# The electrodynamic vacuum field theory approach and the electron inertia problem revisiting

It is a review of some new electrodynamics models of interacting charged point particles and related with them fundamental physical aspects, motivated by the classical A.M.Amper's magnetic and H.Lorentz force laws, as well as O. Jefimenko electromagnetic field expressions. Based on the suitably devised vacuum field theory approach the Lagrangian and Hamiltonian reformulations of some alternative classical electrodynamics models are analyzed in details. A problem closely related to the radiation reaction force is analyzed aiming to explain the Wheeler and Feynman reaction radiation mechanism, well known as the absorption radiation theory, and strongly dependent on the Mach type interaction of a charged point particle in an ambient vacuum electromagnetic medium. There are discussed some relationships between this problem and the one derived within the context of the vacuum field theory approach. The R.Feynman's "heretical" approach to deriving the Lorentz force based Maxwell electromagnetic equations is also revisited, its complete legacy is argued both by means of the geometric considerations and its deep relation with the devised vacuum field theory approach. Based on completely standard reasonings, we reanalyze the Feynman's derivation from the classical Lagrangian and Hamiltonian points of view and construct its nontrivial relativistic generalization compatible with the vacuum field theory approach. The electron inertia problem is reanalyzed within the Lagrangian-Hamiltonian formalisms and the related Feynman proper time paradigm. The validity of the Abraham-Lorentz electromagnetic electron mass origin hypothesis within the shell charged model is argued. The electron stability in the framework of the electromagnetic tension-energy compensation principle is analyzed.

Cet article présente un réexamen de certains nouveaux modèles électrodynamiques d'interaction entre particules chargées ponctuelles, et en lien avec des aspects physiques fondamentales, motivés par les lois magnétiques classiques de A.M. Ampère et par les forces classiques de H.Lorentz, ainsi que par les formulations du champ électromagnétique décrites par O. Jefimenko. Sur la base d'une formulation adéquate de la théorie des champs en vide quantique, les reformulations Lagrangiennes et Hamiltoniennes de certains modèles alternatifs de l'électrodynamique classique sont analysés en profondeur. Un problème étroitement lié à la force de réaction de rayonnement est analysé pour objectif d'expliquer le mécanisme de Wheeler et Feynman de réaction au rayonnement, bien connu comme la théorie d'amortissement de radiation, et dépend fortement de l'interaction de type Mach de particules ponctuelles chargées dans un milieu électromagnétique en vide ambiant. Certains rapports entre ce problème et celui obtenu dans le cadre de l'approche de la théorie des champs en vide quantique sont examinés. L'approche "hérétique" de R.Feynman qui consiste à dériver la force de Lorentz depuis équations électromagnétiques de Maxwell est également revisitée, et son approche est justifiée à la fois par des considérations géométriques et sa relation profonde avec l'approche de la théorie des champs à vide quantique. Sur la base de raisonnements complètement standards, nous réanalysons la dérivation de Feynman des points de vue Lagrangiens et Hamiltoniens classiques et construisons sa généralisation relativiste non triviale compatible avec l'approche de la théorie des champs en vide quantique. Le problème de l'inertie des électrons est réanalysé dans les formalismes de Lagrange et Hamilton et dans le paradigme de Feynman en temps propre correspondant. La validité de l'hypothèse d'Abraham-Lorentz sur l'origine électromagnétique de la masse de l'électron dans le modèle de couche électronique est soutenu. La stabilité de l'électron dans le cadre du principe de compensation tension-énergie électromagnétique est analysée.

*Keywords*: Amper law, Lorentz type force, Lorenz constraint, Vacuum field theory approach, Maxwell electromagnetic equation,Lagrangian and Hamiltonian formalisms, Fock many-temporal approach, Jefimenko equations, Quantum self-interactifermi model, Radiation theory, Feynman's proper time approach, Abraham-Lorentz electron mass problem.

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## 1. Classical relativistic electrodynamics models revisiting: Lagrangian and Hamiltonian analysis

### 1.1. Introductory setting

The Maxwell's equations serve as foundational [1] [2] [3] [4] [5] to the whole modern classical and quantum electromagnetic theory and electrodynamics. They are the cornerstone of a myriad of technologies and are basic to the understanding of innumerable effects. Yet there are a few effects or physical phenomena that cannot be explained [6] [7] [8] [9] [10] [11] [12] [13] within the conventional Maxwell theory. It is important to note here that in [8] [14] [15] [16] [17] there is argued that the Maxwell equations as themselves do not determine causal related to each other electric and magnetic fields, which prove, in reality, to be generated independently by an external charge and current distributions: "There is a widespread interpretation of Maxwell's equations indicating that spatially varying electric and magnetic fields can cause each other to change in time, thus giving rise to a propagating electromagnetic wave... However, Jefimenko's equations show an alternative point of view [3]. Jefimenko says: "...neither Maxwell's equations nor their solutions indicate an existence of causal links between electric and magnetic fields. Therefore, we must conclude that an electromagnetic field is a *dual entity* always having an electric and a magnetic component simultaneously *created by their common sources*: time-variable electric charges and currents." .... Essential features of these equations are easily observed which are that the right hand sides involve "retarded" time which reflects the "causality" of the expressions. In other words, the left side of each equation is actually "caused" by the right side, unlike the normal differential expressions for Maxwell's equations, where both sides take place simultaneously. In the typical expressions for Maxwell's equations there is no doubt that both sides are equal to each other, but as Jefimenko notes [3], "... since each of these equations connects quantities simultaneous in time, none of these equations can represent a causal relation." The second feature is that the expression for (electric field) $E$ does not depend upon (magnetic field) $B$ and vice versa. Hence, it is impossible for $E$ and $B$ fields to be "creating" each other. Charge density and current density are creating them both." As the Jefimenko's equations for the electric field $E$ and the magnetic field $B$ directly follow from the classical retarded Lienard-Wiechert potentials, generated by physically real external charge and current distributions, one naturally infers that these potentials also present suitably interpreted physical field entities mutually related to their sources. This way of thinking proved to be, from the physical point of view, very fruitful, having brought about a new vacuum field theory approach [18] [19] to alternative explaining the nature of the fundamental Maxwell equations and related electrodynamic phenomena.

We start from detailed revisiting the classical A.M. Ampere's law in electrodynamics and show that main inferences suggested by physicists of the former centuries can be strongly extended for them to agree more exactly with many modern both theoretical achievements and experimental results concerning the fundamental relationship of electrodynamic phenomena with the physical structure of vacuum as their principal carrier.

