

BOUND STATE SOLUTIONS OF THE KLEIN-GORDON EQUATION WITH MANNING-ROSEN PLUS YUKAWA POTENTIAL USING PEKERIS-LIKE APPROXIMATION OF THE COULOMB TERM AND PARAMETRIC NIKIFOROV-UVAROV

ABSTRACT

The solutions of the Klein-Gordon equation with Manning-Rosen plus Yukawa potential (MRYP) have been presented using the Pekeris-like approximation of the coulomb term and parametric Nikiforov-Uvarov (NU) method. The bound state energy eigenvalues and the corresponding un-normalized eigen functions are obtained in terms of Jacobi polynomials. Also, Yukawa, Manning-Rosen and coulomb potentials have been recovered from the mixed potential and their eigen values obtained.

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1 INTRODUCTION

Interestingly, the study of the relativistic wave equation in recent years, particularly the Klein-Gordon equation, has attracted the attention of many authors because the solutions of this equation plays the rule in getting the relativistic effect. It is well known that when a particle moves in a strong potential field, the relativistic effect yields the correction for non-relativistic quantum mechanics^[1-3]. Taking the relativistic effect into account, one could apply the Klein-Gordon equation to the treatment of a zero-spin particle and apply the Dirac equation to that of a 1/2-spin particle^[3, 4]. It contains of two major objects, the vector potential $V(r)$ and the scalar potential $S(r)$. The Klein-Gordon equation with the vector and scalar potentials can be written as follows:

$$\left[- \left(i \frac{\partial}{\partial t} - V(r) \right)^2 - \nabla^2 + (S(r) + M)^2 \right] \psi(r, \theta, \phi) = 0$$

where M is the rest mass, $i \frac{\partial}{\partial t}$ = energy eigen value, $V(r)$ and $S(r)$ are the vector and scalar potentials respectively^[5-7]. However, the analytical solutions of the Klein-Gordon equation are possible only in the s-wave case with the angular momentum $l = 0$ for some well-known potential. Conversely, when $l \neq 0$, one can only solve approximately the Klein-Gordon equation for some potential using a suitable approximation scheme^[8-10]. Amongst the potentials studied with these techniques are Manning-Rosen Potential^[11-14] Hulthen Potential^[15,16] Kratzer Potential^[17,18] Wood-Saxon Potential^[19,20] Poschl-Teller Potential^[21]. Different methods have been employed to obtain the bound state Klein-Gordon equation for these exponential-type potentials. These methods include the supersymmetric (SUSY) and shape invariance method^[22,23], the asymptotic iteration method (AIM)^[24,25], and the Nikiforov-Uvarov (NU) Method^[26]. The Klein-Gordon equation for the potential under studies is solved by using the parametric NU method to obtain the energy eigenvalues and eigen functions of the bound state.

The purpose of the present paper is to solve the Klein-Gordon equation for the mixed potential MRYP defined as $V(r) = - \left[\frac{Ce^{-\alpha r} + De^{-2\alpha r}}{(1 - e^{-\alpha r})^2} \right] - \frac{V_0 e^{-\alpha r}}{r}$ using the parametric NU method. The paper is organized as follows: After a brief introduction in section 1, the NU method is reviewed in section 2. In section 3, we solve the radial Klein-Gordon equation using the NU method. Finally, we discuss our results in section 4 and a brief conclusion is then advanced in section 5.

2 REVIEW OF PARAMETRIC NIKIFAROV-UVAROV METHOD

The NU method is based on the solutions of a generalized second order linear differential equation with special orthogonal functions. The hypergeometric NU method has shown its power in calculating the exact energy levels of all bound states for some solvable quantum systems.

$$\Psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)} \Psi_n'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)} \Psi_n(s) = 0 \quad (2)$$

Where $\sigma(s)$ and $\bar{\sigma}(s)$ are polynomials at most second degree and $\tilde{\tau}(s)$ is first degree polynomials. The parametric generalization of the N-U method is given by the generalized hypergeometric-type equation

$$\Psi''(s) + \frac{c_1 - c_2 s}{s(1 - c_3 s)} \Psi'(s) + \frac{1}{s^2(1 - c_3 s)^2} [-\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3] \Psi(s) = 0 \quad (3)$$

Thus eqn. (2) can be solved by comparing it with equation (3) and the following polynomials are obtained

$$\tilde{\tau}(s) = (c_1 - c_2 s), \sigma(s) = s(1 - c_3 s), \bar{\sigma}(s) = -\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3 \quad (4)$$

The parameters obtainable from equation (4) serve as important tools to finding the energy eigenvalue and eigenfunctions. They satisfy the following sets of equation respectively

$$c_2 n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n-1)c_3 + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \quad (5)$$

$$(c_2 - c_3)n + c_3n^2 - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + c_7 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \quad (6)$$

