planck's

$$E = \hbar \omega \quad (2)$$

with infinite nonlocal and nontemporal harmonic plane waves, spreading over the entirety of time and space

$$\psi = Ae^{i(kx - \omega t)}. \quad (3)$$

From it, by a simple utilization of Fourier analysis

$$f(x) = \int_{-\infty}^{\infty} g(k) e^{ikx} dk, \quad (4)$$

and assuming a Gaussian form for the distribution of the spatial frequency,

$$g(k) = Ae^{-\frac{k^2}{2\sigma_k^2}} \quad (5)$$

One gets, by substitution,

$$f(x) \propto Ae^{-\frac{x^2}{2/\sigma_k^2}} \quad (6)$$

or

$$f(x) \propto Ae^{-\frac{x^2}{2\sigma_x^2}} \quad (7)$$

implying that

$$\sigma_x \sigma_k = 1. \quad (8)$$

For the time parameter, and following the same procedure, we would get

$$\sigma_t \sigma_\omega = 1. \quad (9)$$

Now, by taking in consideration de Broglie and Planck formulas, (1) and (2), we get by substitution and for the ideal case:

$$\Delta x \Delta p_x = \hbar \text{ and } \Delta E \Delta t = \hbar, \quad (10)$$

In the general situation, we will have

$$\Delta x \Delta p_x \geq \hbar \text{ and } \Delta E \Delta t \geq \hbar, \quad (10')$$

which constitute the habitual mathematical form for the uncertainty relations.

These relations were derived assuming that the only waves that have single spatial ($\Delta k = 0$) and single temporal ($\Delta \omega = 0$) frequencies are harmonic plane waves, that is, pure sine and cosine waves spreading over the entirety of the universe, along the whole space and time, which in fact is the kernel of Fourier analysis.

One immediate consequence of these implicit assumptions, contained in Heisenberg relations, is that if a quantum particle, a neutron for instance, has a pure single energy, $\Delta E = 0$, and single momentum, $\Delta p_x = 0$, then, it is omnipresent in space and time. On the contrary, if one wishes to know the position and time with absolute precision, $\Delta x = 0$ and $\Delta t = 0$, then, the particle has an infinite number of momenta and energies, $\Delta p_x = \infty$ and $\Delta E = \infty$. Therefore, in this mathematical and conceptual framework we are logically prohibited from simultaneously knowing the exact position and the exact momentum of the neutron.

Bohr, with considerable philosophical insight, concluded that quantum behavior was not fully addressable using a causal space-time framework, as done until that time for macroscopic systems. At the Como conference, in 1927, he presented to an audience of physicists his own views about quantum phenomena, formulating his famous Complementary Principle, that must be expressed mathematically in terms of Fourier analysis.

The Complementary Principle states, that at the quantum level the possibility of a space-time coordinate description is complementary with the possibility of a causal description. This can be seen from the uncertainty relations if one takes linear momentum and energy to be major concepts in a dynamical causal description of an atomic entity. This description will of course be masked by any effort to coordinate such an entity in space and time. Thus, in Bohr's initial and most fundamental views on complementarity, space-time coordination is mutual exclusive to dynamical causal modeling.

By the same time, Max Born proposed a probabilistic interpretation for quantum waves associated with corpuscles. Thus, it is not difficult to accept that Born's probabilistic interpretation, plus the epistemological standpoint of Bohr, has eventually taken the theory to a very extreme idealistic form. In fact, into what is today commonly known as the Copenhagen interpretation. And, in a way, very much due to Bohr's latest thoughts on the matter, we now have infinite elusive probability waves devoid of any physical content, representing the state of knowledge of conscious observers about the world. These waves, although not being real physical entities, are nevertheless capable of interacting with slits and other physical entities, and are readily disposable for all practical purposes by means of a "collapse", once an observation has taken place. In fact, the metaphysical nature of the so-called psi waves implies that atomic objects are only potential entities. Not really existing before being observed and only coming to existence due to the observer's consciousness powers.