The important theoretical physical principles, characterizing the related electrodynamic vacuum field structure, we discuss subject to different charged point particle dynamics, based on the fundamental least action principle. In particular, the main classical relativistic relationships, characterizing the charge point particle dynamics, we obtain by means of the least action principle within the Feynman's approach to the Maxwell electromagnetic equations and the related Lorentz type force derivation. Moreover, for each of the least action principles constructed in the work, we describe the corresponding Hamiltonian pictures and present the related energy conservation laws. The elementary point charged particle, like electron, mass problem was inspiring many physicists [20] from the past as J. J. Thompson, G.G. Stokes, H.A. Lorentz, E. Mach, M. Abraham, P.A. M. Dirac, G.A. Schott and others. Nonetheless, their studies have not given rise to a clear explanation of this phenomenon that stimulated new researchers to tackle it from different approaches based on new ideas stemming both from the classical Maxwell-Lorentz electromagnetic theory, as in [1] [12] [21] [22] [23] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35] [36] [37] [38] [39], and modern quantum field theories of Yang-Mills and Higgs type, as in [40] [41] [42] [43] and others, whose recent and extensive review is done in [44].

We will mostly concentrate on detail analysis and consequences of the Feynman proper time paradigm [1] [22] [45] [46] subject to deriving the electromagnetic Maxwell equations and the related Lorentz like force expression considered from the vacuum field theory approach, developed in works [47] [48] [49] [50] [51], and further, on its applications to the electromagnetic mass origin problem. Our treatment of this and related problems, based on the least action principle within the Feynman proper time paradigm [1], has allowed to construct the respectively modified Lorentz type equation for a moving in space and radiating energy charged point particle. Our analysis also elucidates, in particular, the computations of the self-interacting electron mass term in [29], where there was proposed a not proper solution to the well known classical Abraham-Lorentz [52] [53] [54] [55] and Dirac [56] electron electromagnetic "4/3-electron mass" problem. As a result of our scrutinized studying the classical electromagnetic mass problem we have stated that it can be satisfactory solved within the classical H. Lorentz and M. Abraham reasonings augmented with the additional electron stability condition, which was not taken before into account yet appeared to be very important for balancing the related electromagnetic field and mechanical electron momenta. The latter, following recent enough works [31] [35], devoted to analyzing the electron charged shell model, can be realized within there suggested *pressure-energy compensation principle,* suitably applied to the ambient electromagnetic energy fluctuations and the own electrostatic Coulomb electron energy.

In our investigation, we were in part inspired by works [35] [39] [43] [44] [57] [58] [59] to solving the classical problem of reconciling gravitational and electrodynamic charges within the Mach-Einstein ether paradigm. First, we will revisit the classical Mach-Einstein type relativistic electrodynamics of a moving charged point particle, and second, we study the resulting electrodynamic theories associated with our vacuum potential field dynamical equations (31) and (32), making use of the fundamental Lagrangian and Hamiltonian formalisms which were specially devised in [50] [51].

### 1.2. Classical Maxwell equations and their electromagnetic potentials form revisiting

As the classical Lorentz force expression with respect to an arbitrary inertial reference frame is related with many theoretical and experimental controversies, such as the relativistic potential energy impact into the charged point particle mass, the Aharonov-Bohm effect [60] and the Abraham-Lorentz-Dirac radiation force [2] [5] [6] expression, the analysis of its structure subject to the assumed vacuum field medium structure is a very interesting and important problem, which was discussed by many physicists including E. Fermi, G. Schott, R. Feynman, F. Dyson [1] [45] [46] [61] [62] [63] and many others. To describe the essence of the electrodynamic problems related with the description of a charged point particle dynamics under external electromagnetic field, let us begin with analyzing the classical Lorentz force expression

$$dp/dt = F_L := \xi E + \xi u \times B, \tag{1}$$

where $\xi \in \mathbb{R}$ is a particle electric charge, $u \in T(\mathbb{R}^3)$ is its velocity [47] [64] vector, expressed here in the light speed $c$ units,

$$E := -\partial A/\partial t - \nabla \varphi \tag{2}$$

is the corresponding external electric field and

$$B := \nabla \times A \tag{3}$$

is the corresponding external magnetic field, acting on the charged particle, expressed in terms of suitable vector $A : M^4 \to \mathbb{E}^3$ and scalar $\varphi : M^4 \to \mathbb{R}$ potentials. Here, as before, the sign "$\nabla$" is the standard gradient operator with respect to the spatial variable $r \in \mathbb{E}^3$, "$\times$" is the usual vector product in three-dimensional Euclidean vector space $\mathbb{E}^3 := (\mathbb{R}^3, <\cdot,\cdot>)$, which is naturally endowed with the classical scalar product $<\cdot,\cdot>$. These potentials are defined on the Minkowski space $M^4 \simeq \mathbb{R}\times\mathbb{E}^3$, which models a chosen laboratory reference frame $\mathrm{K}_t$. Now, it is a well known fact [1] [5] [37] [65] that the force expression (1) does not take into account the dual influence of the charged particle on the electromagnetic field and should be considered valid only if the particle charge $\xi \to 0$. This also means that expression (1) cannot be used for studying the interaction between two different moving charged point particles, as was pedagogically demonstrated in classical manuals [1] [5]. As the classical Lorentz force expression (1) is a natural consequence of the interaction of a charged point particle with an ambient electromagnetic field, its corresponding derivation based on the general principles of dynamics, was deeply analyzed by R. Feynman and F. Dyson [1] [45] [46].

Taking this into account, it is natural to reanalyze this problem from the classical, taking only into account the Maxwell-Faraday wave theory aspect, specifying the corresponding vacuum field medium. Other questionable inferences from the classical electrodynamics theory, which strongly motivated the analysis in this work, are related both with an alternative interpretation of the well-known *Lorenz condition*, imposed on the four-vector of electromagnetic observable potentials $(\varphi, A): M^4 \rightarrow T^*(M^4)$ and the classical Lagrangian formulation [5] of charged particle dynamics under external electromagnetic field. The Lagrangian approach latter is strongly dependent on an important Einstein notion of the proper reference frame $\mathrm{K}_\tau$ and the related least action principle, so before explaining it in more detail, we first to analyze the classical Maxwell electromagnetic theory from a strictly dynamical point of view.

