”Uncertainty relations” in the group-theoretic scheme of quantum mechanics

Abstract

Non-commutativity and uncertainty relation in quantum mechanics are considered here from the group-theoretic point of view. It is shown that uncertainty relation is connected with one of unit vector of orthogonal basis of spinor transformations space.

The group-theoretic approach also demonstrates existence of relationship between non-commutativity and irreversibility.

Keywords: quantum mechanics; group theory; uncertainty; Noether

1 Introduction

”Uncertainty principle” is one of the well-known milestones of quantum mechanics, some authors reckon this among most significant initial principles of quantum theory, compared with superposition principle, in spite of the uncertainty principle has, in accordance with Landau, even ”negative content” [Landau & Lifshitz (1963)]. Quantum theory asserts that some pairs of variables can not be measured exactly *and* simultaneously, whereas they may be measured exactly *or* simultaneously individually by means of the same tools, so as some other variables or their combinations. It means, that such particularity of some pairs of last ones does not depend on experimental tools in some areas, and thus it is defined only by their theoretical content.

Conservation laws are another universally recognized principle of physics, not only quantum one, they have the role of foundation stone of any physical phenomena. It was a cause that French Academy had rejected to consider any project of perpetuum mobile, however this decision was unlawful up to appearance of the Noether theorems.

It seems to be extremely important to investigate compatibility of uncertainty principle with conservation laws, as far as any assertion of physical theory has to be in accordance with them.

The Noether theorems are recognized as a mathematical tool which provides fulfillment of conservation laws in physical theory. These theorems operate over a set named groups. In turn, it means that any fundamental physical theory has to be constructed as a mathematical group theory.

The group theory significance for the problem mentioned above is connected beforehand with the Noether theorems. Last ones establish one-to-one correspondence of solution's transformations groups for equation describing physical phenomenon with necessary conservation laws [Olver (1986)]. It is also important that the group theory describes symmetries of scalar and vector variables which may be connected with energy, linear and angular momentums measured in experiment. These circumstances allow one to be sure that the physical theory constructed as consecutive group-theoretic scheme will satisfy all necessary requirements to observables.

From the physical point of view, the theoretical results have to be compared with experimental measurements, therefore it is necessary to construct mathematical variables which are measurable in principle, on the one hand, and which may be associated with variables observed in experiment, on the other hand. The Hermitian forms constructed on the base of the Schroedinger equation solutions or its spinor representation are usually used as observables. For example, a "probability density" $\psi\psi^*$ mentioned above is such variable.

Ascertainment of transformation properties of observables, their connections among themselves, conservation laws fulfillment and comparison of the theoretical conclusions with experimental data in general are *unthinkable* without definition of the Hermitian forms complete system. In particular it means that the Hermitian forms have to satisfy some algebraic completeness condition.

The group-theoretic approach is based on definition of propagators group-theoretic belonging, ascertainment of topological properties of propagators transformations space for the Schroedinger equation spinor representation and definition of the Hermitian forms complete system constructed either on the base of wave function and its derivative or on the base of spinor components [Lunin (2008)], [Lunin (2012)].

This paper is devoted to investigation of compatibility of the uncertainty relation with group theory.

2 Peculiarities of the group-theoretic approach

Let us set forth some known facts on a group-theoretic approach essential for the problem to be considered.

Aiming the purpose to approach a group-theoretic description, let us at first go over, accordingly to [Sokolov (1962), [Kolkunov (1969)], [Kolkunov (1970)], from the unidimensional stationary Schroedinger equation for complex functions

$$\psi^{''}(z) + k^2(z)\psi(z) = 0 \tag{2.1}$$

to pair of first order equations for functions $\Phi_\pm$ connected with wave function and its derivative by means of following equalities

$$\Phi_\pm(z) = \frac{k^{1/2}(z)}{\sqrt{2}}[\psi(z)\pm\frac{1}{ik(z)}\psi^{'}(z)]. \tag{2.2}$$

These equations may be written in matrix form as

$$\Phi^{'}(z) = [ik(z)\sigma_3 + \frac{k^{'}(z)}{2k(z)}\sigma_1]\Phi(z) \tag{2.3}$$

for column

$$\Phi(z) = \left|\left| \begin{array}{c} \Phi_+(z) \\ \Phi_-(z) \end{array} \right|\right| = \left|\left| \begin{array}{c} ae^{i\alpha} \\ be^{i\beta} \end{array} \right|\right| \tag{2.4}$$

with arbitrary conditions at the initial point z_0

$$\Phi_+(z_0) = a_0 e^{i\alpha_0}, \quad \Phi_-(z_0) = b_0 e^{i\beta_0}. \tag{2.5}$$

The equation (2.3) is a spinor representation of the Schroedinger equation, it allows one to use matrix representations of groups to investigate transformation properties of propagators for solutions and conservation laws accordingly to the Noether theorems [Olver (1986)].

