and $(\tilde{\varphi}^{+}(k), \tilde{A}^{+}(k)) : \tilde{\Phi} \to \tilde{\Phi}^{4}$ remain the same as (156), that is

$$
\begin{aligned}
[\tilde{\varphi}(k), \tilde{\varphi}^{+}(s)] &= -\frac{c\hbar}{2|k|}\delta(k-s), [\tilde{\varphi}(k), \tilde{A}_j(s)] = 0, \\
[\tilde{A}_j(k), \tilde{A}_l^{+}(s)] &= \frac{c\hbar}{2|k|}\delta_{jl}\delta(k-s), \\
[\tilde{\varphi}(k), \tilde{\varphi}(s)] &= 0 = [\tilde{\varphi}^{+}(k), \tilde{\varphi}^{+}(s)], \\
[\tilde{A}_j(k), \tilde{A}_l(s)] &= 0 = [\tilde{A}_j^{+}(k), \tilde{A}_l^{+}(s)],
\end{aligned}
\tag{180}
$$

for all $k, s \in \mathrm{E}^3$ and $j, l \in \overline{1,3}$.

Now, concerning the Hamiltonian operators (177) and (179), the following Heisenberg evolutions equations

$$
\frac{\partial \psi}{\partial \bar{t}} := \frac{i}{\hbar}[\bar{H}_f^{(int)}, \psi], \quad \frac{\partial \psi^{+}}{\partial \bar{t}} := \frac{i}{\hbar}[\bar{H}_f^{(int)}, \psi^{+}] \tag{181}
$$

with respect to the ~~own~~ reference frame $\mathrm{K}_{\bar{t}}$ and the Heisenberg evolution equations

$$
\frac{\partial \tilde{\varphi}}{\partial \tilde{t}} := \frac{i}{\hbar}[\tilde{H}_b, \tilde{\varphi}], \frac{\partial \tilde{A}}{\partial \tilde{t}} := \frac{i}{\hbar}[\tilde{H}_b, \tilde{A}] \tag{182}
$$

with respect to the ~~own~~ reference frame $\mathrm{K}_{\tilde{t}}$ hold. Being further interested in the evolution equations (173), suitably rewritten in the transformed Fock space $\tilde{\Phi}$ with respect to the common temporal parameter $t \in \mathrm{R}$, we need to take into account [116] that the following functional relationships

$$
\psi(t) := \psi(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t}, \quad \tilde{A}(t) := \tilde{A}(\bar{t}, \tilde{t})|_{\bar{t}=\tilde{t}=t} \tag{183}
$$

hold. In particular, from (183) the following evolution expressions follow

$$
\begin{aligned}
\partial \psi(t)/\partial t &= \partial \psi(\bar{t}, \tilde{t})/\partial \bar{t}\,|_{\bar{t}=\tilde{t}=t} + \partial \psi(\bar{t}, \tilde{t})/\partial \tilde{t}\,|_{\bar{t}=\tilde{t}=t}, \\
\partial \tilde{A}(t)/\partial t &= \partial \tilde{A}(\bar{t}, \tilde{t})/\partial \bar{t}\,|_{\bar{t}=\tilde{t}=t} + \partial \tilde{A}(\bar{t}, \tilde{t})/\partial \tilde{t}\,|_{\bar{t}=\tilde{t}=t},
\end{aligned}
\tag{184}
$$

for all $t \in \mathrm{R}$. The latter will be useful when deriving the resulting quantum Maxwell electromagnetic equations.

Before doing this, we need to take into account that the weak operator Lorenz constraints (160), rewritten in the transformed Fock space $\tilde{\Phi}$, is compatible with the evolution equations (182):

$$
[\tilde{C}_0(k), \tilde{H}_b] = 0 = [\tilde{C}_0^{+}(k), \tilde{H}_b], \tag{185}
$$

yet they fail to be compatible with the evolution equations (181), that is

$$
[\tilde{C}_0(k), \bar{H}_f^{(int)}] \neq 0 \neq [\tilde{C}_0^{+}(k), \bar{H}_f^{(int)}].
$$

This means that we can not impose on the transformed Fock pace $\tilde{\Phi}$ the constraints

$$\tilde{C}_0(k)\tilde{\Phi} \ : \ = i(<k,\tilde{A}(k)> - |k|\tilde{\varphi}(k))\tilde{\Phi} \neq 0,$$
$$\tilde{C}_0^{+}(k)\tilde{\Phi} \ : \ = -i(<k,\tilde{A}^{+}(k)> - |k|\tilde{\varphi}^{+}(k))\tilde{\Phi} \neq 0 \tag{186}$$

invariantly for all $k \in \mathrm{E}^3$. Notwithstanding, it is easy enough to check that the following slightly perturbed operators

$$\tilde{C}(k) \ : \ = \tilde{C}_0(k) + \frac{i\xi \exp(-ic|k|\bar{t})}{2|k|(2\pi)^{3/2}} \int_{\mathrm{R}^3} \exp(-i<k,y>)\psi^{+}(y)\psi(y)d^3y,$$
$$\tilde{C}^{+}(k) \ : \ = \tilde{C}_0^{+}(k) - \frac{i\xi \exp(ic|k|\bar{t})}{2|k|(2\pi)^{3/2}} \int_{\mathrm{R}^3} \exp(i<k,y>)\psi^{+}(y)\psi(y)d^3y, \tag{187}$$

are commuting both to each other and with the Hamiltonian operators (177) and (179):

$$[\tilde{C}(k),\tilde{C}(s)] \ = \ 0 = [\tilde{C}^{+}(k),\tilde{C}(s)]$$
$$[\tilde{C}(k),\tilde{H}_f^{(int)}] \ = \ 0 = [\tilde{C}^{+}(k),\tilde{H}_f^{(int)}], \tag{188}$$
$$[\tilde{C}(k),\tilde{H}_b] \ = \ 0 = [\tilde{C}^{+}(k),\tilde{H}_b]$$

for all $k,s \in \mathrm{E}^3$. Thus, the related evolution flows (181) and (182) in the transformed Fock space $\tilde{\Phi}$ should be considered under the modified constraints

$$\tilde{C}(k)\tilde{\Phi} = 0 = \tilde{C}^{+}(k)\tilde{\Phi} \tag{189}$$

for all $k \in \mathrm{E}^3$. Taking into account the exact expressions (187), the constraints (189) can be equivalently rewritten as

$$\tilde{C}(\bar{t};\tilde{t},x)\tilde{\Phi} = 0, \tag{190}$$

where for all $x \in \mathrm{R}$ and the corresponding temporal parameters $\bar{t}$ and $\tilde{t} \in \mathrm{R}$