Louis de Broglie³, in the celebrated fifth Solvay Conference of September 1927, presented an alternative view to Bohr's platonic beliefs about quantum waves, based upon some of his own earlier work. de Broglie thought that the quantum waves associated with atomic phenomena would be real physical perturbations in a subquantum medium, guiding or piloting in a nonlinear way, the trajectory of corpuscles. This was a causal nonlinear theory where the statistically empirical aspects were only a consequence of the experimenter's lack of knowledge about the initial velocities of particles. Although de Broglie's theory was quite sophisticated, offering domain for further improvement, it had a major drawback. These de Broglie waves, although taken as real physical entities, were still infinite Fourier, sine and cosine, oscillations spreading throughout the whole of space and time. Consequently,

Heisenberg indetermination scheme would still hold, although de Broglie reasoned that such restriction would be caused only by the measuring devices, imprinting a perturbation on the measured systems. The overall positivist view of Bohr and the Copenhagen school would however see this as a highly suggestive evidence for the presence of human knowledge constraints. And, as even more evidence of the incomplete nature of atomic objects, metaphysically held from existence in their potential state. As a result, de Broglie's real pilot wave theory was refused to be only later and partially reinstated by David Bohm pilot-wave theory, preserving the causal part, dispensing the "collapse" by a conscious observer, but still having to place the quantum waves in a metaphysical configuration space of Fourier ontology.

3. Wavelet formalism applied to quantum mechanics

At the beginning of the present century and giving sequence to de Broglie's research program, a nonlinear proposal for understanding quantum phenomena was presented⁴. This theory, included formally traditional quantum mechanics as a particular case, describing the same phenomena as the orthodoxy, but where the Schrödinger wave equation was substituted by a nonlinear master wave equation. Since we are dealing with a nonlinear equation, the sum of two different solutions is not, in general, a solution to the same equation. Relevant to the present discussion is the fact that within the same approach for understanding Nature, and following de Broglie original suggestion, a quantum object is perceived as one complex inter-relational structure constituted both by a corpuscular part and by an extended undulatory part. The corpuscular part of the quantum particle, the acron, contains practically all the energy, while the extended finite real wave, named the theta wave, subquantum wave, de Broglie wave, pilot wave or vacuum wave, is practically devoid of energy. Still, due to a nonlinear process, the subtle wave guides the high energetic corpuscle to the regions where the intensity of the theta wave is higher.

In this non-linear approach to quantum phenomenology, and here is the crucial point, the extended undulatory part is mathematically represented, not by a nonlocal harmonic infinite plane wave, but by a finite gaussian Morlet wavelet⁵,

$$\theta = Ae^{-\frac{(x-vt)^2}{2\sigma^2}+i(kx-\omega t)} \quad (11)$$

In this expression, v stands for the wave average velocity of the finite wave, which is in general the same as the particles velocity, k is the spatial angular frequency, ω is the angular frequency and σ is the wavelet's spatial extension parameter, giving a value of how much the wave spreads throughout space before decaying.

We stress, that in this model, there will be no infinite waves, because real physical perturbations need to be finite.

Such a choice for a quantum wave representation would have two major consequences:

The first, is that there wouldn't be an absolute need for wave packaging in order to form a system localized in space and time, still presenting, however, both corpuscular and undulatory properties. The two kinds of properties would be intrinsically given from first principles.

The second, consequence is that these same two kinds of properties would migrate to larger systems composed through wavelet packaging, allowing, naturally, the derivation of a more general set of uncertainty relations.

Having said this, we now proceed to compare the two kinds of uncertainty relations. Those, resultant from nonlocal Fourier ontology, championed by orthodox quantum mechanics, to which Bohr gave so much philosophical emphasis, and the ones coming from local analysis by wavelets, resulting from the above nonlinear quantum physics description.

First of all, it is interesting to see that the mathematical formulation in (11) contains the nonlocal and nontemporal harmonic plane wave used in orthodox quantum mechanics as a particular case. It is thus heuristically richer. In fact, as the size of the wavelet increases, when $\sigma \rightarrow \infty$, one obtains:

$$\theta = Ae^{-\frac{(x-vt)^2}{2\sigma_{0x}^2} + i(kx - \omega t)} \xrightarrow{\sigma \rightarrow \infty} Ae^{i(kx - \omega t)} \quad (12)$$

the harmonic plane wave.

By following a process similar to the one followed by Bohr, it is possible to derive at a more general set of uncertainty relations, going beyond Fourier ontology, assuming that a finite wave may have a single pure frequency

$$\theta(x) = e^{-\frac{x^2}{2\sigma_{0x}^2} + ik_0 x} \quad (13)$$

Since any regular function, may be written in terms of wavelets, we have

$$\theta(x) = \int_{-\infty}^{\infty} g(k) e^{-\frac{x^2}{2\sigma_{0x}^2} + ikx} dk. \quad (14)$$

Where σ_{0x} represents the spatial spreading of the mother wavelet used to make the wavelet packaging.