To compare conclusions of any scheme of quantum theory with experimental data one needs to construct a real measurable variables based on complex solutions of the Schroedinger equation. A combinatorial analysis leads to conclusion that only four Hermitian forms may be constructed on basis of wave function together with its derivative, or in terms of two spinor components, coupled with their complex conjugate ones, of course. Here we accept them in the following form

$$j_s(z) = \Phi^+(z)\, \sigma_s \Phi(z), \tag{2.6}$$

where $\sigma_s,\ s = 0, 1, 2, 3$ are the Pauli matrices including the identity one σ_0, they form the basis of any transformations of spinor. Using spinor $\Phi(z)$ (2.4) and its Hermitian conjugate $\Phi^+ = \|\Phi_+^*, \Phi_-^*\|$, one has obvious form of observables [Lunin (2002)], [Lunin (2008)]

$$\begin{array}{ll} j_0 = \Phi_+^*\Phi_+ + \Phi_-^*\Phi_- = a^2 + b^2, & j_1 = \Phi_+^*\Phi_- + \Phi_-^*\Phi_+ = 2ab\cos(\beta - \alpha), \\ j_3 = \Phi_+^*\Phi_+ - \Phi_-^*\Phi_- = a^2 - b^2, & j_2 = -i(\Phi_+^*\Phi_- - \Phi_-^*\Phi_+) = 2ab\sin(\beta - \alpha). \end{array} \tag{2.7}$$

As far as spinor is defined up to a phase factor, the Hermitian forms are dependent on only three real variables $a, b, (\beta - \alpha)$.

It is obviously that the Hermitian forms (2.7) satisfy the identity

$$j_0^2 = j_1^2 + j_2^2 + j_3^2, \tag{2.8}$$

containing all of them, it is valid everywhere and under any conditions, therefore it may be considered as the completeness condition for the set of these Hermitian forms.

Such Hermitian forms may also be constructed on the basis of wave function and its derivative. Taking into account relations (2.2), one may express the Hermitian forms to be found as following

$$\begin{array}{ll} j_0 = k\psi\psi^* + (\psi')(\psi^{*'})/k, & j_1 = k\psi\psi^* - (\psi')(\psi^{*'})/k, \\ j_2 = \psi\psi^{*'} + \psi^*\psi', & j_3 = i(\psi\psi^{*'} - \psi^*\psi'), \end{array} \tag{2.9}$$

they coincide with expressions (2.7) and satisfy the same identity (2.8).

A quantum particle moving under different conditions is described by the Schroedinger equation. Its solution is the complex wave function, therefore it can not be observed directly, and one needs to use the Hermitian forms mentioned above to compare a theory with experiment. Some of them are conserving due to conservation laws corresponding to the equation, other of them are changing in different processes, and all of them together form the complete set of observables at any time and in any point.

Then the question arises: does some equation or its system for observables, i.e. an Hermitian forms, which exclude unobservables and which may be used for description of quantum particle, exist?

Let us find a differential relations for the set of Hermitian forms (2.9). Beforehand, differentiating the last expression for j_3 and taking into account the Schroedinger equation together with its complex conjugate, one has an ordinary equation for a "probability density current" $j_3' = 0$, or $\nabla \boldsymbol{j} = 0$ for the stationary partial differential equation [Landau & Lifshitz (1963)] which, however, is not considered here.

Applying the same procedure to the rest expressions in (2.9), we obtain a set of four equations for the complete set of Hermitian forms

$$j_0' = \frac{k'}{k} j_1, \quad j_1' = 2kj_2 + \frac{k'}{k} j_0, \quad j_2' = -2kj_1, \quad j_3' = 0. \tag{2.10}$$

Of course, the same set may be carried out for the Hermitian forms expressed in terms of spinor components, using the spinor representation of the Schroedinger equation (2.3).