$$\tilde{C}(\bar{t};\tilde{t},x) := \frac{1}{(2\pi)^{3/2}} \int_{\mathrm{R}^3} d^3k \tilde{C}(k) \exp(i<k,x> - i|k|\tilde{t}) +$$
$$+ \frac{1}{(2\pi)^{3/2}} \int_{\mathrm{R}^3} d^3k \tilde{C}^{+}(k) \exp(-i<k,x> + i|k|\tilde{t}) = \tag{191}$$
$$= <\nabla,\tilde{A}> + \frac{1}{c}\frac{\partial\tilde{\varphi}}{\partial\tilde{t}} - \frac{\xi}{2\pi}\int_{\mathrm{R}^3} d^3y \Theta(c(\bar{t}-\tilde{t}),|x-y|)\psi^{+}(y)\psi(y)$$

in which we put, by definition, the relativistic generalized function

$$\Theta(c(t-t),|x-y|) := \frac{\delta(|x-y|+c(\bar{t}-\tilde{t})) - \delta(|x-y|-c(\bar{t}-\tilde{t}))}{2|x-y|}, \tag{192}$$

dual to the well known generalized solution [123] [124]

$$\delta(|x-y|^2 - c^2(\bar{t}-\tilde{t})^2) = \frac{\delta(|x-y|+c(\bar{t}-\tilde{t})) + \delta(|x-y|-c(\bar{t}-\tilde{t}))}{2|x-y|}$$

to the relativistic wave equation.

Remark 5. *It is here worthy to mention that the above defined operator* $\tilde{C}(\bar{t}):\tilde{\Phi}\to\tilde{\Phi}$, *depending parameterically on the bosonic temporal parameter* $\bar{t}\in\mathbb{R}$, *satisfies the relativistic wave equation*

$$\frac{1}{c^2}\frac{\partial^2\tilde{C}}{\partial\bar{t}^2}-\Delta\tilde{C}=0, \qquad (193)$$

that can be easily checked, ~~*if to*~~ by *make* ing *use of the rewritten in the Fock space wave equations (167):*

$$\frac{1}{c^2}\frac{\partial^2\tilde{A}}{\partial\bar{t}^2}-\Delta\tilde{A}=0,\frac{1}{c^2}\frac{\partial^2\tilde{\varphi}}{\partial\bar{t}^2}-\Delta\tilde{\varphi}=0. \qquad (194)$$

Moreover, as ~~*it*~~ *can be shown by means of direct calculations, the transformed bosonic Hamiltonian operator (179) on the reduced via the modified Lorenz type constraints (190) Fock space* $\tilde{\Phi}$ *persists to be, as before, non-negative definite.*

Now we can proceed to deriv~~ing~~e the quantum Maxwell equations starting from the operator equations *(194)* and the suitably transformed to the Fock space $\tilde{\Phi}$ electromagnetic fields definitions (158) and (159):

$$(\nabla\times\tilde{B}-\frac{1}{c}\frac{\partial\tilde{E}}{\partial\bar{t}})\tilde{\Phi}=\frac{\xi}{2\pi}\nabla\int_{\mathbb{R}^3}{}^{3}y\Theta(c(\bar{t}-\bar{t}),|x-y|)\psi^+(y)\psi(y)\tilde{\Phi} \qquad (195)$$

and

$$<\nabla,\tilde{E}>\tilde{\Phi}=-\frac{\xi}{2\pi}\frac{\partial}{\partial\bar{t}}\int_{\mathbb{R}^3}d^3y\Theta(c(\bar{t}-\bar{t}),|x-y|)\psi^+(y)\psi(y)\tilde{\Phi}, \qquad (196)$$

which are considered in the weak operator sense. Taking now into account the relationships (182) and (184), one can obtain strong operator relationships for the electrical and magnetic fields

$$\tilde{E}=-\frac{1}{c}\frac{\partial\tilde{A}}{\partial\bar{t}}-\nabla\tilde{\varphi}=-\frac{1}{c}\frac{\partial\tilde{A}}{\partial\bar{t}}-\nabla\tilde{\varphi},\tilde{B}=\nabla\times\tilde{A}. \qquad (197)$$

with respect to the common reference frame K_t. Similarly one can easily calculate the weak operator relationship

$$(\frac{1}{c}\frac{\partial\tilde{\varphi}}{\partial t}+<\nabla,\tilde{A}>)\tilde{\Phi}=0, \qquad (198)$$

which holds for the common temporal parameter $t\in\mathbb{R}$. Now we will calculate the weak Maxwell type operator relationships (195) and (196) with respect to the common reference frame K_t :

$$(\nabla\times\tilde{B}-\frac{1}{c}\frac{\partial\tilde{E}}{\partial\bar{t}})|_{\bar{t}=\bar{t}=t}\tilde{\Phi}=\frac{\xi}{2\pi}\nabla\int_{\mathbb{R}^3}d^3y\Theta(c(\bar{t}-\bar{t}),|x-y|)\psi^+(y)\psi(y)|_{\bar{t}=\bar{t}=t}\tilde{\Phi}=0 \qquad (199)$$

and

$$<\nabla,\tilde{E}>\tilde{\Phi}=-\frac{\xi}{2\pi}\frac{\partial}{\partial\bar{t}}\int_{\mathbb{R}^3}d^3y\Theta(c(\bar{t}-\bar{t}),|x-y|)\psi^+(y)\psi(y)|_{\bar{t}=\bar{t}=t}\tilde{\Phi}=\xi\psi^+\psi\tilde{\Phi}, \qquad (200)$$

where we ~~there was~~ used the known [121] [124] generalized function relationship

$$\frac{1}{c}\frac{\partial}{\partial s}\Theta(cs,|z|)|_{s=0} = -2\pi\delta(z) \tag{201}$$

for all $z \in \mathbb{R}^3$. To calculate further the expression (199), we need to make use of the strong operator relationships (184) and find that

$$\frac{\partial \tilde{E}}{\partial \tilde{t}}|_{\tilde{t}=\tilde{t}=t} = \frac{\partial \tilde{E}}{\partial t} - \frac{\partial \tilde{E}}{\partial \tilde{t}}|_{\tilde{t}=\tilde{t}=t} = \frac{\partial \tilde{E}}{\partial t} - \frac{i}{\hbar}[\tilde{H}_f^{(int)}, \tilde{E}] = \frac{\partial \tilde{E}}{\partial t} + \xi\psi^+ \alpha\psi. \tag{202}$$

Thus, from (202) and (199) one can obtain that

$$(\nabla \times \tilde{B} - \frac{1}{c}\frac{\partial \tilde{E}}{\partial t})\tilde{\Phi} = \xi\psi^+ \alpha\psi\tilde{\Phi} \tag{203}$$

with respect to the common reference frame K_t. The combined ~~together~~ weak operator relationships (200) and (203)

$$(\nabla \times \tilde{B} - \frac{1}{c}\frac{\partial \tilde{E}}{\partial t})\tilde{\Phi} = \xi\psi^+ \alpha\psi\tilde{\Phi}, <\nabla, \tilde{E}> \tilde{\Phi} = \xi\psi^+ \psi\tilde{\Phi} \tag{204}$$

in the reduced by the weak constraint (198) Fock space $\tilde{\Phi}$ jointly with the evident strong operator relationships

$$\nabla \times \tilde{E} + \frac{1}{c}\frac{\partial \tilde{B}}{\partial \tilde{t}} = 0, \nabla \times \tilde{B} = 0 \quad , \tag{205}$$

compile the complete system of the quantum Maxwell equations with respect to the common reference frame K_t.