While the wave function is given as

$$\Psi_n(s) = N_{n,l} S^{c_{12}} (1 - c_3 s)^{-c_{12} - \frac{c_{13}}{c_3}} P_n^{(c_{10}-1, \frac{c_{11}}{c_3} - c_{10} - 1)} (1 - 2c_3 s) \quad (7)$$

Where

$$\begin{aligned} c_4 &= \frac{1}{2}(1 - c_1), c_5 = \frac{1}{2}(c_2 - 2c_3), c_6 = c_5^2 + \epsilon_1, c_7 = 2c_4c_5 - \epsilon_2, c_8 = c_4^2 + \epsilon_3, \\ c_9 &= c_3c_7 + c_3^2c_8 + c_6, c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}) \\ c_{12} &= c_4 + \sqrt{c_8}, c_{13} = c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}) \end{aligned} \quad (8)$$

and P_n is the orthogonal polynomials.

$$\text{Given that } P_n^{(\alpha, \beta)} = \sum_{r=0}^n \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(\alpha+r+1)\Gamma(n+\beta-r+1)(n-r)!r!} \left(\frac{x-1}{2}\right)^r \left(\frac{x+1}{2}\right)^{n-r} \quad (9)$$

This can also be expressed in terms of the Rodriguez's formula

$$P_n^{(\alpha, \beta)}(x) = \frac{1}{2^n n!} (x-1)^{-\alpha} (x+1)^{-\beta} \left(\frac{d}{dx}\right)^n ((x-1)^{n+\alpha} (x+1)^{n+\beta})$$

3. SOLUTIONS OF THE RADIAL PART OF THE KLEIN-GORDON EQUATION WITH MRYP POTENTIAL

The radial part of the Klein-Gordon Equation with vector $V(r)$ potential = scalar $S(r)$ potential in atomic units ($\hbar = c = 1$) is given as

$$\frac{d^2 R(r)}{dr^2} + [(E^2 - M^2) - 2(E + M)V(r)]R(r) = 0 \quad (11)$$

Substituting potential of Eq. (1) into the Klein-Gordon equation of eq. (11), we obtain

$$\frac{d^2 R(r)}{dr^2} + \left[(E^2 - M^2) - 2(E + M) \left(- \left[\frac{C e^{-\alpha r} + D e^{-2\alpha r}}{(1 - e^{-\alpha r})^2} \right] - \frac{V_0 e^{-\alpha r}}{r} \right) \right] R(r) = 0 \quad (12)$$

Where $\lambda = l(l + 1)$ and $V(r)$ is the Mixed potential energy function

Since the Klein-Gordon equation with above combine potentials rarely has exact analytical solution, an approximation to the centrifugal term has to be made. The good approximation for $1/r^2$ in the centrifugal barrier is taken as

$$\frac{1}{r^2} = \frac{4\alpha^2}{(1 + e^{2\alpha r})^2}, \quad (13)$$

Similar to other related work^[1]

Making the transformation $s = e^{-\alpha r}$ equation (1) becomes

$$V(s) = - \left[\frac{CS + DS^2}{(1-s)^2} \right] - \frac{\alpha V_0 s}{1-s} \quad (14)$$

To solve Eq.(12) by the present method, we need to recast Eq. (13) and applying the transformation given as $s = -e^{2\alpha r}$

$$\frac{d^2 R(s)}{ds^2} + \frac{(1-s)}{(1-s)s} \frac{dR(s)}{ds} + \frac{1}{(1-s)^2 s^2} [-(\beta^2 - F + B)s^2 + (2\beta^2 + A + B)s - (\beta^2)] R(s) = 0, \quad (15)$$

Where

$$-\beta^2 = \frac{E^2 - M^2}{4\alpha^2}; \quad B = 2 \left(\frac{E+M}{\alpha} \right) V_0; \quad A = 2 \left(\frac{E+M}{\alpha^2} \right) C; \quad F = 2 \left(\frac{E+M}{\alpha^2} \right) D \quad (16)$$

Comparing equation (12) with equation (3) yields the following parameters

$$\begin{aligned} c_1 = c_2 = c_3 = 1, c_4 = 0, c_5 = -\frac{1}{2}, c_6 = \frac{1}{4} + \beta^2 + B - F, c_7 = -2\beta^2 - A - B, c_8 = \beta^2, c_9 = \frac{1}{4} - \\ (A + F), c_{10} = 1 + 2\sqrt{\beta^2}, c_{11} = 2 + 2 \left(\sqrt{\frac{1}{4} - A - F} + \sqrt{\beta^2} \right), c_{12} = \sqrt{\beta^2}, c_{13} = -\frac{1}{2} - \left(\sqrt{\frac{1}{4} - A - F} + \sqrt{\beta^2} \right), \\ \epsilon_1 = \beta^2 + B - F, \epsilon_2 = 2\beta^2 + A + B, \epsilon_3 = \beta^2, \end{aligned} \quad (17)$$