It is interesting to note that these equations allow one to express the parameter $k(z)$ in the Schroedinger equation (2.1) only in terms of observables j_s and their derivatives due to the third equation in (2.10). This circumstance may be used to set a geometric content of the parameter $k(z)$ in the Schroedinger equation, and also the same for the parameter k'/k due to the first equation in (2.10). It would be useful, in turn, for clarification of spatial behaviour of quantum particle described by the complete set of observables.

It is relevant to put a question: is the set (2.10) complete, or not? Evidently, its completeness may take place only under the j_s set completeness. Then, differentiating completeness condition (2.8), one has $j_0 j_0' = j_1 j_1' + j_2 j_2' + j_3 j_3'$. Substituting equations (10) into this expression, one has also an identity. Thus, a use of the Hermitian forms (2.6) or (2.9) leads not only to obvious completeness condition for them. It leads also to the similar condition for its increments. It means that they are consistent for the Schroedinger equation also with external potential, i.e. under constant or changing $k(z)$. More certainly, the set of equations (2.10), being also complete one as the set of the Hermitian forms (2.6) or (2.9), contains all possible conservation laws for observables of the Schroedinger equation under different conditions, if one will put $j_s' = 0$ for any s. However, investigation of the conservation law $j_1' = 0$ goes out of the paper.

Let us return to the spinor representation of the Schroedinger equation (2.3). The Hermitian forms expressed in terms of wave function and its derivative on the one hand, and those expressed by means of spinor components on the other hand, are the same under connections (2.2). Furthermore, the equation (2.3) leads to the same its increments, therefore both approaches lead to the same observables dependence on coordinates and problem parameters. Nevertheless, equation (3), being a pair of first order equations, is more preferable with respect to the Schroedinger equation due to opportunity of groups representations use to investigate a group-theoretic properties of propagators transformations, so as conservation laws.

Two ways may be used to obtain spinor representation (2.3) of the Schroedinger equation.

The first one is a substitution of ψ and ψ' from expression (2.2) into the Schroedinger equation to obtain pair of first order equations for $\Phi_\pm$.

The second one is connected with physical content of function $k^2(z)$ in the Schroedinger equation (2.1). Usually this function is supposed to be a difference between kinetic and potential energy of particle. It allows one to use the method [Sokolov (1962)], [Kolkunov (1969)], [Kolkunov (1970)]. based on division of potential into sequence of small stepwise segments and requirements of $\psi(z)$ and $\psi'(z)$ continuity at common points of neighboring infinitesimal steps. Such procedure leads to the matrix sewing ψ and ψ' between such small segments continuously, moreover, both of them as a functions of coordinates and also as a function of parameter k. Significance of the last circumstance will be discussed below. Limit of consecutive products of these, almost unit under $\Delta z \to 0$ (and also $\Delta k \to 0$) but non-commutative in general case matrices, leads to the product integral [Gantmakher (1988)] introduced by Volterra in 1887. Then one has a solution for spinor $\Phi(z_f) = Q(z_f, z_i)\Phi(z_i)$, where z_i and z_f are initial and final points respectively, and where matrix $Q(z_f, z_i)$ is expressed as

$$Q = \lim_{\substack{N\to\infty \\ \Delta z \to 0}} \prod_{m=1}^{N} \exp[ik_m \Delta z_m \sigma_3 + (\Delta k_m / 2k_m)\sigma_1] \equiv T \exp \int_{z_i}^{z_f} [ikdz\sigma_3 + \frac{dk}{2k}\sigma_1]. \tag{2.11}$$

An analysis of last expression [Sokolov (1962)], [Kolkunov (1969)], [Kolkunov (1970)], [Lunin (2002)] shows that $detQ = 1, Q_{21} = Q_{12}^*, Q_{22} = Q_{11}^*$, i.e. matrix propagator belongs to the group $SU(1,1)$.

It is necessary to emphasize that this propagator leads not only to the spinor components continuity as a function of z and k everywhere due to sewing procedure mentioned above, or the same for ψ and ψ', that leads in turn to continuity of all Hermitian forms, i.e. of all observables. Besides, the propagator (2.11) belonging to the group $SU(1,1)$ leads, in accordance with the Noether theorems, to fulfillment of all conservation laws for the Schroedinger equation.