~~Really,~~ From the Heisenberg evolution equations (181) one easily obtains the strong operator charge conservative flow relationship

$$\frac{\partial}{\partial t}(\xi\psi^+ \psi) + <\nabla, \xi\psi^+\alpha\psi> = 0, \tag{206}$$

in which the quantity

$$\rho := \xi\psi^+ \psi \tag{207}$$

is interpreted as the operator charge density and the quantity

$$J := \xi\psi^+ c\alpha\psi \tag{208}$$

is naturally interpreted as the operator current density in the space $\mathbb{R}^3$. Whence the weak operator equations (204) can be rewritten, taking into account the definitions (207) and (208), in the standard Maxwell equations weak form of the

$$(\nabla \times \tilde{B} - \frac{1}{c}\frac{\partial \tilde{E}}{\partial t})\tilde{\Phi} = \frac{J}{c}\tilde{\Phi}, <\nabla, \tilde{E}> \tilde{\Phi} = \rho\tilde{\Phi} \tag{209}$$

under the Fock space $\tilde{\Phi}$ constraint (198). Moreover, based on the weak operator Maxwell equations (209) and the Lorenz constraint (198), one can derive ~~easily~~ the following weak operator linear wave equations

$$(\frac{1}{c^2}\frac{\partial^2 \tilde{\varphi}}{\partial t^2} - \Delta\tilde{\varphi})\tilde{\Phi} = \rho\tilde{\Phi}, (\frac{1}{c^2}\frac{\partial^2 \tilde{A}}{\partial t^2} - \Delta\tilde{A})\tilde{\Phi} = \frac{J}{c}\tilde{\Phi} \tag{210}$$

with ~~in~~ respect to the common laboratory reference frame K_t, allowing to calculate the induced by the charged fermionic field causal quantum bosonic potentials $(\tilde{\varphi}_\xi, \tilde{A}_\xi) : \tilde{\Phi} \to \tilde{\Phi}^4$ in the analytical form:

$$\tilde{\varphi}_{\xi} = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\rho(t', y)d^3y}{|x-y|}, \tilde{A}_{\xi} = \frac{1}{4\pi c}\int_{\mathbb{R}^3}\frac{J(t', y)d^3y}{|x-y|}, \tag{211}$$

where the "retarded" temporal parameter $t' := t - |x-y|/c \in \mathbb{R}$, making the equations (210) exactly satisfied modulo the solutions to their uniform forms. Moreover, owing to (206), the expressions (211) satisfy exactly the strong operator Lorenz constraint

$$\frac{1}{c}\frac{\partial\tilde{\varphi}_{\xi}}{\partial t} + <\nabla, \tilde{A}_{\xi}> = 0 \tag{212}$$

with respect to the laboratory reference frame K_t.

From the analysis of the quantum charged particle fermionic field model, interacting with the self-generated quantum bosonic electromagnetic field, one can infer the following important consequences:

- the physical effective evolution of the fermionic-bosonic system with respect to the common reference frame K_t is governed by the reduced fermionic Hamiltonian operator (177), acting on the canonically transformed Fock space $\tilde{\Phi}$, reduced by means of the weak Lorenz type operator constraint (198);

- the compatibility of evolutions of the quantum fermionic and bosonic fields with respect to the common temporal reference frame K_t entails the reciprocal influence of the fermionic field on the bosonic one and vice versa, being clearly demonstrated both by the weak field potentials operator equations (210) and the Lorentz type weak constraint (198) imposed on the Fock space $\tilde{\Phi}$;

- subject to the basic self-interacting fermionic-bosonic system described by the joint Hamiltonian operator (172) in the transformed Fock space $\tilde{\Phi}$, one can claim that the bosonic electromagnetic impact into the quantum charged particle dynamics is decisive, as owing to it the fermionic system can realize its charge interaction property through the physical vacuum deformation, caused by the related deformation of the weak Lorenz type operator constraint (190), and resulting into the weak operator potential equations (211).

The consequences formulated above subject to the quantum fermionic-bosonic self-interacting phenomenon, as it was shown in ~~**Error! Reference source not found.**~~, appeared to be very important from a classical point of view, especially for physical understanding the inertial properties of a charged particle under action of the self-generated electromagnetic field.

6. Classical reduction of the quantum charged particle and electromagnetic field evolutions

Let's consider the vector position operator $\hat{x} : \tilde{\Phi} \to \tilde{\Phi}^3$ and its weak in the reduced Fock space $\tilde{\Phi}$ evolution with respect to the complete and suitably renormalized charged particle

Hamiltonian operator (177). Taking into account that the Hamiltonian operator $\tilde{H}_f^{(int)}:\tilde{\Phi}\to\tilde{\Phi}$ can be represented as

$$\tilde{H}_f^{(int)}=\int_{\mathbb{R}^3}d^3x\psi^+ < c\alpha,\hat{p}_x > \psi+\int_{\mathbb{R}^3}d^3x(\xi\psi^+\psi\tilde{\varphi}_\xi-\xi\psi^+ < c\alpha,\tilde{A}_\xi > \psi), \tag{213}$$

within which the operators $(\tilde{\varphi}_\xi,\tilde{A}_\xi):\tilde{\Phi}\to\tilde{\Phi}^3$ are given by the nonlocal integral expressions (211) and $\hat{p}_x:\tilde{\Phi}\to\tilde{\Phi}^3$ is the locally defined charged particle ξ momentum operator $\hat{p}_x:=\frac{\hbar}{i}\nabla_x$, canonically conjugated [71] to the position operator $\hat{x}:\tilde{\Phi}\to\tilde{\Phi}^3$, that is

$$[\hat{p}_y,\hat{x}]=\frac{\hbar}{i}\delta(x-y) \tag{214}$$

for any $x,y\in\mathbb{R}^3$. This also, in particular, means that the position operator $\hat{x}:\tilde{\Phi}\to\tilde{\Phi}^3$ is *a priori* given in the diagonal representation: $\hat{x}\tilde{f}:=x\tilde{f}$ for any vector $\tilde{f}\in\tilde{\Phi}$.

