Original Research Article

THE COHERENT STATES AND LUCAS NUMBERS

ABSTRACT. In this paper we consider the coherent states which play an important role in quantum optics, especially in laser physics and much work in this field. Here we connect the coherent states with the Lucas numbers and Fibonacci numbers.

1. INTRODUCTION

The term coherent state, also called Glauber state, has been introduced by Roy J. Glauber [4] in 1963 year. It is not strongly related to the classical term coherence, and refers to a special sort of pure quantum mechanical state of the light field corresponding to a single resonator mode.

We describe a dynamical system in terms of a pair of complex operators a and $a^\dagger$, which we call them as the annihilation and creation operators. These operators, which obey the following commutation relation

$$[a, a^\dagger] = 1,$$

play a fundamental role in descriptions of systems of harmonic oscillators and quantized fields. It is obvious from the algebraic properties of the operators a and $a^\dagger$ that we may construct a sequence of states for the harmonic oscillator system. These states labeled by $|n\rangle$ satisfy the identity

$$\begin{aligned} a|n\rangle &= \sqrt{n}|n-1\rangle, \\ a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \\ a^\dagger a|n\rangle &= n|n\rangle \end{aligned} \tag{1}$$

for an nonnegative integer n. They are generated from the state $|0\rangle$ by the rule

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{2}$$

Let us now define for each complex number α the displacement operator

$$D(\alpha) = \exp(\alpha a^\dagger - \bar{\alpha} a), \tag{3}$$

which is unitary and obeys the relation

$$D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha).$$

When a and b commute with their commutator $c := [a, b]$ we have the well-known Kermack-McCrae identity

$$\exp(a+b) = \begin{cases} \exp(-\frac{1}{2}c)\exp(a)\exp(b), & \text{if ab-ordered,} \\ \exp(\frac{1}{2}c)\exp(b)\exp(a), & \text{if ba-ordered,} \end{cases}$$

therefore we are led to

$$D(\alpha) = \exp(-\frac{|\alpha|^2}{2})\exp(\alpha a^\dagger)\exp(-\bar{\alpha} a). \tag{4}$$

For each complex number α the coherent state $|\alpha\rangle$ is defined by

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(5) $$|\alpha\rangle = D(\alpha)|0\rangle.$$

We note that the state $|\alpha\rangle$ is an eigenstate of the operator a with eigenvalue α,

(6)
$$a|\alpha\rangle = \alpha|\alpha\rangle \qquad \text{and} \qquad \langle\alpha|a^\dagger = \langle\alpha|\bar{\alpha}.$$

By using Eqs. (2), (4), (5), and the fact $a|0\rangle = 0$, we may relate the coherent states to the states $|n\rangle$:

(7)
$$\begin{aligned}
&|\alpha\rangle \\
&= D(\alpha)|0\rangle \\
&= \exp(-\frac{|\alpha|^2}{2})\exp(\alpha a^\dagger)\exp(-\bar{\alpha}a)|0\rangle \\
&= \exp(-\frac{|\alpha|^2}{2})\exp(\alpha a^\dagger)|0\rangle \\
&= \exp(-\frac{|\alpha|^2}{2})\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle
\end{aligned}$$

(see [3, (2.23)]).

In this paper we consider the coherent states as the special states, that is, the eigenvalues α and β of the eigenstates $|\alpha\rangle$ and $|\beta\rangle$ respectively, satisfy the coefficients conditions appearing in Lucas numbers as follows :

(8) $$\alpha+\beta = \frac{1}{2}(1+\sqrt{5})+\frac{1}{2}(1-\sqrt{5}) = 1,$$

(9) $$\alpha\beta = \frac{1}{2}(1+\sqrt{5})\cdot\frac{1}{2}(1-\sqrt{5}) = -1,$$

(10)
$$\alpha^2 = 1+\alpha, \qquad \text{and} \qquad \beta^2 = 1+\beta.$$

We can set the Lucas numbers [5] by

(11) $$L_n = \alpha^n + \beta^n,$$

where

(12)
$$\alpha = \frac{1}{2}(1+\sqrt{5}), \qquad \beta = \frac{1}{2}(1-\sqrt{5}).$$

Depending on the above properties we obtain :

Lemma 1.1. *Let α and β be in (12). And let the operator a apply to the coherent states $|\alpha\rangle$ and $|\beta\rangle$. Then*

(a)
$$\langle\beta|\alpha\rangle = \exp(-\frac{5}{2}),$$

(b)
$$\langle\beta|\exp(\frac{5}{2}a^\dagger)\exp(\frac{5}{2}a)|\alpha\rangle = 1.$$