An integrand under sign of the product integral in expression (2.11) allows geometric interpretation of propagators for the Schroedinger equation in its spinor representation. Being written as matrix two by two in the basis of Pauli matrices including the identity σ_0, an integrand may be considered as some vector in the space of propagators logarithms. The Pauli matrices may be said to be analogous to the unit vectors of orthogonal basis at the same time [Rostokin & Kolkunov (1969)], [Casanova (1976)]. Taking into account also continuity of the propagator Q from (2.11) as a function both of coordinates and parameter k, a length of vector ds squared of infinitesimal transformation defined by integrand, may be written in form [Lunin (1994)], [Lunin (2002)], [Lunin (2012)]

$$ds^2 = -k^2 dz^2 + \frac{dk^2}{4k^2}. \tag{2.12}$$

This expression may be considered as the metric of the space of propagators logarithms transformations. Such kind of metric, which is compatible with other its forms accepted in literature and leading to the same Gaussian curvature, means that the propagators logarithms space is the plane with constant negative Gaussian curvature, i.e. the Lobachevsky plane, with $C_G = -4$.

Besides, it was shown in [Lunin (2002)], [Lunin (2012)] that only this special value of Gaussian curvature among all, which may have spaces with constant negative Gaussian curvature, leads to the wave equations similar to the Schroedinger or Helmholtz ones. In addition, nonzero Gaussian curvature of this space represents also non-commutativity of transformations of the Schroedinger equation solutions.

Thus, the space of propagators logarithms of the Schroedinger equation is the Lobachevsky plane with unique Gaussian curvature $C_G = -4$. It should be noted that possibility of identification of such kind propagators space with the Lobachevsky one is closely connected with isomorphism of the groups $SU(1,1)$ and $SL(2,R)$ [Vilenkin (1965)].

Having determined the metric and the Gaussian curvature of the space, one may find an appropriate geometric image for an integrand, and further, for a propagator in the expression (2.11). Taking into account orthogonality of the Pauli matrices [Casanova (1976)], both terms in integrand may be mapped on the Lobachevsky plane as the oriented orthogonal segments of geodesic lines in accordance with [Lunin (1994)], [Lunin (1999)]. Furthermore, one may note that consideration of propagators as a geodesic lines segments in the Lobachevsky space allows one to solve geometric problems of such kind geometry. In turn, it may be found to be useful for physical problems.

One may note that even if only two of the Pauli matrices entered the integrand in expressions (2..11) evidently, the product integral includes all these matrices together with σ_0 since its expression is product of similar matrices. It means that dimension of the space mentioned above is defined by all Pauli matrices which together with σ_0 form the basis of all possible transformations described by matrices two by two. For example, a dimension of this space is the same, both in the case of the unidimensional Schroedinger equation or in the case of non-unidimensional one, including the time Schroedinger equation.

The geometric mapping of matrix propagators into the Lobachevsky space had allowed one to establish the non-Euclidean superposition principle for alternative propagators which takes into account their non-commutativity [Lunin (1994)], [Lunin (2002)], [Lunin (2012)]. It contains four binary compositions of non-commutative matrix propagators, all of them belong to the same group as both entered the compositions and have necessary properties with respect to permutations and inversions, and go to the ordinary Euclidean superposition principle under corresponding conditions. Two of four compositions contain irreversibility, although each of two non-commutative propagators entered these two compositions, is the solutions of the reversible Schroedinger equation [Lunin & Kogan (2004)], [Lunin & Kogan, (2009)].

Besides geometric interpretation of propagators, it is extremely important for physical purposes to determine a space of observables j_s.

One may often listen that a particle being described by the unidimensional Schroedinger equation, for example with only z-dependence of potential (and may be its derivatives) in the equation, would be moved strictly along the same axis. In particular, the authors of [Landau & Lifshitz (1963)], considering the problem on quantum particle moving above unidimensional potential step, supposed conserving "probability density current" $j_3 = i(\psi\psi^{*'} - \psi^*\psi')$ to be directed along such axis, transmitted and reflected particles are moving along this current, i.e. along z - axis.

This point of view seems to be hardly satisfactory. A quantum particle motion is defined by all Hermitian forms of its complete set which may be constructed on the basis of the Schroedinger equation solutions. This set includes four Hermitian forms, only three of them are independent due to the identity (2.8). The number of these forms does not depend on if equation is defined by one or more variables. It is defined by the dimensionality of the group, transforming solutions* , i.e. $SU(1,1)$ in this case.