As a result of a simple calculation one finds the expression

$$d\hat{x}/dt=\psi^+c\alpha\psi, \tag{215}$$

which can be used for obtaining the classical charged particle ξ velocity $u(t,x)\in T(\mathbb{R}^3)$ as

$$u(t,x):=(\Omega,d\hat{x}/dt\Omega)=(\Omega,\psi^+c\alpha\psi\Omega), \tag{216}$$

where the vector $\Omega\in\tilde{\Phi}$ is the ground state of the Hamiltonian operator (213) acting in the Lorenz type reduced and suitably renormalized [71] [88] [121] [123] Fock space $\tilde{\Phi}$. Substituting (215) and (207) into the Hamiltonian expression (213) one obtains the expression

$$\tilde{H}_f^{(int)}=\int_{\mathbb{R}^3}d^3x < d\hat{x}/dt,\hat{p}_x > +\int_{\mathbb{R}^3}d^3x(\rho\tilde{\varphi}_\xi-\frac{1}{c}J,\tilde{A}_\xi >), \tag{217}$$

whose classical counterpart looks as

$$\bar{H}_f^{(int)}=\int_{\mathbb{R}^3}d^3x(\rho\bar{\varphi}_\xi- < \frac{1}{c}J,\bar{A}_\xi >), \tag{218}$$

within which there was taken into account the previously assumed quantum massless charged particle ξ fermionic field. The expression (218) jointly with the renormalized bosonic field Hamiltonian (162) gives rise to the complete classical Hamiltonian function

$$\bar{H}_{f-b}^{(int)}=\int_{\mathbb{R}^3}d^3x[\frac{1}{2}(|\bar{E}|^2+|\bar{B}|^2)+\rho\bar{\varphi}_\xi- < \frac{1}{c}J,\bar{A}_\xi >], \tag{219}$$

governing the temporal evolution both of the charged particle ξ and of the electromagnetic fields with respect to the laboratory reference frame K_t. The obtained ~~above~~ Hamiltonian function and its corresponding Lagrangian form *(166)* have been effectively used before in [125] for describing the classical self-interacting charged particle dynamics and its inertial properties.

Being experienced with the analysis of a self-interacting charged quantum particle fermionic field with the self-generated quantum bosonic electromagnetic field, we understand well that the influence of the electromagnetic field on the charged particle should be considered as crucial, strongly modifying the related fermionic Hamiltonian operator, describing the charged particle dynamics. As the simultaneously modified bosonic electromagnetic operator depends, owing to the self-interaction, on the charge and current particle field densities, the joint impact on the charged particle dynamics can be effectively classically modeled by means of its inertial mass parameter. In the quantum operator case the physical charged particle mass parameter $m_{ph} \in \mathrm{R}_+$ can be naturally defined by means of the least quantum renormalized Hamiltonian (172) eigenvalue

$$m_{ph} := c^{-2} \inf_{\tilde{f} \in \tilde{\Phi}, \|\tilde{f}\|=1} (\tilde{f}, \tilde{H}^{(int)}_{f-b} \tilde{f}), \tilde{H}^{(int)}_{f-b} := \tilde{H}^{(int)}_{f} + \tilde{H}_b, \tag{220}$$

in the suitably transformed ~~and~~ reduced by means of the operator Lorenz type constraint (198) Fock space $\tilde{\Phi}$ with respect to the common reference frame K_t. As the quantum spectral problem (220) is very complicated, new tools are needed to be developed for its successful analysis.

7. Classical self-interacting charged particle dynamics and its inertial properties

Being experienced with the analysis of a self-interacting charged quantum particle fermionic field with the self-generated quantum bosonic electromagnetic field, we understand well that the influence of the electromagnetic field on the charged particle should be considered as crucial, strongly modifying the related fermionic Hamiltonian operator, describing the charged particle dynamics. As the simultaneously modified bosonic electromagnetic operator depends, owing to the self-interaction, on the charge and current particle field densities, the joint impact on the charged particle dynamics can be effectively classically modeled by means of its inertial mass parameter. In the quantum operator case the physical charged particle mass parameter $m_{ph} \in \mathrm{R}_+$ can be naturally defined by means of the least quantum renormalized Hamiltonian (172) eigenvalue (220) in the suitably transformed and reduced by means of the operator Lorenz type constraint (198) Fock space $\tilde{\Phi}$ with respect to the common reference frame K_t. As the quantum spectral problem (220) is very complicated, we will try below to analyze it from the classical point of view.

repetition

The quantum operator Hamiltonian approach of Section 4. makes it possible to treat analytically the charged particle self-interaction mechanism, which can be described by means of the following two steps. The first one consists in producing the charged particle dynamics governed by the gauge type component of the charged particle Hamiltonian operator (177), and the second one - consists in modifying this dynamics by means of the self-generated electromagnetic field, whose influence is governed by the bosonic Hamiltonian (179), perturbed by the dependence of the electromagnetic field potentials on the related charge and current densities through the differential relationships (210). This mechanism can be classically realized analytically by means of the alternative and already before mentioned Lagrangian least action formalism, following the well known slightly modified [5] Landau-Lifschitz scheme. Namely, the Lagrangian function for the classical charged particle ξ, interacting with the self-generated electromagnetic field, is easily derived from the corresponding Hamiltonian function (219), giving rise to the classical Lagrangian expressions *(166)* in the following slightly extended form:

$$\tilde{\mathrm{L}}_{(f-b)} = \int_{\mathrm{R}^3} d^3x(<-\frac{1}{c}J,\tilde{A}>-\rho\tilde{\varphi})+$$

$$+\frac{1}{2}\int_{\mathrm{R}^3} d^3x(<\nabla\tilde{\varphi}+\frac{1}{c}\frac{\partial\tilde{A}}{\partial t},\nabla\tilde{\varphi}+\frac{1}{c}\frac{\partial\tilde{A}}{\partial t}>- \tag{221}$$

$$-<\nabla\times\tilde{A},\nabla\times\tilde{A}>)-<k,dx/dt>,$$

where vector $k:=k(t,x)\in\mathrm{E}^3$ models the related radiation reaction momentum, caused by the accelerated charged particle ξ with respect to the laboratory reference frame K_t, as well as ~~there implied~~ assuming that the classical Lorenz type constraint (198) is satisfied *a priori*. Here we need to mention that the first part of the Lagrangian (221) is responsible for the internal gauge type charged particle self-interaction and the second one is responsible for the external charged particle self-interaction induced by the suitably perturbed electromagnetic field, depending on the particle charge and current densities. The physical difference between these two phenomena proves to be especially important for calculation of an effective Lagrangian function for the related dynamical properties of the self-interacting charged particle.