Choosing, as before, a gaussian form for the height function, $g(k)$, after some easy calculation we arrive at the general uncertainty relations:

$$\Delta x^2 = \frac{\hbar^2}{\Delta p_x^2 + \frac{\hbar^2}{\sigma_{0x}^2}} \quad \text{and} \quad \Delta t^2 = \frac{\hbar^2}{\Delta E^2 + \frac{\hbar^2}{\sigma_{0t}^2}}. \quad (15)$$

The plot of the first relation is shown in Fig.1, for different values of σ_{0x}

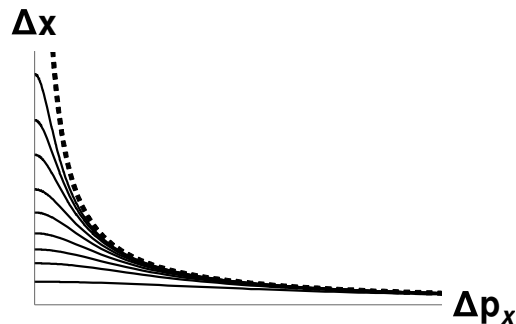


Fig.1 - First general uncertainty relation for different values of σ_{0x}

In Fig.1 we show several instances of the first general uncertainty relation in (15). Each line reaches the vertical Δx axis on a different value of σ_{0x} , which increases upwards. The horizontal axis represents Δp_x . The usual uncertainty relations, as is well known, would be here represented by a hyperbola, the dashed line in the plot. It is clear, that the largest value of Δx will be equal to σ_{0x} when $\Delta p_x = 0$, so that σ_{0x} also represents the maximum value of Δx .

From these general uncertainty relations, one sees that when the size of the mother wavelet is relatively large ($\sigma_{0x} \rightarrow \infty$), the general uncertainty relations transform into the traditional Heisenberg relations:

$$\Delta x \Delta p_x \geq \hbar \text{ and } \Delta E \Delta t \geq \hbar.$$

The general uncertainty relations (15) may also be written in the form,

$$\Delta p_x = \frac{\hbar}{\Delta x} \sqrt{1 - \Delta x^2 / \sigma_{0x}^2}, \quad (16)$$

$$\Delta E = \frac{\hbar}{\Delta t} \sqrt{1 - \Delta t^2 / \sigma_{0t}^2}, \quad (17)$$

in which we can consider $\sigma_{0x} = v \sigma_{0t}$.

Now, from (16) and (17) one notes that if the space coordinate uncertainty Δx (or inversely understood, the space coordinate accuracy) is of same magnitude as the spatial spreading of the mother wavelet σ_{0x} , one can have a linear momentum accuracy as large as one wants, independently of the spatial coordinate accuracy. This happens, even if the spatial required accuracy Δx is very high, in which case we must make σ_{0x} very small. That is:

$$\text{If } \Delta x \simeq \sigma_{0x} \quad \text{then} \quad \Delta p_x \simeq 0, \quad \forall \Delta x \quad (18)$$

Similarly,

$$\text{If } \Delta t \simeq \sigma_{0t} \quad \text{then} \quad \Delta E \simeq 0, \quad \forall \Delta t. \quad (18')$$

This implies, (18) and (18'), that for the particular limit situation, in which we have, $\Delta x \simeq \sigma_{0x} \simeq 0$, we may also have, $\Delta p_x = 0$. In this natural way, we have regained the possibility of describing a physical situation using both a space and time description. We have done a quantum measurement, with an ideal zero error in the determination of both the position and momentum, that is, $\Delta x = 0 \wedge \Delta p_x = 0$. In other words, we have thus gone beyond Niels Bohr Complementary Principle. This principle, claims that if $\Delta x = 0$ then $\Delta p_x = \infty$, or if $\Delta p_x = 0$, then $\Delta x = \infty$, that is, we must always have, $\Delta x = 0 \vee \Delta p_x = 0$.

On the other hand, and once again, if one takes the spatial accuracy Δx in (16) to be of an order of magnitude smaller than the spatial spreading of the mother wavelet σ_{0x} , one gets the Heisenberg standard relation. That is,

$$\text{If } \Delta x \simeq 0 \text{ and } \Delta x \ll \sigma_{0x} \quad \text{then} \quad \Delta p_x \rightarrow \infty, \quad (19)$$

and similarly,

$$\text{If } \Delta t \simeq 0 \text{ and } \Delta t \ll \sigma_{0t} \quad \text{then} \quad \Delta E \rightarrow \infty \quad (19')$$

This, of course, means that in this situation there will be an inverse proportional relation between error and accuracy measurement of dynamical (undulatory) physical quantities versus space-time physical quantities.