In particular, it means that the tangent to the line along which all conservation laws are fulfilled is defined by all Hermitian forms and may be by their derivatives.Therefore, the condition (2.8) may be considered not only as completeness condition but also as the circumstance that j_s form some vector in the Euclidean space [Lunin (2008)], [Lunin (2012)], or, more rigorously, in the space with zero Gaussian curvature.The requirements of ψ and ψ' continuity fulfilled for them as for solutions of the second order differential equation lead to continuity of all Hermitian forms as well. The group-theoretic approach provides fulfillment of necessary conservation laws in accordance with the Noether theorems. Both these circumstances allow one to suppose that a consequence of the points where these conservation laws are fulfilled form the continuous line. This line may be considered as the quantum particle trajectory.

If it is not so, then one assumes that a particle may be found at the points where conservation laws had been violated.

Consideration of observables j_s as orthogonal components of (path) velocity (j_0 is its absolute value in such interpretation) in the Euclidean space allows one to attain a second, along with probabilistic, interpretation of the Hermitian forms in quantum mechanics [Lunin (2008)], [Lunin (2012)]. Furthermore, connection of j_s and its derivatives with curvature and torsion leads to set the line for point-like object described by the Schroedinger equation, along which all conservation laws are fulfilled, under known of all initial conditions, of course. It is well known from differential geometry [Poznyak & Shikin (1990)] that these two parameters, the curvature and the torsion, define the spatial line to within a position in space.

In particular, free particle under $k = const$ and arbitrary initial conditions is moving along spiral line having the curvature and the torsion to be fixed as far as all necessary conservation laws are fulfilled along this line. The last circumstance leads to an opportunity to consider free quantum particle trajectory as the Euclidean straight line on the Euclidean plane with zero Gaussian curvature which is rolled up into the cylinder surface with the same Gaussian curvature.

Such behaviour of free quantum particle allows one to propose a qualitative explanation [Lunin (2008)], [Lunin (2012)] double-slit experiment under extremely low intensity of a particles source [Biberman et al. (1949)], [Tonomura et al. (1989)].

Obviously, the particle at the potential step may be considered similar to described above but the trajectory in this case would be disposed onto the conical surface with the same zero Gaussian curvature due to the identity (2.8) as well. The last one is also fulfilled at the step when propagator and corresponding Hermitian forms are varying together with variation of k under conditions $j_2 = const, j_3 = const$, [Lunin (2008)], [Lunin (2012)].

*Since the time Schroedinger equation contains only first order time derivative, it has the same Hermitian forms complete set as the stationary one, i.e. the set (2.9).

3 "Uncertainty"

Completeness of set of the Hermitian forms forces to look for another basis for probability concept, which is more compatible with conservation laws then it takes place now, for example in the simplest case of potential step, as it was shown in [Lunin (2015)].

Ad interim, let us investigate the uncertainty principle from the group-theoretic point of view.

Transformations of solutions of the Schroedinger equation (2.1), or its spinor representation (2.3), belong to the group $SU(1,1)$. Matrix representations of this three-parameter group may be expressed as

$$Q = exp(\boldsymbol{a\sigma}) = exp(a_1\sigma_1 + a_2\sigma_2 + ia_3\sigma_3) \tag{3.1}$$

with real parameters $a_1, a_2, a_3, a^2 = a_1^2 + a_2^2 - a_3^2$, or, for example, as

$$Q = exp(iM\sigma_3)exp(L\sigma_1)exp(iN\sigma_3) \tag{3.2}$$

with also real N, L, M.

Basis of arbitrary transformations of two-component spinor (2.4) consists of four matrices σ_s, but expression for propagator (2.11), just as (3.2), contains only matrices σ_3 and σ_1.

Let us analyze these expressions from standpoint of basis completeness, on the one hand, and from approximate calculations of propagators under Δz_m and Δk_m, which are supposed to be small in some sense, often even in the experimental one, and also under large Δz_m, Δk_m, when exact calculations are valid, on the other hand.