Before proceeding further we need to make an important comment concerning the least action properties of the classical relativistic self-interacting Lagrangian (221). Namely, taking into account a deep quantum vacuum origin [121] of the electromagnetic field and its effective measuring only with respect to the common laboratory reference frame K_t, we can state that the related Maxwell equations should be naturally derived from the following least action principle: the variation $\delta\tilde{S}^{(t)}_{f-b}=0,$ where by definition, the action functional

$$\tilde{S}^{(t)}_{f-b} := \int_{t_1}^{t_2}\tilde{\mathrm{L}}_{(f-b)}dt \tag{222}$$

is calculated with respect to the laboratory reference frame K_t on a fixed temporal interval $[t_1,t_2]\subset\mathrm{R}$. Yet, as it is easy to check, the above action functional (222) fails to derive the corresponding Lorentz type dynamical equations for the self-interacting charged particle ξ, if to take into account that the related charged particle is considered to be pointwise, located at point $x(t)\in\mathrm{E}^3$ for $t\in\mathrm{R}$ and endowed with the current density vector $J=\rho dx(t)/dt\in\mathrm{E}^3$ and the charge density $\rho:=\xi\delta(x-x(t)),x\in\mathrm{E}^3$. This, evidently, means that the action functional (222) should be suitably modified with respect to the [1] [51] Feynman proper time reference frame paradigm, owing to which the action functional for the charged particle dynamics has a physical sense if and only if it is considered with respect to the proper time reference frame K_τ:

$$\tilde{S}^{(\tau)}_{f-b} := \int_{\tau_1}^{\tau_2}\tilde{\mathrm{L}}_{(f-b)}\sqrt{(1+|\dot{x}|^2/c^2)}d\tau \tag{223}$$

on a fixed temporal interval $[\tau_1,\tau_2]\subset\mathrm{R},$ where we took into account, that $dt:=\sqrt{(1+|\dot{x}|^2/c^2)}d\tau$ and, by definition, the velocity $\dot{x}:=dx/d\tau$ with respect to the proper temporal parameter $\tau\in\mathrm{R}$. Then from the least action condition $\delta\tilde{S}^{(\tau)}_{f-b}=0$ on the fixed temporal interval $[\tau_1, \tau_2] \subset \mathbb{R}$ one easily obtains the well known classical Lorentz dynamical equation

$$\frac{d}{dt}(mu) = \xi \tilde{E} + \xi u \times \tilde{B}, \tag{224}$$

written with respect to the laboratory reference frame K_t. When deriving (224) ~~there was put,~~ we defined ~~by definition,~~ the inertial mass by ~~definition~~ $m := -\bar{\varphi}/c^2$. The reasonings presented above will be in part employed below when analyzing a suitably reduced Lagrangian function (221).

For the self-interacting charged particle to be physically specified by the mentioned ~~above~~ phenomena in detail, we will consider below a so-called shell model particle, whose charge is uniformly distributed on a sphere of a very small yet fixed radius. Then, following the similar calculations from [5], one can obtain from (221) that

$$\begin{aligned}
\tilde{\mathrm{L}}_{(f-b)} &= \frac{1}{2}\int_{\mathbb{R}^3} d^3x \Big(\tilde{\varphi} < \nabla, \tilde{E} > + \frac{1}{c} < \tilde{A}, \frac{\partial \tilde{E}}{\partial t} > - \frac{1}{c} < \tilde{A}, J + \frac{\partial \tilde{A}}{\partial t} > \Big) - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3x < \tilde{A}, \tilde{E} > + \int_{\mathbb{R}^3} d^3x \Big(< \frac{1}{c} J, \tilde{A} > - \rho \bar{\varphi}\Big) - \\
&- \frac{1}{2}\lim_{r\to\infty}\int_{S_r^2} < \tilde{\varphi}\tilde{E} + \tilde{A} \times \tilde{B}, dS_r^2 > - < k, dx/dt > = \\
&= -\int_{\Omega_-(\xi)} d^3x \Big(< \frac{1}{c} J, \tilde{A} > - \rho \bar{\varphi}\Big) + \int_{\Omega_-(\xi) \cup \Omega_+(\xi)} d^3x \Big(< \frac{1}{c} J, \tilde{A} > - \rho \bar{\varphi}\Big) - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3x < \tilde{A}, \tilde{E} > - < k, dx/dt > = \\
&= \frac{1}{2}\int_{\Omega_-(\xi)} d^3x \Big(< \frac{1}{c} J, \tilde{A} > - \rho \bar{\varphi}\Big) + \frac{1}{2}\int_{\Omega_-(\xi) \cup \Omega_+(\xi)} d^3x \Big(< \frac{1}{c} J, \tilde{A} > - \rho \bar{\varphi}\Big) - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3x < \tilde{A}, \tilde{E} > - < k, dx/dt >,
\end{aligned} \tag{225}$$

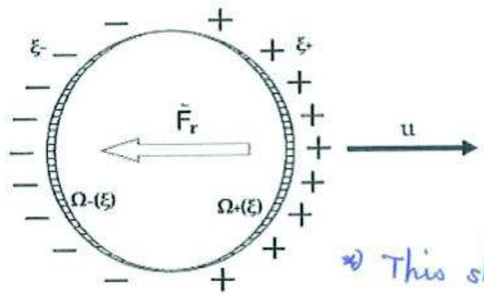

Fig. 1. The courtesy picture from [31]

*) This should be written in a more simple way with shorter sentences. It is nearly not readable.

where we took into account that $\lim_{r\to\infty}\int_{S_r^2} <\tilde{\varphi}\tilde{E}+\tilde{A}\times\tilde{B}, dS_r^2> = 0,$ meaning ~~the~~ vanishing of ~~the~~ radiated ~~by the accelerated charged particle~~ energy. Also ~~as well as~~ we denoted by $\Omega_-(\xi) :=$ supp $\xi_- \subset \mathrm{S}^2$ and by $\Omega_+(\xi) :=$ supp $\xi_+ \subset \mathrm{S}^2$ the corresponding charge ξ parts supports, located on the electromagnetic field shadowed rear and on the electromagnetic field exerted on semi-spheres of the charged particle spherical shell $\Omega(\xi) := \Omega_-(\xi) \cup \Omega_+(\xi)$, respectively (see Fig 1.) to its motion with velocity $u := dx/dt \in \mathrm{E}^3$ with respect to the laboratory reference frame K_t. *)

The expression (225) demonstrates explicitly that during the charged particle motion the self-generated electromagnetic field interacts effectively only with its frontal part $\Omega_+(\xi) \subset \mathrm{S}^2$ of the particle spherical shell S^2, as the rear part $\Omega_-(\xi) \subset \mathrm{S}^2$ of the particle shell enters during its motion into the shadowed interior region of the sphere, where the net electric field $\tilde{E} \in \mathrm{E}^3$ is vanishing owing to the charged particle spherical symmetry. To proceed further we need to calculate the electromagnetic potentials $(\tilde{\varphi}, \tilde{A}): M^4 \to \mathrm{R}\times\mathrm{E}^3$, using the determining expressions (211) as $1/c \to 0$:

$$\tilde{\varphi}=\int_{\mathbb{R}^3} d^3y \frac{\rho(t',y)}{|x-y|}\bigg|_{t'=t-|x-y|/c} = \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^3} d^3y \frac{\rho(t-\varepsilon,y)}{|x-y|}+$$

$$+\lim_{\varepsilon\downarrow 0}\frac{1}{2c^2}\int_{\mathbb{R}^3} d^3y\,|x-y|\,\partial^2\rho(t-\varepsilon,y)/\partial t^2+$$

$$+\lim_{\varepsilon\downarrow 0}\frac{1}{6c^3}\int_{\mathbb{R}^3} d^3y\,|x-y|^2\,\partial\rho(t-\varepsilon,y)/\partial t+O(1/c^4)= \tag{226}$$