Theorem 1.2. *Let $n \in \mathbb{N}$. Then*

$$\langle\beta|\left(a^n + (a^\dagger)^n\right)|\alpha\rangle = L_n\exp(-\frac{5}{2}).$$

Theorem 1.3. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned}
&\sum_{m=0}^{n}\langle\alpha|\left(a^{n-m}+(a^\dagger)^{n-m}\right)|\beta\rangle \\
&\quad\times\langle\beta|\left(a^m+(a^\dagger)^m\right)|\alpha\rangle \\
&= e^{-10}\left((n+1)L_n + 2F_{n+1}\right),
\end{aligned}$$

where $F_n := \dfrac{\alpha^n-\beta^n}{\alpha-\beta}$ is the Fibonacci number.

2. Proofs of Lemma 1.1, Theorem 1.2, and Theorem 1.3

Let $\mathbb{N}$ be the set of positive integers. Then we define the Lucas numbers, L_n with $n \in \mathbb{N}$, by

$$L_0 = 2, \qquad L_1 = 1,$$

and

$$L_{n+2} = L_{n+1} + L_n.$$

The very general functions studied by Lucas and generalized by Bell [1], [2], are essentially the L_n defined by (11) with α, β being the roots of the quadratic equation $x^2 = Px - Q$ so that $\alpha+\beta = P$ and $\alpha\beta = Q$.

Proof of Lemma 1.1. (a) First from the definition of α and β in (12) we note that

$$\bar{\alpha} = \alpha \qquad \text{and} \qquad \bar{\beta} = \beta.$$

Then by (7), (8), (9), and (10) we have

$$\begin{aligned}
&\langle \beta | \alpha \rangle \\
&= \langle m | \exp(-\frac{|\beta|^2}{2}) \sum_{m=0}^{\infty} \frac{\bar{\beta}^m}{\sqrt{m!}} \\
&\quad \times \exp(-\frac{|\alpha|^2}{2}) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\
&= \exp(-\frac{\beta^2}{2} - \frac{\alpha^2}{2}) \\
&\quad \times \sum_{m,n=0}^{\infty} \frac{\beta^m \alpha^n}{\sqrt{m!n!}} \langle m | n \rangle \\
&= \exp(-\frac{1}{2}(\beta + 1 + \alpha + 1)) \\
&\quad \times \sum_{m,n=0}^{\infty} \frac{\beta^m \alpha^n}{\sqrt{m!n!}} \delta_{m,n} \\
&= \exp(-\frac{3}{2}) \sum_{n=0}^{\infty} \frac{(\alpha\beta)^n}{n!} \\
&= \exp(-\frac{3}{2}) \exp(-1) \\
&= \exp(-\frac{5}{2}).
\end{aligned}$$

(b) By (6) and Lemma 1.1 (a) we have

$$\begin{aligned}
&\langle \beta | \exp(\frac{5}{2} a^\dagger) \exp(\frac{5}{2} a) | \alpha \rangle \\
&= \exp(\frac{5}{2} \bar{\beta}) \exp(\frac{5}{2} \alpha) \langle \beta | \alpha \rangle \\
&= \exp(\frac{5}{2} (\beta + \alpha)) \langle \beta | \alpha \rangle \\
&= \exp(\frac{5}{2}) \exp(-\frac{5}{2}) \\
&= 1.
\end{aligned}$$

□

Proof of Theorem 1.2. From (6) and Lemma 1.1 (a) we obtain

$$\begin{aligned}
&\langle \beta | \left(a^n + (a^\dagger)^n \right) | \alpha \rangle \\
&= \langle \beta | a^n | \alpha \rangle + \langle \beta | (a^\dagger)^n | \alpha \rangle \\
&= \alpha^n \langle \beta | \alpha \rangle + \bar{\beta}^n \langle \beta | \alpha \rangle \\
&= (\alpha^n + \beta^n) \langle \beta | \alpha \rangle \\
&= L_n \exp(-\frac{5}{2}).
\end{aligned}$$

□

In Figure 1, Figure 2, and Figure 3 we depict

$$\langle \beta | \left(a^n + (a^\dagger)^n \right) | \alpha \rangle = L_n \exp(-\frac{5}{2})$$

in Theorem 1.2. Here we can know that as n approaches to a large positive integer, the value $\langle \beta | \left(a^n + (a^\dagger)^n \right) | \alpha \rangle$ is bigger. And a transition from the $|\alpha\rangle$ state to $|\beta\rangle$ state behaves like a step function. If $\langle \beta | \left(a^n + (a^\dagger)^n \right) | \alpha \rangle$ stands for the probability then physically we should restrict $n = 0, 1, 2, 3, 4, 5$ since the probability is greater or equal to 0 and less than or equal to 1.