Now using equations (5), (13) and (14) we obtain the energy eigen spectrum of the MRYP as

$$\beta^2 = \left[\frac{A+B - \left(n^2 + n + \frac{1}{2} \right) - (2n+1) \sqrt{\frac{1}{4} - A - F}}{(2n+1) + 2 \sqrt{\frac{1}{4} - A - F}} \right]^2 \quad (18)$$

Equation (15) can be solved explicitly and the energy eigen spectrum of MRYP becomes

$$E^2 - M^2 = -4 \alpha^2 \left[\frac{2 \left(\frac{E+M}{\alpha^2} \right) C + 2 \left(\frac{E+M}{\alpha} \right) V_0 - \left(n^2 + n + \frac{1}{2} \right) - (2n+1) \sqrt{\frac{1}{4} - 2 \left(\frac{E+M}{\alpha^2} \right) C - 2 \left(\frac{E+M}{\alpha^2} \right) D}}{(2n+1) + 2 \sqrt{\frac{1}{4} - 2 \left(\frac{E+M}{\alpha^2} \right) C - 2 \left(\frac{E+M}{\alpha^2} \right) D}} \right]^2, \quad (19)$$

We now calculate the radial wave function of the MRYP as follows

The weight function $\rho(s)$ is given as

$$\rho(s) = s^{c_{10}-1} (1 - c_3 s)^{\frac{c_{11}-c_{10}-1}{c_3}}, \quad (20)$$

Using equation (14) we get the weight function as

$$\rho(s) = s^U (1 - s)^V, \quad (21)$$

Where $U = 2\sqrt{\beta^2}$ and $V = 2 \sqrt{\frac{1}{4} - A - F}$

Also we obtain the wave function $\chi(s)$ as

$$\chi(s) = P_n^{c_{10}-1, \frac{c_{11}}{c_3}-c_{10}-1} (1 - 2c_3s), \quad (22)$$

Using equation (14) we get the function $\chi(s)$ as

$$\chi(s) = P_n^{(U,V)} (1 - 2s), \quad (23)$$

Where $P_n^{(U,V)}$ are Jacobi polynomials

Lastly,

$$\varphi(s) = s^{c_{12}} (1 - c_3s)^{-c_{12}-\frac{c_{13}}{c_3}}, \quad (24)$$

And using equation (14) we get

$$\varphi(s) = s^{U/2} (1 - s)^{V-1/2}, \quad (25)$$

We then obtain the radial wave function from the equation

$$R_n(s) = N_n \varphi(s) \chi_n(s), \quad (26)$$

As

$$R_n(s) = N_n s^{U/2} (1 - s)^{(V-1)/2} P_n^{(U,V)} (1 - 2s), \quad (27)$$

Where n is a positive integer and N_n is the normalization constant.

4 DISCUSSION:

We have solved the radial Schrödinger equation and obtained the energy eigen values for the Manning-Rosen plus Yukawa potential (MRYP) in equation (16).

The following cases are considered:

Case 1: If $C = D = 0$ in equation (10), the potential turns back into the Yukawa potential and equation (16) yields the energy eigen values of the Yukawa potential as

$$E^2 - M^2 = -4 \alpha^2 \left[\frac{2 \left(\frac{E+M}{\alpha} \right) V_0 - (n+1)^2}{2(n+1)} \right]^2, \quad (28)$$

$$E^2 - M^2 = -4 \frac{(E+M)}{(n+1)^2} + 4\alpha(E+M)V_0 - \alpha^2 (n+1)^2$$

Case 2: If $\alpha \rightarrow 0$ in equation (28), the energy eigen values for Coulomb potential becomes

$$E^2 - M^2 = -4 \frac{(E+M)}{(n+1)^2} \quad (29)$$

Case 3: If $V_0 = 0$ the potential in equation (10) yields the Manning-Rosen potential with energy eigen values given as

$$E^2 - M^2 = -4 \alpha^2 \left[\frac{2 \left(\frac{E+M}{\alpha^2} \right) C - \left(n^2 + n + \frac{1}{2} \right) - (2n+1) \sqrt{\frac{1}{4} - 2 \left(\frac{E+M}{\alpha^2} \right) C - 2 \left(\frac{E+M}{\alpha^2} \right) D}}{(2n+1) + 2 \sqrt{\frac{1}{4} - 2 \left(\frac{E+M}{\alpha^2} \right) C - 2 \left(\frac{E+M}{\alpha^2} \right) D}} \right]^2 \quad (30)$$

5 CONCLUSION:

We have obtained the energy eigen values and the corresponding un-normalized wave function using the parametric NU method for the Schrödinger equation with MRYP. Special cases of the potential have also been considered.

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