$$=\int_{\Omega_+(\xi)} d^3y \frac{\rho(t,y)}{|x-y|}+\frac{1}{2c^2}\int_{\Omega_+(\xi)} d^3y\,|x-y|\,\partial^2\rho(t,y)/\partial t^2+$$

$$+\frac{1}{6c^3}\int_{\Omega_+(\xi)} d^3y\,|x-y|^2\,\partial\rho(t,y)/\partial t+O(1/c^4),$$

$$\tilde{A}=\frac{1}{c}\int_{\mathbb{R}^3} d^3y \frac{J(t',y)}{|x-y|}\bigg|_{t'=t-|x-y|/c} = \lim_{\varepsilon\downarrow 0}\frac{1}{c}\int_{\mathbb{R}^3} d^3y \frac{J(t-\varepsilon,y)}{|x-y|}-$$

$$-\lim_{\varepsilon\downarrow 0}\frac{1}{c^2}\int_{\mathbb{R}^3} d^3y\,\partial J(t-\varepsilon,y)/\partial t+$$

$$+\lim_{\varepsilon\downarrow 0}\frac{1}{2c^3}\int_{\mathbb{R}^3} d^3y\,|x-y|\,\partial^2 J(t-\varepsilon,y)/\partial t^2+O(1/c^4)=$$

$$=\frac{1}{c}\int_{\Omega_+(\xi)} d^3y \frac{J(t,y)}{|x-y|}-\frac{1}{c^2}\int_{\Omega_+(\xi)} d^3y\,\partial J(t,y)/\partial t+$$

$$+\frac{1}{2c^3}\int_{\Omega_+(\xi)} d^3y\,|x-y|\,\partial^2 J(t,y)/\partial t^2+O(1/c^4),$$

where the limit $lim_{\varepsilon\downarrow 0}(\ldots)$ was treated physically, that is taking into account the assumed ~~above~~ spherical shell model of the charged particle ξ and its corresponding self-interaction during its motion. Now, as a result of ~~simple enough~~ calculations based on the electromagnetic potentials (226), the effective expression for the classical Lagrangian (225) can be equivalently rewritten up to $O(1/c^4)$ accuracy with respect to the laboratory reference frame K_t as

$$\mathrm{L}^{(t)}_{(f-b)}=\frac{\mathrm{E}_{es}}{2c^2}|u|^2, \tag{227}$$

where we have made

~~If to make~~ use of the following integral expressions:

$$\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3x \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3y\rho(t,y)\rho(t,y) := \xi^2,$$

$$\frac{1}{2}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3x \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3y \frac{\rho(t,y)\rho(t,y)}{|x-y|} := \mathrm{E}_{es},$$

$$\int_{\Omega_+(\xi)} d^3x\rho(t,x)\int_{\Omega_+(\xi)} d^3y \frac{\rho(t;y)}{|y-x|} = \frac{1}{2}\mathrm{E}_{es},$$

$$\int_{\Omega_-(\xi)} d^3x\rho(t,x)\int_{\Omega_-(\xi)} d^3y \frac{\rho(t;y)}{|y-x|} = \frac{1}{2}\mathrm{E}_{es}, \tag{228}$$

$$\int_{\Omega_-(\xi)} d^3x\rho(t,x)\int_{\Omega_+(\xi)} d^3y \frac{\rho(t;y)}{|x-y|} |\frac{<y-x,u>}{|y-x|}|^2 > := \frac{\mathrm{E}_{es}}{6}|u|^2,$$

$$\int_{\Omega_+(\xi)} d^3x\rho(t,x)\int_{\Omega_+(\xi)} d^3y \frac{\rho(t;y)}{|x-y|} |\frac{<y-x,u>}{|y-x|}|^2 > := \frac{\mathrm{E}_{es}}{6}|u|^2,$$

obtained owing to the reasonings similar to those in [2] [126]. Now, to derive from the reduced Lagrangian function (227) the corresponding dynamic equation for the charged shell model particle ξ, we need ~~within the discussed above~~ the Feynman proper time paradigm to transform this Lagrangian with respect to the charged particle proper time reference frame K_τ:

$$\mathrm{L}^{(t)}_{(f-b)} \to \mathrm{L}^{(\tau)}_{(f-b)} = \frac{\bar{m}_{es}}{2}|\dot{x}|^2 - <k,\dot{x}>, \tag{229}$$

where we denoted by

$$\bar{m}_{es} := m_{es}\sqrt{1-|u|^2/c^2} \tag{230}$$

the so-called relativistic rest mass of the charged particle with respect to the proper time reference frame K_τ, and by

$$m_{es} := \mathrm{E}_{es}/c^2 \tag{231}$$

the so-called charged particle electromagnetic mass with respect to the laboratory reference frame K_t. Based on the Lagrangian function (229) one can construct up to $O(1/c^2)$ the generalized charged particle inertial momentum

$$\tilde{\pi}_f := m_{ph}u - k \tag{232}$$

as

$$\tilde{\pi}_f = \partial\mathrm{L}^{(\tau)}_{(f-b)}/\partial\dot{x} = m_{es}u - k, \tag{233}$$

satisfying, with respect to the proper time reference frame K_τ, the evolution equation

$$d\tilde{\pi}_f/d\tau = \partial\mathrm{L}^{(\tau)}_{(f-b)}/\partial x = 0, \tag{234}$$

which is equivalent to the Lorentz type equation

$$d(m_{es}u)/dt = dk(t)/dt := \tilde{F}_r \tag{235}$$

with respect to the laboratory reference frame K_r, where the right hand side of (235) means, by definition, the corresponding radiation reaction force $\tilde{F}_r$. Having applied to the Lagrangian function (229) the standard Legendre transformation, one ~~easily~~ finds the quasi-classical conserved Hamiltonian function

$$H_{f-b}^{(t)} := <\tilde{\pi}_f, \dot{x}> - \tilde{L}_{(f-b)}^{(\tau)} = \frac{m_{es}|u|^2}{2}(1+\frac{1}{2}|u|^2/c^2), \tag{236}$$

satisfying, with respect to the laboratory reference frame K_r, the condition $dH_{f-b}^{(t)}/dt = 0$ for all $t \in R$. Yet, the most interesting and important consequence from (236)and the dynamic equation (235), consists in coinciding the electromagnetic mass parameter $m_{es} \in R_+$:

$$m_{phys} := m_{es}, \tag{237}$$

defined by (231), with the naturally related and physically observed inertial mass $m_{phys} \in R_+$, as it was conceived by H. Lorentz and M. Abraham more than one hundred years ago.