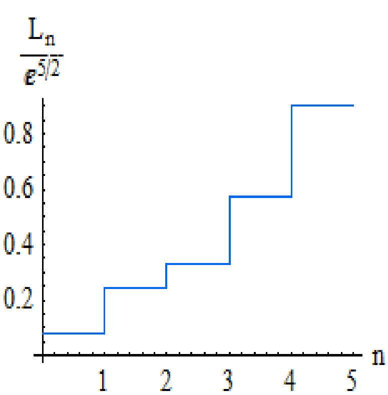

FIGURE 1. $L_n e^{-5/2}$ versus n $(0 \le n \le 5)$

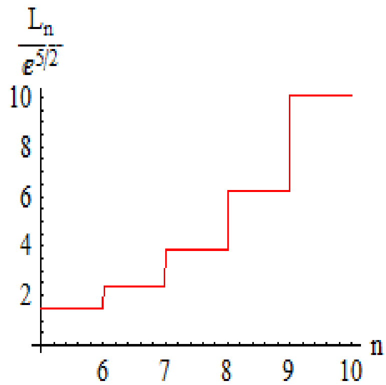

FIGURE 2. $L_n e^{-5/2}$ versus n $(5 \le n \le 10)$

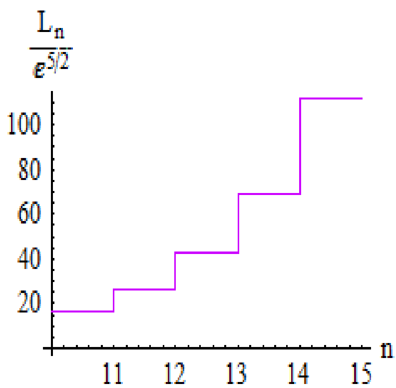

FIGURE 3. $L_n e^{-5/2}$ versus n $(10 \le n \le 15)$

To obtain the sums of the coherent states

$$\langle\alpha|\left(a^{n-m}+(a^{\dagger})^{n-m}\right)|\beta\rangle \times \langle\beta|\left(a^{m}+(a^{\dagger})^{m}\right)|\alpha\rangle$$

in Theorem 1.3 we request the following identity :

(13)
$$\sum_{m=0}^{n} L_m L_{n-m} = (n+1)L_n + 2F_{n+1}$$

(see [6]).

Proof of Theorem 1.3. From Theorem 1.2 and (13) we have

$$\begin{aligned}
&\sum_{m=0}^{n}\langle\alpha|\left(a^{n-m}+(a^{\dagger})^{n-m}\right)|\beta\rangle \\
&\qquad \times \langle\beta|\left(a^{m}+(a^{\dagger})^{m}\right)|\alpha\rangle \\
&= \sum_{m=0}^{n}\langle\alpha|\left(a^{n-m}+(a^{\dagger})^{n-m}\right)|\beta\rangle \\
&\qquad \times L_m \exp(-\frac{5}{2}) \\
&= \sum_{m=0}^{n} L_{n-m}\exp(-\frac{5}{2})\cdot L_m \exp(-\frac{5}{2}) \\
&= e^{-10}\sum_{m=0}^{n} L_m L_{n-m} \\
&= e^{-10}\left((n+1)L_n + 2F_{n+1}\right).
\end{aligned}$$

□

In Figure 4, Figure 5, Figure 6 , and Figure 7 we draw

$$\begin{aligned}
&\sum_{m=0}^{n}\langle\alpha|\left(a^{n-m}+(a^{\dagger})^{n-m}\right)|\beta\rangle \\
&\qquad \times \langle\beta|\left(a^{m}+(a^{\dagger})^{m}\right)|\alpha\rangle \\
&= e^{-10}\left((n+1)L_n + 2F_{n+1}\right)
\end{aligned}$$

in Theorem 1.3. In a similar manner to Figure 1, they are bigger as n is larger and the pictures jump abruptly at integer position but they grow linearly at non-integer spot. And if

$$\sum_{m=0}^{n}\langle\alpha|\left(a^{n-m}+(a^{\dagger})^{n-m}\right)|\beta\rangle \times \langle\beta|\left(a^{m}+(a^{\dagger})^{m}\right)|\alpha\rangle$$

implies the sum of transition probabilities then physically we should choose $n = 0, 1, 2, \cdots, 14$ because the sum of probabilities is greater or equal to 0 and less than or equal to 1. Furthermore if the number of transition occurs many times then the probability variation becomes smoothly compared to Figure 1, Figure 2, and Figure 3.