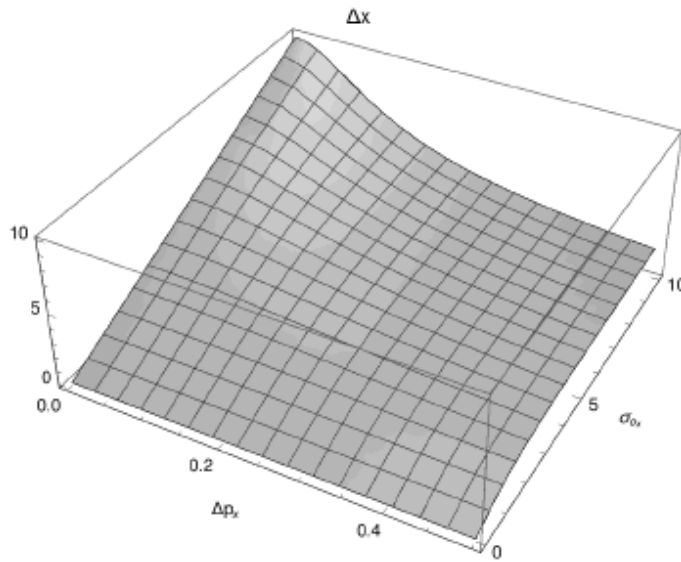


Fig.2 – Plot of Δx as a function of σ_{0x} and Δp_x .

Fig. 2 shows a 3D plot of Δx , accordingly to the first generalized relation in (15). The case shown in (18) and (18'), that is, for the situation where we have low values for Δx and Δp_x (i.e. high accuracy for both physical measures) corresponds to the left low corner in the graphic, where we also have a low value for σ_{0x} . Note that it will always be $\Delta x \leq \sigma_{0x}$.

Inspired by what Niels Bohr thought, about a century ago, giving his own interpretation to the Heisenberg's relations, we will now argue philosophically in a quite different way, giving our own interpretation to (16) and (17).

Since Heisenberg relations, may be seen as a particular case of the general uncertainty relations, one may infer that if we change from nonlocal Fourier representation to local wavelet representation, then it stands to reason that the Complementary Principle will lose its universal status. This occurs, even in the quantum case, because for the Complementary Principle to hold, it would be necessary to guarantee that the case considered in (18) and (18') never occurs at the atomic level, when in fact there are good reasons to suppose it does⁶.

From this, it seems reasonable to invoke a first alternative general principle stating that:

Nature, is an ontological complete structure, without any logical mutual exclusiveness, i.e., an ontological unified structure, which is consistent and intelligible.

Furthermore, we wish to remark that expressions (18) and (18') define a situation where a causal dynamical description and space-time description of phenomena are simultaneously feasible. Something very much akin to what we are used to at the macroscopic scale.

Since for (16) and (17), we still assume the basic phenomenological fundamental formulas, (1) and (2):

$$\begin{aligned} p_x &= \hbar k, \\ E &= \hbar \omega, \end{aligned}$$

and given that we have just sustained a sort of unification statement about Nature, we now go a step further, hypothesizing the existence of only one description scheme similarly applicable to both scales, atomic and macroscopic.

We thus also propose the further principle that:

Nature, can be physically described at all scales using both a causal undulatory scheme and a space-time local scheme, with variable accuracy dependence between the two descriptions.

The accuracy dependence between those descriptions will be of course given by (16) and (17), to which we may well call the relations of completeness, while the two posited statements appear as a kind of Completeness Principles about Nature. It should be noted that as Niels Bohr Complementary Principle is accepted only on an epistemological basis, and therefore not as an absolute statement about Nature, it will be a correct particular case of the second Completeness Principle.

An obvious critique to the second statement is that things do not seem to adopt undulatory behaviors at the macroscopic scale. To this we will reply that this is not entirely true. Even if in most cases, macroscopically observable undulatory behaviors do not express themselves under our presently normal use of objects, that does not mean that they aren't observable under other conditions. This is, for instance, the case of the so-called Doubochinski's pendulum and similar coupled oscillatory systems [9]. Furthermore, Croca *et al.*⁸ have found that the Titius Bode planetary regularity can be adequately addressed using an undulatory formalism involving stationary pilot-waves.

4. Conclusion

We have seen how different mathematical descriptions can lead to quite different philosophical conclusions about Nature. This seems to be indeed the case where wavelet local analysis, once applied to quantum description, has proven itself richer than the former nonlocal Fourier scheme. Independently of the how far long the reach of our proposed insights and interpretations may be, it seems undoubtable that the local wavelet representation of undulatory phenomena is far richer from a mathematical and heuristic point of view than the Fourier approach.

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