8. The radiation reaction force analysis

To calculate the radiation reaction force (235) one can make use of the classical Lorentz type force expression (224) and obtain in the case of the charged particle shell model, similarly to [2] [126], up to $O(1/c^4)$ accuracy, the resulting self-interacting Abraham-Lorentz type force expression with respect to the laboratory reference frame K_r. Owing to the zero net force condition, we have that

$$d\tilde{\pi}_f/dt + \tilde{F}_s = 0, \tag{238}$$

where, by definition, $\tilde{\pi}_f := m_{ph}u$, the Lorentz force can be rewritten in the following form:

$$\begin{aligned} \tilde{F}_s \; = \; & -\frac{1}{2c}\int_{\Omega_-(\xi)} d^3x\rho(t,x)\frac{d}{dt}\tilde{A}(t,x) - \\ & -\frac{1}{2c}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3x\rho(t,x)\frac{d}{dt}\tilde{A}(t,x) - \\ & -\frac{1}{2}\int_{\Omega_-(\xi)} d^3x\rho(t,x)\nabla\tilde{\varphi}(t,x)\,(1-|u/c|^2) - \\ & -\frac{1}{2}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3x\rho(t,x)\nabla\tilde{\varphi}(t,x)\,(1-|u/c|^2). \end{aligned} \tag{239}$$

Based on calculations similar to those of [2] [126], from (239) and (226) one can obtain, within the charged particle shell model, for small $|u/c| = 1$ and slow enough acceleration that

$$\tilde{F}_s = \sum_{n\in Z_+} \frac{(-1)^{n+1}}{2n!c^n}(1-|u/c|^2)[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+$$

$$+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\int_{\Omega_+(\xi)}d^3y\frac{\partial^n}{\partial t^n}\rho(t,y)\nabla|x-y|^{n-1} +$$

$$+\sum_{n\in Z_+}\frac{(-1)^{n+1}}{2n!c^{n+2}}[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+$$

$$+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\int_{\Omega_+(\xi)}d^3y|x-y|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}J(t,y]=$$

$$= \sum_{n\in Z_+} \frac{(-1)^{n+1}}{2n!c^{n+2}}(1-|u/c|^2)[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+ \tag{240}$$

$$+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\int_{\Omega_+(\xi)}d^3y\frac{\partial^{n+2}}{\partial t^{n+2}}\rho(t,y)\nabla|x-y|^{n+1} +$$

$$+\sum_{n\in Z_+}\frac{(-1)^{n+1}}{2n!c^{n+2}}[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+$$

$$+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\int_{\Omega_+(\xi)}d^3y|x-y|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}J(t,y).$$

The relationship above can be rewritten, owing to the charge continuity equation (206)-(208) and the rotational symmetry property, giving rise to the radiation force differential-integral expression:

$$\tilde{F}_s = \frac{d}{dt}[\sum_{n\in Z_+}\frac{(-1)^{n+1}}{6n!c^{n+2}}[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\times$$

$$\times\int_{\Omega_+(\xi)}d^3y|x-y|^{n-1}\frac{\partial^n}{\partial t^n}J(t,y)-\sum_{n\in Z_+}\frac{(-1)^n|u|^2}{6n!c^{n+4}})[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+ \tag{241}$$

$$+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\int_{\Omega_+(\xi)}d^3y|x-y|^{n-1}\frac{\partial^n}{\partial t^n}J(t,y)].$$

~~From the latter,~~ Taking into account the integral expressions (228), one finds from (241) up to the $O(1/c^4)$ accuracy the final radiation reaction force expression

$$\tilde{F}_s = -\frac{d}{dt}\left(\frac{E_{es}}{c^2}u\right) + \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}) = \tag{242}$$
$$= -\frac{d}{dt}(m_{es}u) + \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} = -\frac{d}{dt}(m_{es}u - \frac{2\xi^2}{3c^3}\frac{du}{dt})$$

holds. We mention here that following the reasonings from [7] [31] [35] [105] [106], in the expressions above there is taken into account an additional hidden and ~~the~~ velocity $u \in T(\mathbb{R}^3)$ directed electrostatic Coulomb surface self-force, acting only on the *front half part* of the spherical electron shell. As a result, from (238), (239) and the relationship (232) one obtains that the generalized charged particle momentum

$$\tilde{\pi}_p := m_{es}u - \frac{2\xi^2}{3c^3}\frac{du}{dt} = m_{es}u - k, \tag{243}$$

thereby defining both the radiation reaction momentum $k(t) = \frac{2\xi^2}{3c^3}\frac{du(t)}{dt}$ for all $t \in \mathbb{R}$ and the corresponding radiation reaction force

$$\tilde{F}_r = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}, \tag{244}$$

which it

coincides exactly with the classical Abraham-Lorentz--Dirac expression. From (243) ~~one easily~~ follows that the observable physical charged particle shell model inertial mass

$$m_{ph} = m_{es} = E_{es}/c^2 \tag{245}$$

is of ~~the~~ electromagnetic origin, coinciding exactly with the result (237) obtained above. Moreover, (243) ensues the final force expression

$$\frac{d}{dt}(m_{es}u) = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} + O(1/c^4). \tag{246}$$

The latter means, in particular, that the real physically observed " inertial" mass m_{ph} of the charged shell model particle ξ is strongly determined by its electromagnetic self-interaction energy E_{es} with respect to the laboratory reference frame K_t. A similar statement ~~there~~ was recently discussed in [31] [35], based on the vacuum Casimir effect type considerations. Moreover, the assumed ~~above~~ boundedness of the electrostatic self-energy E_{es} appears to be completely equivalent both to the presence of the so-called intrinsic Poincaré type " *tensions*" , analyzed in [7] [31] [118], and to the existence of a special compensating Coulomb " *pressure*" , suggested in [35], guaranteeing the assumed electron stability in the works of H. Lorentz and M. Abraham.

8.1. Comments

The electromagnetic mass origin problem was reanalyzed in details within the Feynman proper time paradigm and related vacuum field theory approach by means of the fundamental least action principle and the Lagrangian and Hamiltonian formalisms. The resulting electron inertia appeared to coincide in part, in the quasi-relativistic limit, with the momentum expression obtained more than one hundred years ago by M. Abraham and H. Lorentz [53] [54] [55] [64], yet it proved to contain an additional hidden impact owing to the imposed electron stability constraint, which was taken into account in the original action functional as some preliminarily undetermined constant component. As it was demonstrated in [31] [35], this stability constraint can be successfully realized within the charged shell model of electron at rest, if to take into account the existing ambient electromagnetic " dark" energy fluctuations, whose inward directed spatial pressure on the electron shell is compensated by the related outward directed electrostatic Coulomb spatial pressure as the electron shell radius satisfies some limiting compatibility condition. The latter also allows to compensate simultaneously the corresponding electromagnetic energy fluctuations deficit inside the electron shell, thereby forbidding the external energy to flow into the electron. In contrary to the lack of energy flow inside the electron shell, during the electron movement the corresponding internal momentum flow is not vanishing owing to the nonvanishing hidden electron momentum flow caused by the surface pressure flow and compensated by the suitably generated surface electric current flow. As it was shown, this backward directed hidden momentum flow makes it possible to justify the corresponding self-interaction electron mass expression and to state, within the electron shell model, the fully electromagnetic electron mass origin, as it has been conceived by H. Lorentz and M. Abraham and strongly supported by R. Feynman in his Lectures [1]. This consequence is also independently supported by means of the least action approach, based on the Feynman proper time paradigm and the suitably calculated regularized retarded electric potential impact into the charged particle Lagrangian function.