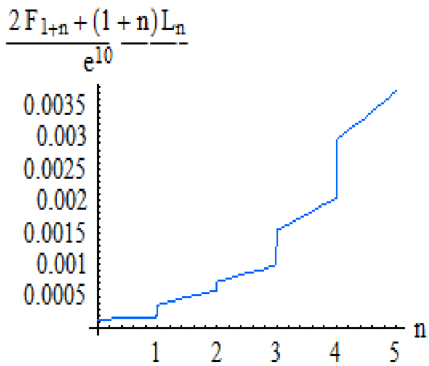

FIGURE 4. $e^{-10}\left((n+1)L_n+2F_{n+1}\right)$ versus n $(0 \leq n \leq 5)$

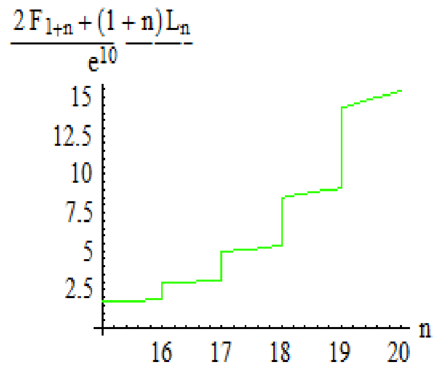

FIGURE 7. $e^{-10}\left((n+1)L_n+2F_{n+1}\right)$ versus n $(15 \leq n \leq 20)$

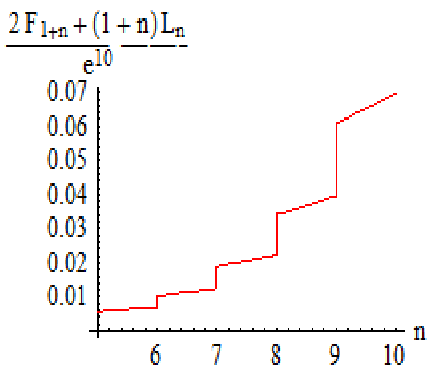

FIGURE 5. $e^{-10}\left((n+1)L_n+2F_{n+1}\right)$ versus n $(5 \leq n \leq 10)$

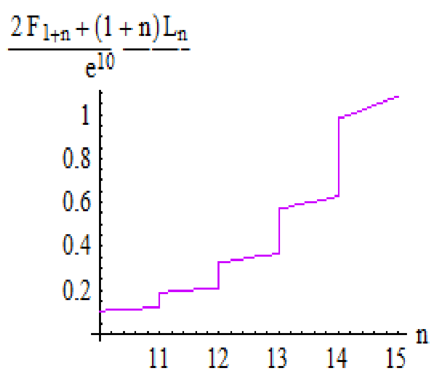

FIGURE 6. $e^{-10}\left((n+1)L_n+2F_{n+1}\right)$ versus n $(10 \leq n \leq 15)$

Next we analogize the coherent state $|\alpha^2\rangle$ and estimate the occupation number in Lemma 2.1. In advance by referring to (7) we note that

$$|\alpha^2\rangle = \exp\left(-\frac{|\alpha^2|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{\sqrt{n!}}|n\rangle \tag{14}$$

is adequate since

$$\begin{aligned}
&\langle\alpha^2|\alpha^2\rangle \\
&= \langle m|\exp\left(-\frac{|\alpha^2|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\bar{\alpha}^{2m}}{\sqrt{m!}} \\
&\qquad \times \exp\left(-\frac{|\alpha^2|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{\sqrt{n!}}|n\rangle \\
&= \exp(-\alpha^4)\sum_{m,n=0}^{\infty}\frac{\alpha^{2m}\alpha^{2n}}{\sqrt{m!n!}}\delta_{m,n} \\
&= \exp(-\alpha^4)\sum_{n=0}^{\infty}\frac{\alpha^{4n}}{n!} \\
&= \exp(-\alpha^4+\alpha^4) \\
&= 1.
\end{aligned}$$