Parameters N and M may be calculated exactly in areas with $k = const$, in just the same way as parameter L on the potential wall directly [Lunin (2008)]. These circumstance allows one to consider both small and large N, L, M (but we do not consider turning points here). Taking (3.1) and (3.2) into account, one may calculate them as matrix identity

$$\begin{gathered} Q = \sigma_0 \cosh a + (\boldsymbol{a\sigma}) \sinh a/a = \sigma_0 \cosh L \cos(M+N) + \\ + i\sigma_3 \cosh L \sin(M+N) + \sigma_1 \sinh L \cos(M-N) - \sigma_2 \sinh L \sin(M-N), \end{gathered} \tag{3.3}$$

from where all components of geodesic vector a_1, a_2, a_3, and its length a may be calculated exactly.

One may see that all matrices σ_s enter propagator Q in general case. As far as this matrix belongs to the $SU(1,1)$ group, then all conservation laws, arisen from the Schroedinger equation, are satisfied also exactly in accordance with the Noether theorems. In spite of the circumstance that the component a_2 goes to zero under small parameters N, L, M more quickly then others, neglect of this term immediately leads to violation of conservation laws due to failure of the condition $detQ = 1$, i.e. one of the group-theoretic requirements in this case.

Moreover, one may see, that $a_1^2 - a_3^2 = 0$ in expression (3.1) under condition of $\tanh L = \sin(M+N)/\cos(M-N)$, and the length of vector $\boldsymbol{a}$ is defined only by the component a_2 in this case. In other words, even if a_1 and a_3 are large, there may appear conditions, when the solution is defined by a_2, and the last term with σ_2 in the expression (3.3) has to be taken into account in general case.

One may also see that dimensionless coefficient $k\Delta z$ before σ_3 in expression (2.11) has a sense of action measured in $\hbar$-units along z-axis under constant value of k. It means that coefficient $\Delta k/(2k)$, dimensionless in any units, is also an action on the wall directly, but $\hbar$ enters only the first term. This circumstance allows one to compare actions of these two kinds, one of which contains the Planck constant, but another does not.

As far as expression (3.3) is valid under arbitrary values of parameters N, L, M, let us consider this matrix under small its absolute values. Since our aim is to examine not only accuracy of calculations but also completeness of transformations basis in (2.11), we shall retain lowest order terms before each matrix of basis. Then the expression (3.3) goes over to

$$\sigma_0 \cosh a + (a_1\sigma_1 + a_2\sigma_2 + ia_3\sigma_3)\sinh a/a \simeq \sigma_0 + i(M+N)\sigma_3 + L\sigma_1 - L(M-N)\sigma_2, \tag{3.4}$$

where the last term has the second order in this case.

One may see that matrix Q from (3.1)-(3.3) contains all matrices of basis, $detQ \equiv 1, Q \in SU(1,1)$ and conservation laws are fulfilled under any values of parameters N, L, M. Approximate expression (3.4) for Q under small ones also contains all matrices of basis, but $detQ \neq 1$, then Q does not belong to the group $SU(1,1)$, and conservation laws are violated.

Comparing expression (3.3) and (3.4), one may also see that in spite of the term with σ_2 appears only in the second order under small N, L, M in (3.4), it would be extremely essential to take this term into account for fulfillment conservation laws in general case.

In accordance with above-stated, one may arrive at the conclusion that appearance of the term with σ_2 is equivalent to inclusion of the supplementary dimension, in addition to σ_3 and σ_1, into approximate expression for propagator Q, which is absent in the first order of smallness. Moreover, the terms $ik\Delta z\sigma_3$ and $(\Delta k/2k)\sigma_1$ may be large, but the term with with σ_2 in (3.3) will dominate under condition $k\Delta z \approx \Delta k/(2k)$ even in this case, whereas $a_1^2 - a_3^2 \approx 0$.

As a rule, uncertainty relations are considered to be connected with non-commutativity of operators forming commutators of kind $[A, B] = AB - BA$, belonging to the Lie algebra of the group. Keeping in mind appearance of the third matrix, σ_2, in the general expression (3.1) so as in (3.4), it is interesting to determine its relation with such kind commutator.

Using the Baker-Campbell-Hausdorff formula $\exp A \exp B \cong \exp(A+B+(1/2)[A, B])$, conserving the group properties of matrices, and restricting with only one commutator of Lie algebra of the group supposing that commutators of higher order are negligible, one may carry out that the integrand in expression (2.11) has to be replaced with

$$ik_m\Delta z_m\sigma_3 + (\Delta k_m/2k_m)\sigma_1 - (\Delta k_m\Delta z_m/2)\sigma_2, \tag{3.5}$$

where sign minus before σ_2 appears due to what first term in sum of matrices was taken as the first one in their product.