Let us consider with respect to a laboratory reference frame $\mathrm{K}_t$ the additional *Lorenz condition*

$$\partial\varphi / \partial t + <\nabla, A> = 0, \tag{4}$$

*a priori* assumed the Lorentz invariant wave scalar field equation

$$\partial^2\varphi / \partial t^2 - \nabla^2\varphi = \rho \tag{5}$$

and the charge continuity equation

$$\partial\rho / \partial t + <\nabla, J> = 0, \tag{6}$$

where $\rho: M^4 \rightarrow \mathrm{R}$ and $J: M^4 \rightarrow \mathrm{E}^3$ are, respectively, the charge and current densities of the ambient matter. Then one can derive [18] [51] that the Lorentz invariant wave equation

$$\partial^2 A / \partial t^2 - \nabla^2 A = J \tag{7}$$

and the classical electromagnetic Maxwell field equations [1] [2] [5] [65] [66]

$$\begin{aligned} \nabla \times E + \partial B / \partial t &= 0, <\nabla, E> = \rho, \\ \nabla \times B - \partial E / \partial t &= J, \quad <\nabla, B> = 0, \end{aligned} \tag{8}$$

hold for all $(t, r) \in M^4$ with respect to the chosen laboratory reference frame $\mathrm{K}_t$. As it was shown by O.D. Jefimenko [3] [4], the corresponding solutions to (8) for the electric $E: M^4 \rightarrow \mathrm{E}^3$ and magnetic $B: M^4 \rightarrow \mathrm{E}^3$ fields can be represented (in the light speed $c = 1$ units) by means of the following causally independent to each other field expressions

$$E(t,r) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\left[\left(\frac{\rho(t_r,r')}{|r-r'|^3}+\frac{1}{|r-r'|^2}\frac{\partial\rho(t_r,r')}{\partial t}\right)(r-r')-\right.$$

$$\left.-\frac{1}{|r-r'|^2}\frac{\partial J(t_r,r')}{\partial t}\right]d^3r', \tag{9}$$

$$B(t,r) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\left[\frac{J(t_r,r')}{|r-r'|^3}+\frac{1}{|r-r'|^2}\frac{\partial J(t_r,r')}{\partial t}\right]\times(r-r')d^3r',$$

where $(t,r)\in M^4$ and $t_r = t-|r-r'|$ is the retarded time. The result (9) was based on direct derivation from the classical Lienard-Wiechert potentials [2] [3] solving the field equations (5) and (7), causally depending on the corresponding charge and current distributions. Based strongly on this fact in [3] [4] there was argued from physical point of view that related with equations (5) and (7) electric and magnetic potentials really constitute some suitably interpreted physical entities, in contrast to the usual statements [1] [2] [5] about their pure mathematical origin.

It is worth to notice here that, inversely, Maxwell's equations (8) do not directly reduce, via definitions (2) and (3), to the wave field equations (5) and (7) without the Lorenz condition (4). This fact and reasonings presented above are very important: they suggest that, when it comes to choose main governing equations, it proves to be natural replacing the Maxwell's equations (8) with the electric potential field equation (5), the Lorenz condition (4) and the charge continuity equation (6). To make the equivalence statement, claimed above, more transparent we formulate it as the following proposition.

**Proposition 1.** *The Lorentz invariant wave equation (5) together with the Lorenz condition (4) for the observable potentials* $(\varphi,A):M^4\to T^*(M^4)$ *and the charge continuity relationship (6) are completely equivalent to the Maxwell field equations (8).*

*Proof.* Substituting (4), into (5), one easily obtains

$$\partial^2\varphi/\partial t^2 = -<\nabla,\partial A/\partial t>=<\nabla,\nabla\varphi>+\rho, \tag{10}$$

which implies the gradient expression

$$<\nabla,-\partial A/\partial t-\nabla\varphi>=\rho. \tag{11}$$

Taking into account the electric field definition (2), expression (11) reduces to

$$<\nabla,E>=\rho, \tag{12}$$

which is the second of the first pair of Maxwell's equations (8).

Now upon applying $\nabla\times$ to definition (2), we find, owing to definition (3), that

$$\nabla\times E+\partial B/\partial t=0, \tag{13}$$

which is the first pair of the Maxwell equations (8). Having differentiated with respect to the temporal variable $t\in\mathbb{R}$ the equation (5) and taken into account the charge continuity equation (6), one finds that

$$<\nabla,\partial^2 A/\partial t^2-\nabla^2 A-J>=0. \tag{14}$$

The latter is equivalent to the wave equation (7) if to observe that the current vector $J:M^4\to \mathrm{E}^3$ is defined by means of the charge continuity equation (6) up to a vector function $\nabla\times S:M^4\to \mathrm{E}^3$. Now applying operation $\nabla\times$ to the definition (3), owing to the wave equation (7) one obtains

$$\begin{aligned}\nabla\times B &= \nabla\times(\nabla\times A)=\nabla<\nabla,A>-\nabla^2 A=\\ &=-\nabla(\partial\varphi/\partial t)-\partial^2 A/\partial t^2+(\partial^2 A/\partial t^2-\nabla^2 A)=\\ &=\frac{\partial}{\partial t}(-\nabla\varphi-\partial A/\partial t)+J=\partial E/\partial t+J,\end{aligned} \tag{15}$$

leading directly to

$$\nabla\times B=\partial E/\partial t+J,$$

which is the first of the second pair of the Maxwell equations (8). The final *"no magnetic charge*" equation

$$<\nabla,B>=<\nabla,\nabla\times A>=0,$$

in (8) follows directly from the elementary identity $<\nabla,\nabla\times>=0,$ thereby completing the proof.

This proposition allows us to consider the observable potential functions $(\varphi,A):M^4\to T^*(M^4)$ as fundamental ingredients of the ambient *vacuum field medium*, by means of which we can try to describe the related physical behavior of charged point particles imbedded in space-time $M^4$. As there was still written by J.K. Maxwell [67]: "The conception of such a quantity, on the changes of which, and not on its absolute magnitude, the induction currents depends, occurred to Faraday at an early stage of his researches. He observed that the secondary circuit, when at rest in an electromagnetic field which remains of constant intensity, does not show any electrical effect, whereas, if the same state of the field had been suddenly produced, there would have been a current. Again, if the primary circuit is removed from the field, or the magnetic forces abolished, there is a current of the opposite kind. He therefore recognized in the secondary circuit, when in the electromagnetic field, a " peculiar electrical condition of matter' to which he gave the name of Electrotonic State." The following observation provides a strong support of this reasonings within this vacuum field theory approach:

**Observation.** *The Lorenz condition* (4) *actually means that the scalar potential field* $\varphi:M^4\to \mathrm{R}$ *continuity relationship, whose origin lies in some new field conservation law, characterizes the deep intrinsic structure of the vacuum field medium.*

To make this observation more transparent and precise, let us recall the definition [1] [5] [65] [66] of the electric current $J:M^4\to \mathrm{E}^3$ in the dynamical form

$$J:=\rho u, \tag{16}$$

where the vector $u\in T(\mathrm{R}^3)$ is the corresponding charge velocity. Thus, the following continuity relationship

$$\partial\rho/\partial t+<\nabla,\rho u>=0 \tag{17}$$

holds, which can easily be rewritten **Error! Reference source not found.** as the integral conservation law

$$\frac{d}{dt}\int_{\Omega_t}\rho(t,r)d^3r=0 \qquad (18)$$

for the charge inside of any bounded domain $\Omega_t \subset \mathrm{E}^3$, moving in the space-time $M^4$ with respect to the natural evolution equation for the moving charge system

$$dr/dt := u. \qquad (19)$$

Following the above reasoning, we obtain the following result.