Lemma 2.1. *Let $n \in \mathbb{N}$. Then*

(a)

$$\langle\beta|a|\alpha^2\rangle = \alpha^2\exp(-2\alpha-2),$$

(b)

$$\langle\beta|a^\dagger a|\alpha^2\rangle = -\alpha\exp(-2\alpha-2).$$

Proof. (a) By (1) and (14) we observe that

$$
\begin{aligned}
&\langle\beta|a|\alpha^2\rangle\\
&= \langle m|\exp(-\frac{|\beta|^2}{2})\sum_{m=0}^{\infty}\frac{\bar{\beta}^m}{\sqrt{m!}}\cdot a\\
&\quad\times\exp(-\frac{|\alpha^2|^2}{2})\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{\sqrt{n!}}|n\rangle\\
&= \exp(-\frac{\beta^2}{2}-\frac{\alpha^4}{2})\\
&\quad\times\sum_{m,n=0}^{\infty}\frac{\beta^m\alpha^{2n}}{\sqrt{m!n!}}\langle m|a|n\rangle\\
&= \exp(-\frac{\beta^2}{2}-\frac{\alpha^4}{2})\\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\beta^m\alpha^{2n}}{\sqrt{m!n!}}\langle m|\sqrt{n}|n-1\rangle\\
&= \exp(-\frac{\beta^2+\alpha^4}{2})\\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{\beta^m\alpha^{2n}}{\sqrt{m!n!}}\sqrt{n}\delta_{m,n-1}\\
&= \exp(-\frac{\beta^2+\alpha^4}{2})\\
&\quad\times\sum_{n=1}^{\infty}\frac{\beta^{n-1}\alpha^{2n}}{\sqrt{(n-1)!n!}}\sqrt{n}\\
&= \exp(-\frac{\beta^2+\alpha^4}{2})\alpha^2\sum_{n=1}^{\infty}\frac{(\beta\alpha^2)^{n-1}}{(n-1)!}\\
&= \alpha^2\exp(-\frac{\beta^2+\alpha^4}{2}+\beta\alpha^2)
\end{aligned}
$$

then by (8), (9), (10) and the fact

$$
\begin{aligned}
\alpha^4 &= (\alpha^2)^2\\
&= (\alpha+1)^2\\
&= \alpha^2+2\alpha+1\\
&= 3\alpha+2,
\end{aligned}
\tag{15}
$$

the above equation shows that

$$
\begin{aligned}
&\langle\beta|a|\alpha^2\rangle\\
&= \alpha^2\exp(-\frac{\beta+1+3\alpha+2}{2}-\alpha)\\
&= \alpha^2\exp(-\frac{\beta+5\alpha+3}{2})\\
&= \alpha^2\exp(-2\alpha-2).
\end{aligned}
$$

(b) In a similar style, by (1), (8), (9), (10), (14), and (15) we interpret

$$
\begin{aligned}
&\langle\beta|a^\dagger a|\alpha^2\rangle\\
&= \langle m|\exp(-\frac{|\beta|^2}{2})\sum_{m=0}^{\infty}\frac{\bar{\beta}^m}{\sqrt{m!}}\cdot a^\dagger a\\
&\quad\times\exp(-\frac{|\alpha^2|^2}{2})\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{\sqrt{n!}}|n\rangle\\
&= \exp(-\frac{\beta^2}{2}-\frac{\alpha^4}{2})\\
&\quad\times\sum_{m,n=0}^{\infty}\frac{\beta^m\alpha^{2n}}{\sqrt{m!n!}}\langle m|a^\dagger a|n\rangle\\
&= \exp(-\frac{\beta^2}{2}-\frac{\alpha^4}{2})\\
&\quad\times\sum_{m,n=0}^{\infty}\frac{\beta^m\alpha^{2n}}{\sqrt{m!n!}}\langle m|n|n\rangle\\
&= \exp(-\frac{\beta^2+\alpha^4}{2})\\
&\quad\times\sum_{m,n=0}^{\infty}\frac{\beta^m\alpha^{2n}}{\sqrt{m!n!}}\cdot n\delta_{m,n}\\
&= \exp(-\frac{\beta^2+\alpha^4}{2})\sum_{n=0}^{\infty}\frac{\beta^n\alpha^{2n}}{\sqrt{n!n!}}\cdot n\\
&= \exp(-\frac{\beta^2+\alpha^4}{2})\beta\alpha^2\\
&\quad\times\sum_{n=1}^{\infty}\frac{(\beta\alpha^2)^{n-1}}{(n-1)!}\\
&= -\alpha\exp(-\frac{\beta^2+\alpha^4}{2}+\beta\alpha^2)\\
&= -\alpha\exp(-2\alpha-2).
\end{aligned}
$$

$\square$

3. Conclusion

A product of quantum fields, or equivalently their creation and annihilation operators, is usually said to be normal ordered, also called Wick order, when all creation operators($= a^\dagger$) are to the left of all annihilation operators($= a$) in the product. On the other hand, if the annihilation operators are placed to the left of the creation operators then we define antinormal order. Even though the Wick order and Lucas numbers are recursively, they are strictly different. Wick order gives operators a sequence but Lucas numbers do not provide an order, instead they present an eigenvalue, an expectation value, etc., as a sort of scalar quantity.

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