The last term, arisen as commutator of the Lie algebra of the Lie group, was established without any assumptions on smallness of $\Delta k, \Delta z$ in any sense, theoretical or experimental. Moreover, they may be arbitrary small or large, as it is seen from consideration of the step-wall potential, where propagator may be calculated exactly. This term in (3.5) is similar to the same, entered the Heisenberg uncertainty relation.

It should be noted that the third term in (3.5) was obtained exclusively theoretically, without any references on experimental causes, similar to accuracy and so on. But this approach does not exclude such interpretation of Δk and Δz. More definitely, the approach allows consideration of these values as consequence of experiment, but does not accept their origin as exceptionally experimental.

As a rule, Δk and Δz in the uncertainty relation are considered to be small at least in some sense. However, non-commutativity is exclusively important property of pairs of conjugate variables in quantum mechanics, therefore it is difficult to imagine, that such discrete symmetry may become apparent mainly under small values of these parameters. For example, non-commutativity appears clearly under long paths in navigation, but vanishes under small ones. This circumstance engages us to find some kind of significant physical problem, in which non-commutativity becomes apparent more forcibly then it takes place under ordinary consideration of the uncertainty relation.

Let us pay attention to irreversibility in quantum mechanics, the problem, considered to be unsolved up to now [Ginzburg (1999)]. It is appropriate to bring here merely a brief description of the subject, only with an aim to show connection between non-commutativity and irreversibility, more detailed description may be found in [Lunin & Kogan (2004)], [Lunin (2012)] and, especially, in [Lunin & Kogan, (2009)].

The Schroedinger equation is reversible, therefore its solution, presented by propagator Q from (2.11), is also reversible. It means, that vector $\boldsymbol{a}$ goes over to $-\boldsymbol{a}$ under time inversion. This conclusion is valid up to any solution, but different propagators are non-commutative in general case.

In accordance with [Lunin & Kogan, (2009)], let us multiply two matrices from (3.1):

$$\begin{aligned}\exp(\boldsymbol{c\sigma}) = \exp(\boldsymbol{b\sigma})\exp(\boldsymbol{a\sigma}) = \sigma_0[\cosh b\cosh a + (\boldsymbol{n}_b\boldsymbol{n}_a)\sinh b\sinh a] + \\ +\boldsymbol{\sigma}[\boldsymbol{n}_b\sinh b\cosh a + \boldsymbol{n}_a\sinh a\cosh b + i[\boldsymbol{n}_b\boldsymbol{n}_a]\sinh b\sinh a]\end{aligned} \tag{3.6}$$

This is exact expression. Using BCH-formula, one may also derive

$$\exp(\boldsymbol{c\sigma}) = \exp(\boldsymbol{b\sigma})\exp(\boldsymbol{a\sigma}) \approx \boldsymbol{b\sigma} + \boldsymbol{a\sigma} + (1/2)[\boldsymbol{b\sigma}, \boldsymbol{a\sigma}] = \boldsymbol{b\sigma} + \boldsymbol{a\sigma} + i\boldsymbol{\sigma}[\boldsymbol{ba}], \tag{3.7}$$

whence one may see, that commutator in (3.7) is directly connected with vector product. The terms in (3.6) or (3.7), which are linear with respect to $\boldsymbol{n}_a$ and $\boldsymbol{n}_b$, change sign under time inversion, they do not describe irreversible evolution. The last term does not change under time inversion, it describes irreversible phenomena.

4 Discussion and conclusion

The non-commutativity, lying in the basis of uncertainty relations, is an impressive manifestation of the nonzero Gaussian curvature of the space of solutions transformations. As usual, ones connect these relations with an accuracy of measurements, then one compares product of some pairs of physical parameters with the Planck constant. As far as accuracy of such measurements is near a confidence level, both from experimental and also theoretical viewpoints, there is doubt on origin and interpretation of this relations, see for example, [Belinsky & Lapshin (2017)]. As a rule, non-commutati-
vity and smallness of values, entered relations, are considered together and their origin becomes too unclear.