**Proposition 2.** *The Lorenz condition (4) is equivalent to the integral conservation law*

$$\frac{d}{dt}\int_{\Omega_t}\varphi(t,r)d^3r=0, \qquad (20)$$

*where* $\Omega_t \subset \mathrm{E}^3$ *is any bounded domain, moving with respect to the charged point particle* $\xi$ *evolution equation*

$$dr/dt = u(t,r), \qquad (21)$$

*which represents the velocity vector of the related local potential field changes propagating in the Minkowski space-time* $M^4$. *Moreover, for a particle with the distributed charge density* $\rho: M^4 \to \mathrm{R}$, *the following Umov type local energy conservation relationship*

$$\frac{d}{dt}\int_{\Omega_t}\frac{\rho(t,r)\varphi(t,r)}{(1-|u(t,r)|^2)^{1/2}}d^3r=0 \qquad (22)$$

*holds for any* $t\in\mathrm{R}$.

*Proof.* Consider first the corresponding solutions to potential field equations (5), taking into account condition (16). Owing to the standard results from [1] [5], one finds that

$$A=\varphi u, \qquad (23)$$

which gives rise to the following form of the Lorenz condition (4):

$$\partial\varphi/\partial t + <\nabla,\varphi u>=0, \qquad (24)$$

This obviously can be rewritten [68] as the integral conservation law *(20)*, so the expression *(20)* is stated.

To state the local energy conservation relationship *(22)* it is necessary to combine the conditions (17), (24) and find that

$$\partial(\rho\varphi)/\partial t + <u,\nabla(\rho\varphi)> + 2\rho\varphi <\nabla,u>=0. \qquad (25)$$

Taking into account that the infinitesimal volume transformation $d^3r=\chi(t,r)d^3r_0$, where the Jacobian $\chi(t,r):=|\partial r(t;r_0)/\partial r_0|$ of the corresponding transformation $r:\Omega_{t_0}\to\Omega_t$, induced by the Cauchy problem for the differential relationship *(21)* for any $t\in\mathrm{R}$, satisfies the evolution equation

$$d\chi/dt = <\nabla,u>\chi, \qquad (26)$$

easily following from *(21)*, and applying to the equality (25) the operator $\int_{\Omega_{t_0}}(\ldots)\chi^2d^3r_0$, one obtains that

$$0=\int_{\Omega_{t_0}}\frac{d}{dt}(\rho\varphi\chi^2)d^3r_0=\frac{d}{dt}\int_{\Omega_{t_0}}(\rho\varphi\chi)Jd^3r_0=$$
$$=\frac{d}{dt}\int_{\Omega_t}(\rho\varphi\chi\ )d^3r:=\frac{d}{dt}\mathrm{E}(\xi;\Omega_t). \tag{27}$$

Here we denoted the conserved charge $\xi:=\int_{\Omega_t}\rho(t,r)d^3r$ and the local energy conservation quantity $\mathrm{E}(\xi;\Omega_t):\ =\int_{\Omega_t}(\rho\varphi\chi\ )d^3r=\mathrm{E}(\xi;\Omega_{t_0}), t\in\mathrm{R}$. The latter quantity can be simplified, owing to the infinitesimal Lorentz invariance four-volume measure relationship $d^3r(t,r_0)\wedge dt=d^3r_0\wedge dt_0$, where variables $(t,r)\in\mathrm{R}_t\times\Omega_t\subset M^4$ are, within the present context, taken with respect to the moving reference frame $\mathrm{K}_t$, related to the infinitesimal charge quantity $d\xi(t,r):=\rho(t,r)d^3r$, and variables $(t_0,r_0)\in\mathrm{R}_{t_0}\times\Omega_{t_0}\subset M^4$ are taken with respect to the laboratory reference frame $\mathrm{K}_{t_0}$, related to the infinitesimal charge quantity $d\xi(t_0,r_0)=\rho(t_0,r_0)d^3r_0$, satisfying the charge conservation invariance $\int_{\Omega_t}d\xi(t,r)=\int_{\Omega_{t_0}}d\xi(t_0,r_0)$. The mentioned above infinitesimal Lorentz invariance relationships make it possible to calculate the local energy conservation quantity $\mathrm{E}(\xi;\Omega_0)$ as

$$\mathrm{E}(\xi;\Omega_t)\ =\ \int_{\Omega_t}(\rho\varphi\chi\ )d^3r=\int_{\Omega_t}(\rho\varphi\frac{d^3r}{d^3r_0}\ )d^3r=$$
$$=\ \int_{\Omega_t}(\rho\varphi\frac{d^3r\wedge dt}{d^3r_0\wedge dt}\ )d^3r=\int_{\Omega_t}(\rho\varphi\frac{d^3r_0\wedge dt_0}{d^3r_0\wedge dt}\ )d^3r= \tag{28}$$
$$=\ \int_{\Omega_t}(\rho\varphi\frac{dt_0}{dt})\ d^3r=\int_{\Omega_t}\frac{\rho\varphi d^3r}{(1-|u|^2)^{1/2}},$$

where we took into account that $dt=dt_0(1-|u|^2)^{1/2}$. Thus, owing to (27) and (28) the local energy conservation relationship *(22)* is satisfied, proving the proposition.

The constructed above local energy conservation quantity (28) can be rewritten as

$$\mathrm{E}(\xi;\Omega_t)=\int_{\Omega_t}\frac{d\xi(t,r)\varphi(t,r)}{(1-|u|^2)^{1/2}}=\int_{\Omega_{t_0}}d\xi(t_0,r_0)\varphi(t_0,r_0):=\int_{\Omega_{t_0}}d\mathrm{E}(\xi;r_0)=\mathrm{E}(\xi;\Omega_{t_0}), \tag{29}$$

where $d\mathrm{E}(t_0,r_0)=d\xi(t_0,r_0)\varphi(t_0,r_0)$ is the distributed in vacuum electromagnetic field energy density, related with the electric charge $d\xi(t_0,r_0)$, located initially at point $(t_0,r_0)\in M^4$.

The above proposition suggests a physically motivated interpretation of electrodynamic phenomena in terms of what should naturally be called *the vacuum potential field*, which determines the observable interactions between charged point particles. More precisely, we can *a priori* endow the ambient vacuum medium with a scalar potential energy field density function $W := \xi\varphi : M^4 \to \mathrm{R}$, where $\xi \in \mathrm{R}_+$ is the value of an elementary charge quantity, and satisfying the governing *vacuum field equations*

$$\partial^2 W / \partial t^2 - \nabla^2 W = \rho\xi, \quad \partial W / \partial t + <\nabla, \mathrm{A}> = 0,$$
$$\partial^2 \mathrm{A} / \partial t^2 - \nabla^2 \mathrm{A} = \xi\rho v, \quad \mathrm{A} = Wv, \tag{30}$$

taking into account the external charged sources, which possess a virtual capability for disturbing the vacuum field medium. Moreover, this vacuum potential field function $W : M^4 \to \mathrm{R}$ allows the natural potential energy interpretation, whose origin should be assigned not only to the charged interacting medium, but also to any other medium possessing interaction capabilities, including for instance, material particles, interacting through the gravity.