The charged particle radiation problem, revisited in this Section, allowed to conceive the explanation of the charged particle mass as that of a compact and stable object which should be exerted by a vacuum field self-interaction energy. The latter can be satisfied if to impose on the intrinsic charged particle structure [30] some nontrivial geometrical constraints. Moreover, as follows from the physically observed particle mass expressions (245), the electrostatic potential energy being of the self-interaction origin, contributes into the inertial mass as its main relativistic mass component.

There exist different relativistic generalizations of the force expression (246), which suffer the common physical inconsistency related to the no radiation effect of a charged particle in uniform motion.

Another deeply related problem to the radiation reaction force analyzed above is the search for an explanation to the Wheeler and Feynman reaction radiation mechanism, called the absorption radiation theory, strongly based on the Mach type interaction of a charged particle with the ambient vacuum electromagnetic medium. Concerning this problem, one can also observe some of its relationships with the one devised here within the vacuum field theory approach, but this question needs a more detailed and extended analysis.

9. Supplement: the "minimum" interaction principle and its geometric backgrounds

In this Section we will sketch analytical backgrounds of the "minimum" interaction principle widely used in modern theoretical and mathematical physics. For description of a moving point charged particle under an external electromagnetic field, we will make use of the geometric approach [64]. Namely, let a trivial fiber bundle structure $\pi : \mathrm{M} \to \mathbb{R}^3, \mathrm{M} = \mathbb{R}^3 \times G$, with the abelian structure group $G := \mathbb{R} \setminus \{0\}$, equivariantly act on the canonically symplectic coadjoint space $T^*(\mathrm{M})$. The latter possesses the canonical symplectic structure

$$\begin{aligned} \omega^{(2)}(p,z;x,g) \; &:= d(pr_*)^* \alpha^{(1)}(x,g) = < dp, \wedge dx > + \\ &+ \; < dz, \wedge g^{-1}dg >_{\mathrm{G}} + < zdg^{-1}, \wedge dg >_{\mathrm{G}} \end{aligned} \tag{247}$$

for all $(p,z;x,g) \in T^*(\mathrm{M})$, where $\alpha^{(1)}(x,g) := < p, dx > + < z, g^{-1}dg >_{\mathrm{G}} \in T^*(\mathrm{M})$ is the corresponding Liouville form on $T^*(\mathrm{M})$ and $< \cdot, \cdot >$ is the usual scalar product in $\mathbb{E}^3$. On the fibered space M one can define a connection Γ by means of an one-form $\mathrm{A} : M \to T^*(\mathrm{M}) \times \mathrm{G}$, determined as

$$\mathrm{A}(x,g) := g^{-1} < \xi A(x), dx > g + g^{-1}dg \tag{248}$$

with $\xi \in \mathrm{G}^*, (x,g) \in \mathbb{R}^3 \times G$. The corresponding curvature 2-form $\Sigma^{(2)} \in \Lambda^2(\mathbb{R}^3) \otimes \mathrm{G}$ is

$$\Sigma^{(2)}(x) := d\mathrm{A}(x,g) + \mathrm{A}(x,g) \wedge \mathrm{A}(x,g) = \xi \sum_{i,j=1}^{3} F_{ij}(x) dx^i \wedge dx^j, \tag{249}$$

where

$$F_{ij}(x) := \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \tag{250}$$

for $i,j = \overline{1,3}$ is the spatial electromagnetic tensor with respect to the reference frame K_t. For an element $\xi \in \mathrm{G}^*$ to be compatibly fixed, we need to construct the related momentum mapping $l : T^*(\mathrm{M}) \to \mathrm{G}^*$ with respect to the canonical symplectic structure (247) on $T^*(\mathrm{M})$, and put, by definition, $l(x,p) := \xi \in \mathrm{G}^*$ to be constant, $P_\xi := l^{-1}(\xi) \subset T^*(\mathrm{M})$ and $G_\xi = \{ g \in G : Ad_G^* \xi \}$ to be the corresponding isotropy group of the element $\xi \in \mathrm{G}^*$. Next we can apply the standard [47] [64] [96] invariant Marsden-Weinstein-Meyer reduction scheme to the orbit factor space $\tilde{P}_\xi := P_\xi / G_\xi$ subject to the corresponding group G action. Then, as a result of the Marsden-Weinstein-Meyer reduction, one finds that $G_\xi \simeq G$, the factor-space $\tilde{P}_\xi \simeq T^*(\mathbb{R}^3)$ becomes Poisson space with the suitably reduced symplectic structure $\bar{\omega}_\xi^{(2)} \in T^*(\tilde{P}_\xi)$. The corresponding Poisson brackets on the reduced manifold $\tilde{P}_\xi$ equal to

$$\begin{aligned} \{x^i, x^j\}_\xi \; &= 0, \{p_j, x^i\}_\xi = \delta_j^i, \\ \{p_i, p_j\}_\xi \; &= \xi F_{ij}(x) \end{aligned} \tag{251}$$

for $i,j = \overline{1,3}$, being considered with respect to the laboratory reference frame K_t. Based on (251) ~~is worth to~~ one can observe that a new so called "*shifted*" momentum variable

$$\tilde{\pi} := p + \xi A(x) \tag{252}$$

on $\tilde{P}_\xi$ gives rise to the symplectomorphic transformation $\bar{\omega}_\xi^{(2)} \to \tilde{\omega}_\xi^{(2)} := < d\tilde{\pi}, \wedge dx > \in \Lambda^2(T^*(\mathbb{R}^3))$. The latter gives rise to the following important in theoretical physics "*minimal interaction*" canonical Poisson brackets:

$$\{x^i, x^j\}_{\tilde{\omega}^{(2)}_\xi} = 0, \{\tilde{\pi}_j, x^i\}_{\tilde{\omega}^{(2)}_\xi} = \delta^i_j, \{\tilde{\pi}_i, \tilde{\pi}_j\}_{\tilde{\omega}^{(2)}_\xi} = 0 \tag{253}$$

for $i, j = \overline{1,3}$, represented ~~already~~ with respect to some new reference frame $\tilde{K}_{t'}$, characterized by the phase space coordinates $(x, \tilde{\pi}) \in \tilde{P}_\xi$ and a new evolution parameter $t' \in \mathbb{R}$, as the spatial Maxwell field compatibility equations

$$\partial F_{ij} / \partial x_k + \partial F_{jk} / \partial x_i + \partial F_{ki} / \partial x_j = 0 \tag{254}$$

are identically satisfied on $\mathbb{R}^3$ for all $i, j, k = \overline{1,3}$, owing to the electromagnetic curvature tensor (250) definition.

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