Meanwhile, these two factors may be considered separately, and the group-theoretic approach is a sufficient tool for such consideration.

During many years quantum mechanics is accompanied by the probability concept. During the same years the problem of hidden parameters is discussed in the papers devoted to foundations of quantum mechanics. One may note at the same time that during the same period the probability density $\psi\psi^*$ was not accompanied by the value $\psi'\psi^{*'}$ as long as the (convective) probability density current $i(\psi\psi^{*'} - \psi'\psi^*)$ was not accompanied by the (diffusion) probability density current $\psi\psi^{*'} + \psi'\psi^*$.

The group-theoretic approach contains the complete set of observables which presented by the Hermitian forms. Part of them are used in the ordinary schemes of quantum mechanics, as $\psi\psi^*$ and $i(\psi\psi^{'*} - \psi^{'}\psi^*)$ during many years, part of them, $\psi^{'}\psi^*$ and $(\psi\psi^{'*} + \psi^{'}\psi^*)$ are appended in the framework of the group-theoretic approach [Lunin (2015)]. Being together, these four Hermitian forms generate complete set of observables, they also well known as the Stokes parameters during many years.

Besides, the approach, being a consistent mathematical group theory, includes the Noether theorems, which supply quantum theory with fulfillment of conservation laws.

Geometric content is also included into the scheme. The stationary Schroedinger equation in its spinor representation leads to conclusion that the space of spinor transformations is the Lobachevsky one with the Gaussian curvature $C_G = -4$ [Lunin (2002)]. The frame of reference there is formed with three Pauli matrices, unit one, σ_0, inclusive, this is the complete basis of any spinor transformations. Thus, one has a basis for analysis of completeness of the Schroedinger equation in any actual case.

It is necessary to note here that, since the space is curved and transformations are non-commutative, composition of different solutions, or alternative transformations, requires to go over to non-Euclidean superposition principle [Lunin (1994)], [Lunin (2002)], [Lunin (2012)], taking non-commutati-

vity into account.

The non-Euclidean superposition principle is formed with four compositions of two matrices, non-commutative ones, as a rule, with definite properties with respect to permutations and inversions, and conserving the group properties of result. These formulae may be geometrically presented as sum, difference, and vector product of two geodesic vectors in the curved space. The last composition is especially important in the case of uncertainty relations, so as in the irreversibility, of course.

The geodesic vectors of two first compositions are placed at the Lobachevsky plane defined by two initial vectors. The compositions with vector product are orthogonal with respect to this plane. As it was shown above, such compositions lead to product of Δk and Δz, similar to the same entered the uncertainty relations. Therefore one may say that consideration of product $\Delta k \Delta z$ corresponds to inclusion of the third dimension, this is directed along vector product, or σ_2 in the case of simplified consideration, as above.

It should be noted that the subject connected with non-Euclidean superposition principle, and, in particular, one of its composition, containing vector product, requires more attention. In particular, the same mathematical tool is connected with not only "uncertainty", but also with irreversibility in quantum mechanics, as it was shown in [Lunin & Kogan (2004)], [Lunin & Kogan, (2009)]. Moreover, such compositions from non-Euclidean superposition principle, due to equality of terms with Δk and Δz, may turn out to be sufficient tool in the problem of transitions in quantum systems, in particular for consideration of radiation.

All ordinary schemes of quantum theory do not contain a mathematical scheme of irreversibility [Ginzburg (1999)], phenomena, where measurements may be carried out with sufficient confidence level. The group-theoretic scheme, based on the reversible Schroedinger equation and using its reversible solutions, together with non-commutativity, is capable to explain irreversibility in closed systems. It means inclusion of unused dimension in the Lobachevsky space.

This inclusion allows one to explain also the appearance of $\Delta k \Delta z$ in quantum theory, but we can not formulate, being in the framework of mathematical theory, any restrictions, connected with experimental measurements. In opposite case, if one will include some experimental data as a necessary element of physical theory, it would be considered as if one recognize impossibility to construct the fundamental physical theory as a consistent closed mathematical theory, as it takes place now.

Moreover, it was shown in previous section, that the case, when contribution of similar terms, connected with the Lie algebra commutators, may be principal, is possible, it is a peculiarity of the group $SU(1,1)$. Nevertheless, an ordinary interpretation of uncertainty relation as an experimental restriction may be remained, but as experimental, and only as experimental one.

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