The latter leads naturally to the next important step, consisting in deriving the equation governing the corresponding potential field $\bar{W} : M^4 \to \mathrm{R}$, assigned to a charged point particle moving in the vacuum field medium with velocity $u \in T(\mathrm{R}^3)$ and located at point $r(t) := R(t) \in \mathrm{E}^3$ at time $t \in \mathrm{R}$. As can be readily shown [18] [19] [50] [69], the corresponding evolution equation governing the related potential field function $\bar{W} : M^4 \to \mathrm{R}$, assigned to a moving in the space $\mathrm{E}^3$ charged particle $\xi$ under the stationary distributed field sources, has the form

$$\frac{d}{dt}(-\bar{W}u) = -\nabla\bar{W}, \tag{31}$$

where $\bar{W} := W(t,r)|_{r \to R(t)}$, $u(t) := dR(t)/dt$ at point particle location $(t, R(t)) \in M^4$.

Similarly, if there are two interacting charged point particles, located at points $r(t) = R(t)$ and $r_f(t) = R_f(t) \in \mathrm{E}^3$ at time $t \in \mathrm{R}$ and moving, respectively, with velocities $u := dR(t)/dt$ and $u_f := dR_f(t)/dt$, the corresponding potential field function $\bar{W}' : M^4 \to \mathrm{R}$, considered with respect to the reference frame $\mathrm{K}'_{t'}$ specified by Euclidean coordinates $(t', r - r_f) \in \mathrm{E}^4$ and moving with the velocity $u_f \in T(\mathrm{R}^3)$ subject to the laboratory reference frame $\mathrm{K}_t$, should satisfy [18] [19] with respect to the reference frame $\mathrm{K}'_{t'}$ the dynamical equality

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla\bar{W}', \tag{32}$$

where, by definition, we have denoted the velocity vectors $u' := dr/dt', u'_f := dr_f/dt' \in T(\mathrm{R}^3)$. The latter comes with respect to the laboratory reference frame $\mathrm{K}_t$ about the dynamical equality

$$\frac{d}{dt}[-\bar{W}\ (u - u_f)] = -\nabla\bar{W}(1 - |u_f|^2). \tag{33}$$

The dynamical potential field equations (31) and (32) appear to have important properties and can be used as means for representing classical electrodynamic phenomena.

Consequently, we shall proceed to investigate their physical properties in more detail and compare them with classical results for Lorentz type forces arising in the electrodynamics of moving charged point particles in an external electromagnetic field.

### 1.2.1. Classical relativistic electrodynamics revisited

The classical relativistic electrodynamics of a freely moving charged point particle in the Minkowski space-time $M^4$ ; $\mathrm{R}\times\mathrm{E}^3$ is based on the Lagrangian approach [1] [5] [65] [66] [70] with Lagrangian function

$$\mathrm{L}_0 := -m_0(1-|u|^2)^{1/2}, \tag{34}$$

where $m_0 \in \mathrm{R}_+$ is the so-called particle rest mass parameter with respect to the so called proper reference frame $\mathrm{K}_\tau$, parameterized by means of the Euclidean space-time parameters $(\tau, r) \in \mathrm{E}^4$, and $u \in T(\mathrm{R}^3)$ is its spatial velocity with respect to a laboratory reference frame $\mathrm{K}_t$, parameterized by means of the Minkowski space-time parameters $(t, r) \in M^4$, expressed here and in the sequel in light speed units (with light speed $c = 1$). The least action principle in the form

$$\delta S = 0, \, S := -m_0 \int_{t_1}^{t_2} (1-|u|^2)^{1/2}\, dt \tag{35}$$

for any fixed temporal interval $[t_1, t_2] \subset \mathrm{R}$ gives rise to the well-known relativistic relationships for the mass of the particle

$$m = m_0(1-|u|^2)^{-1/2}, \tag{36}$$

the momentum of the particle

$$p := mu = m_0 u (1-|u|^2)^{-1/2} \tag{37}$$

and the energy of the particle

$$\mathrm{E}_0 = m = m_0(1-|u|^2)^{-1/2}. \tag{38}$$

It follows from [5] [65], that the origin of the Lagrangian (34) can be extracted from the action

$$S := -m_0 \int_{t_1}^{t_2} (1-|u|^2)^{1/2}\, dt = -m_0 \int_{\tau_1}^{\tau_2} d\tau, \tag{39}$$

on the suitable temporal interval $[\tau_{1,}\tau_2] \subset \mathrm{R}$, where, by definition,

$$d\tau := dt(1-|u|^2)^{1/2} \tag{40}$$

and $\tau \in \mathrm{R}$ is the so-called, proper temporal parameter assigned to a freely moving particle with respect to the proper reference frame $\mathrm{K}_\tau$. The action (39) is rather questionable from the dynamical point of view, since it is physically defined with respect to the proper reference frame $\mathrm{K}_\tau$, giving rise to the constant action $S = -m_0(\tau_2 - \tau_1)$, as the limits of integrations $\tau_1 < \tau_2 \in \mathrm{R}$ were taken to be fixed from the very beginning. Moreover, considering this particle to have charge $\xi \in \mathrm{R}$ and be moving in the Minkowski space-time $M^4$ under action of an electromagnetic field $(\varphi, A) \in T^*(M^4)$, the corresponding classical (relativistic) action functional is chosen (see [1] [5] [47] [51] [65] [66]) as follows:

$$S := \int_{\tau_1}^{\tau_2} [-m_0 d\tau + \xi < A, \dot{r} > d\tau - \xi\varphi(1-|u|^2)^{-1/2} d\tau], \tag{41}$$

with respect to the *proper reference frame*, parameterized by the Euclidean space-time variables $(\tau, r) \in \mathrm{E}^4$, where we have denoted $\dot{r} := dr/d\tau$ in contrast to the definition $u := dr/dt.$ The action (41) can be rewritten with respect to the laboratory reference frame $\mathrm{K}_t$ as

$$S = \int_{t_1}^{t_2} \mathrm{L}(r, dr/dt)dt, \mathrm{L}(r, dr/dt) := -m_0(1-|u|^2)^{1/2} + \xi < A, u > -\xi\varphi, \tag{42}$$

on the suitable temporal interval $t_1, t_2] \subset \mathrm{R},$ which gives rise to the following [1] [5] [65] [66] dynamical expressions

$$P = p + \xi A, \ \ p = mu, m = m_0(1-|u|^2)^{-1/2}, \tag{43}$$

for the particle momentum and

$$\mathrm{E}_0 = (m_0^2 + |P - \xi A|^2)^{1/2} + \xi\varphi \tag{44}$$

for the charged particle $\xi$ energy, where, by definition, $P \in \mathrm{E}^3$ is the common momentum of the particle and the ambient electromagnetic field at a Minkowski space-time point $(t, r) \in M^4.$ The related dynamics of the charged particle $\xi$ follows [1] [5] [65] [66] from the Lagrangian equation

$$dP/dt := \nabla \mathrm{L}(r, dr/dt) = -\nabla(\xi\varphi - \xi < A, u >). \tag{45}$$

The expression (44) for the particle energy $\mathrm{E}_0$ also appears to be open to question, since the potential energy $\xi\varphi,$ entering additively, has no affect on the particle "inertial" mass $m = m_0(1-|u|^2)^{-1/2}.$ This was noticed by L. Brillouin [21], who remarked that the fact that the potential energy has no affect on the particle mass tells us that "... any possibility of existence of a particle mass related with an external potential energy, is completely excluded". Moreover, it is necessary to stress here that the least action principle (42), formulated with respect to the laboratory reference frame $\mathrm{K}_t$ time parameter $t \in \mathrm{R},$ appears logically inadequate, for there is a strong physical inconsistency with other time parameters of the Lorentz equivalent reference frames. This was first mentioned by R. Feynman in [1] in his efforts to physically argue the Lorentz force expression with respect to the proper reference frame $\mathrm{K}_\tau$. This and other special relativity theory and electrodynamics problems stimulated many prominent physicists of the past [1] [21] [65] [71] [72] and present [7] [23] [24] [25] [26] [44] [57] [59] [60] [73] [74] [75] [76] [77] [78] and [79] [80] [81] [11] [82] [69] [83] [84] [85] [86] [87] to try to develop alternative relativity theories based on completely different space-time and matter structure principles. Some of them prove to be closely related with a virtual relationship between electrodynamics and gravity, based on classical works of H. Lorentz, G. Schott, J. Schwinger, R. Feynman [1] [22] [53] [54] [63] [88] and many others on the so called "electrodynamic mass" of elementary particles. Arguing this way of this mass, one can readily come to a certain paradox: the well-known energy-mass relationship for the particle mass suitably determines the energy of its gravitational field. Yet this energy should lead to an increase in the mass of the particle that in turn should lead to increased gravitational field and so on. In the limit, for instance, an electron must have infinite mass and energy, what we do not really observe. There also is another controversial inference from the action expression (42). As one can easily show, owing to (45), the corresponding expression for the Lorentz force

$$dp/dt = F_L := \xi E + \xi u \times B \tag{46}$$

holds, where we have defined here, as before,

$$E := -\partial A/\partial t - \nabla \varphi \tag{47}$$

the corresponding electric field and

$$B := \nabla \times A \tag{48}$$

the related magnetic field, acting on the charged point particle $\xi$. The expression (46), in particular, means that the Lorentz force $F_L$ depends linearly on the particle velocity vector $u \in T(\mathbb{R}^3)$, and so there is a strong dependence on the reference frame with respect to which the charged particle $\xi$ moves. Attempts to reconcile this and some related controversies [21] [1] [89] [11] [69] [13] forced Einstein to devise his special relativity theory and proceed further to creating his general relativity theory trying to explain the gravity by means of geometrization of space-time and matter in the Universe. Here we must mention that the classical Lagrangian function $\mathrm{L}$ in (42) is written in terms of a combination of terms expressed by means of both the Euclidean proper reference frame variables $(\tau, r) \in \mathrm{E}^4$ and arbitrarily chosen Minkowski reference frame variables $(t, r) \in M^4$.

These problems were recently analyzed using a completely different " *no-geometry*" approach [18] [19] [69], where new dynamical equations were derived, which were free of the controversial elements mentioned above. Moreover, this approach avoided the introduction of the well known Lorentz transformations of the space-time reference frames with respect to which the action functional (42) is invariant. From this point of view, there are interesting for discussion conclusions from [90] [91] [92] [93], in which some electrodynamic models, possessing intrinsic Galilean and Poincaré-Lorentz symmetries, were reanalyzed from diverse geometrical points of view. From completely different point of view the related electrodynamics of charged particles was reanalyzed in [3] [4] [8] [14] [15], where all relativistic relationships were successfully inferred from the classical Lienard-Wiechert potentials, solving the corresponding electromagnetic equations. Subject to a possible geometric space-type structure and the related vacuum field background, exerting the decisive influence on the particle dynamics, we need to mention here recent works [79] [85] [13] and the closely related with their ideas the classical articles [94] [95]. Next, we shall revisit the results obtained in [18] [19] from the classical Lagrangian and Hamiltonian formalisms [47] [64] [66] [96] in order to shed new light on the physical underpinnings of the vacuum field theory approach to the study of combined electromagnetic and gravitational effects.

### 1.3. The vacuum field theory electrodynamics equations: Lagrangian analysis

#### 1.3.2. A moving in vacuum point charged particle - an alternative electrodynamic

**model**

In the vacuum field theory approach to combining electromagnetism and the gravity, devised in [18] [19], the main vacuum potential field function $\bar{W}: M^4 \to \mathrm{R},$ related to a charged point particle $\xi$ under the external stationary distributed field sources, satisfies the dynamical equation (30), namely

$$\frac{d}{dt}(-\bar{W}u) = -\nabla \bar{W} \tag{49}$$

in the case when the external charged particles are at rest, where, as above, $u := dr/dt$ is the particle velocity with respect to some reference system.

To analyze the dynamical equation (49) from the Lagrangian point of view, we write the corresponding action functional as

$$S := -\int_{t_1}^{t_2} \bar{W} dt = -\int_{\tau_1}^{\tau_2} \bar{W}(1+|\dot{r}|^2)^{1/2} d\tau, \tag{50}$$

expressed with respect to the proper reference frame $\mathrm{K}_\tau$. Fixing the proper temporal parameters $\tau_1 < \tau_2 \in \mathrm{R},$ one finds from the least action principle $(\delta S = 0)$ that

$$\begin{aligned} p &:= \partial \mathrm{L} / \partial \dot{r} = -\bar{W}\dot{r}(1+|\dot{r}|^2)^{-1/2} = -\bar{W}u, \\ \dot{p} &:= dp/d\tau = \partial \mathrm{L}/\partial r = -\nabla \bar{W}(1+|\dot{r}|^2)^{1/2}, \end{aligned} \tag{51}$$

where, owing to (50), the corresponding Lagrangian function is

$$\mathrm{L} := -\bar{W}(1+|\dot{r}|^2)^{1/2}. \tag{52}$$

Recalling now the definition of the particle mass

$$m := -\bar{W} \tag{53}$$

and the relationships

$$d\tau = dt(1-|u|^2)^{1/2}, \dot{r}d\tau = udt, \tag{54}$$

from (51) we easily obtain exactly the dynamical equation (49). Moreover, one now readily finds that the dynamical mass, defined by means of expression (53), is given as

$$m = m_0(1-|u|^2)^{-1/2},$$

which coincides with the equation (36) of the preceding section. Now one can formulate the following proposition using the above results.

**Proposition 3.** *The alternative freely moving point particle electrodynamic model (49) allows the least action formulation (50) with respect to the "rest" reference frame variables, where the Lagrangian function is given by expression (52). Its electrodynamics is completely equivalent to that of a classical relativistic freely moving point particle, described in Subsection 1.2.1 .*

#### 1.3.3. A moving in vacuum interacting two charge system - an alternative electrodynamic model

We proceed now to the case when our charged point particle $\xi$ moves in the space-time with velocity vector $u \in T(\mathbb{R}^3)$ and interacts with another external charged point particle $\xi_f$, moving with velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to a common reference frame $\mathrm{K}_t$. As was shown in [18] [19], the respectively modified dynamical equation for the vacuum potential field function $\bar{W}' : M^{4,'} \to \mathbb{R}$ subject to the moving reference frame $\mathrm{K}'_{t'}$ is given by equality (32), or

$$\frac{d}{dt'}[-\bar{W}'(u' - u'_f)] = -\nabla \bar{W}', \tag{55}$$

where, as before, the velocity vectors $u' := dr/dt', u'_f := dr_f/dt' \in T(\mathbb{R}^3)$. Since the external charged particle $\xi_f$ moves in the space-time $M^4$, it generates the related magnetic field $B := \nabla \times A$, whose magnetic vector potentials $A : M^4 \to \mathrm{E}^3$ and $A' : M^{4,'} \to \mathrm{E}^3$ are defined, owing to the results of [18] [19] [69], as

$$\xi A := \bar{W} u_f, \; \xi A' := \bar{W}' u'_f, \tag{56}$$

Whence, taking into account that the field potential

$$\bar{W} = \bar{W}'(1 - |u_f|^2)^{-1/2} \tag{57}$$

and the particle momentum $p' = -\bar{W}' u' = -\bar{W} u$, equality (55) becomes equivalent to

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}', \tag{58}$$

if considered with respect to the moving reference frame $\mathrm{K}'_{t'}$, or to the Lorentz type force equality

$$\frac{d}{dt}(p + \xi A) = -\nabla \bar{W}(1 - |u_f|^2), \tag{59}$$

if considered with respect to the laboratory reference frame $\mathrm{K}_t$, owing to the classical Lorentz invariance relationship (57), as the corresponding magnetic vector potential, generated by the external charged point test particle $\xi_f$ with respect to the reference frame $\mathrm{K}'_{t'}$, is identically equal to zero. To imbed the dynamical equation (59) into the classical Lagrangian formalism, we start from the following action functional, which naturally generalizes the functional (50):

$$S := -\int_{\tau_1}^{\tau_2} \bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} d\tau. \tag{60}$$

Here, as before, $\bar{W}'$ is the respectively calculated vacuum field potential $\bar{W}$ subject to the moving reference frame $\mathrm{K}_{t'}$, $\dot{r} = u' dt'/d\tau, \dot{r}_f = u'_f dt'/d\tau$, $d\tau = dt'(1 - |u' - u'_f|^2)^{1/2}$, which take into account the relative velocity of the charged point particle $\xi$ subject to the reference frame $\mathrm{K}_{t'}$, specified by the Euclidean coordinates $(t', r - r_f) \in \mathbb{R}^4$, and moving simultaneously with velocity vector $u_f \in T(\mathbb{R}^3)$ with respect to the laboratory reference frame $\mathrm{K}_t$, specified by the Minkowski coordinates $(t, r) \in M^4$ and related to those of the reference frame $\mathrm{K}_{t'}$ and $\mathrm{K}_\tau$ by means of the following infinitesimal relationships:

$$dt^2 = (dt')^2 + |dr_f|^2, (dt')^2 = d\tau^2 + |dr - dr_f|^2 . \qquad (61)$$

So, it is clear in this case that our charged point particle $\xi$ moves with the velocity vector $u' - u'_f \in T(\mathbb{R}^3)$ with respect to the reference frame $\mathrm{K}_{t'}$ in which the external charged particle $\xi_f$ is at rest. Thereby, we have reduced the problem of deriving the charged point particle $\xi$ dynamical equation to that before solved in Subsection 1.2.1.

Now we can compute the least action variational condition $\delta S = 0,$ taking into account that, owing to (60), the corresponding Lagrangian function with respect to the proper reference frame $\mathrm{K}_\tau$ is given as

$$\mathrm{L} := -\bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}. \qquad (62)$$

As a result of simple calculations, the generalized momentum of the charged particle $\xi$ equals

$$P \;:= \partial \mathrm{L} / \partial \dot{r} = -\bar{W}'(\dot{r} - \dot{r}_f)(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} =$$

$$= -\bar{W}'\dot{r}(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} + \bar{W}'\dot{r}_f(1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} = \qquad (63)$$

$$= m'u' + \xi A' := p' + \xi A' = p + \xi A,$$

where, owing to (57) the vectors $p' := -\bar{W}'u' = -\bar{W}u = p \in \mathbb{E}^3, \;\; A' = \bar{W}'u'_f = \bar{W}u_f = A \in \mathbb{E}^3,$ and giving rise to the dynamical equality

$$\frac{d}{d\tau}(p' + \xi A') = -\nabla \bar{W}'(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} \qquad (64)$$

with respect to the proper reference frame $\mathrm{K}_\tau$. As $dt' = d\tau(1 + |\dot{r} - \dot{r}_f|^2)^{1/2}$ and $(1 + |\dot{r} - \dot{r}_f|^2)^{1/2} = (1 - |u' - u'_f|^2)^{-1/2},$ we obtain from (64) the equality

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}', \qquad (65)$$

exactly coinciding with equality (58) subject to the moving reference frame $\mathrm{K}_{t'}$. Now, making use of expressions (61) and (57), one can rewrite (65) as that with respect to the laboratory reference frame $\mathrm{K}_t$ :

$$\frac{d}{dt'}(p' + \xi A') = -\nabla \bar{W}' \Rightarrow$$

$$\Rightarrow \frac{d}{dt'}\left(\frac{-\bar{W}u'}{(1+|u_f'|^2)^{1/2}} + \frac{\xi \bar{W} u_f'}{(1+|u_f'|^2)^{1/2}}\right) = -\frac{\nabla \bar{W}}{(1+|u_f'|^2)^{1/2}} \Rightarrow \tag{66}$$

$$\Rightarrow \frac{d}{dt'}\left(\frac{-\bar{W}dr}{(1+|u_f'|^2)^{1/2}dt'} + \frac{\xi \bar{W} dr_f /}{(1+|u_f'|^2)^{1/2}dt'}\right) = -\frac{\nabla \bar{W}}{(1+|u_f'|^2)^{1/2}} \Rightarrow$$

$$\Rightarrow \frac{d}{dt}\left(-\bar{W}\frac{dr}{dt} + \xi \bar{W}\frac{dr_f}{dt}\right) = -\nabla \bar{W}(1-|u_f|^2),$$

exactly coinciding with (59):

$$\frac{d}{dt}(p + \xi A) = -\nabla \bar{W}(1-|u_f|^2). \tag{67}$$

**Remark 1.** *The equation (67) allows to infer the following important and physically reasonable phenomenon: if the test charged point particle velocity* $u_f \in T(\mathrm{R}^3)$ *tends to the light velocity* $c=1,$ *the corresponding acceleration force* $F_{ac} := -\nabla \bar{W}(1-|u_f|^2)$ *is vanishing. Thereby, the electromagnetic fields, generated by such rapidly moving charged point particles, have no influence on the dynamics of charged objects if observed with respect to an arbitrarily chosen laboratory reference frame* $\mathrm{K}_t$.

The latter equation (67) can be easily rewritten as

$$\begin{aligned} dp/dt \quad &= \quad -\nabla \bar{W} - \xi dA/dt + \nabla \bar{W} |u_f|^2 = \\ &= \quad \xi(-\xi^{-1}\nabla \bar{W} - \partial A/\partial t) - \xi < u, \nabla > A + \xi \nabla < A, u_f >, \end{aligned} \tag{68}$$

or, using the well-known **Error! Reference source not found.** identity

$$\nabla < a,b > = < a, \nabla > b + < b, \nabla > a + b \times (\nabla \times a) + a \times (\nabla \times b), \tag{69}$$

where $a,b \in \mathrm{E}^3$ are arbitrary vector functions, in the standard Lorentz type form

$$dp/dt = \xi E + \xi u \times B - \nabla < \xi A, u - u_f > . \tag{70}$$

The result (70), being before found and written down with respect to the moving reference frame $\mathrm{K}_t'$ in [18] [19] [69] makes it possible to formulate the next important proposition.

**Proposition 4.** *The alternative classical relativistic electrodynamic model (58) allows the least action formulation based on the action functional (60) with respect to the proper reference* *frame* $\mathrm{K}_\tau$, *where the Lagrangian function is given by expression (62). The resulting Lorentz type force expression equals (70), being modified by the additional force component* $F_c := -\nabla < \xi A, u - u_f >$, *important for explanation* [97] [98] [99] *of the well known Aharonov-Bohm effect.*

### 1.3.4. A moving charged point particle dynamics formulation dual to the classical relativistic invariant alternative electrodynamic model

It is easy to see that the action functional (60) is written utilizing the classical Galilean transformations of reference frames. If we now consider the action functional (50) for a charged point particle moving with respect the reference frame $\mathrm{K}_\tau$, and take into account its interaction with an external magnetic field generated by the vector potential $A: M^4 \rightarrow \mathrm{E}^3$, it can be naturally generalized as

$$S := \int_{t_1}^{t_2} (-\bar{W} dt + \xi < A, dr >) = \int_{\tau_1}^{\tau_2} [-\bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi < A, \dot{r} >] d\tau, \tag{71}$$

where $d\tau = dt(1 - |u|^2)^{1/2}$.

Thus, the corresponding common particle-field momentum takes the form

$$\begin{aligned} P \quad &:= \partial \mathrm{L} / \partial \dot{r} = -\bar{W} \dot{r} (1 + |\dot{r}|^2)^{-1/2} + \xi A = \\ &= mu + \xi A := p + \xi A, \end{aligned} \tag{72}$$

and satisfies

$$\begin{aligned} \dot{P} \quad &:= dP / d\tau = \partial \mathrm{L} / \partial r = -\nabla \bar{W} (1 + |\dot{r}|^2)^{1/2} + \xi \nabla < A, \dot{r} >= \\ &= -\nabla \bar{W} (1 - |u|^2)^{-1/2} + \xi \nabla < A, u > (1 - |u|^2)^{-1/2}, \end{aligned} \tag{73}$$

where

$$\mathrm{L} := -\bar{W}(1 + |\dot{r}|^2)^{1/2} + \xi < A, \dot{r} > \tag{74}$$

is the corresponding Lagrangian function. Since $d\tau = dt(1 - |u|^2)^{1/2}$, one easily finds from (73) that

$$dP / dt = -\nabla \bar{W} + \xi \nabla < A, u > . \tag{75}$$

Upon substituting (72) into (75) and making use of the identity (69), we obtain the classical expression for the Lorentz force $F$, acting on the moving charged point particle $\xi$:

$$dp / dt := F_L = \xi E + \xi u \times B, \tag{76}$$

where, by definition,

$$E := -\xi^{-1} \nabla \bar{W} - \partial A / \partial t \tag{77}$$

is its associated electric field and

$$B := \nabla \times A \tag{78}$$

is the corresponding magnetic field. This result can be summarized as follows.

**Proposition 5.** *The classical relativistic Lorentz force (76) allows the least action formulation (71)* *with respect to the proper reference frame variables, where the Lagrangian function is given by formula (74). Yet its electrodynamics, described by the Lorentz force (76), is not equivalent to the classical relativistic moving point particle electrodynamics, described by means of the Lorentz force (46), as the inertial mass expression* $m = -\bar{W}$ *does not coincide with that of (36).*

Expressions (76) and (70) are equal up to the gradient like term $F_c := -\nabla < \xi A, u - u_f >,$ which reconciles the Lorentz forces acting on a charged moving particle $\xi$ with respect to different reference frames. This fact is important for our vacuum field theory approach since it uses no special geometry and makes it possible to analyze both electromagnetic and gravitational fields simultaneously by employing the new definition of the dynamical mass by means of the Mach-Einstein type expression (53).

### 1.4. The A.M. Ampere's law in electrodynamics - the classical and modified Lorentz force derivations

The classical ingenious Andre-Marie Ampere's analysis of magnetically interacting to each other two electric currents in thin conductors, as is well known, was based [1] [5] [65] [66] on the following experimental fact: the force between two electric currents depends on the distance between conductors, their mutual spatial orientation and the quantitative values of currents. Having additionally accepted the infinitesimal superposition principle of A.M. Ampere had derived a general analytical expression for the force between two infinitesimal elements of currents under regard:

$$df(r, r^{'}) = II^{'} \frac{(r - r^{'})}{|r - r^{'}|^2} \alpha(s, s^{'}; n) dl dl^{'}, \tag{79}$$

where vectors $r, r^{'} \in \mathrm{E}^3$ point at infinitesimal currents $dr = s dl, dr^{'} = s^{'} dl^{'}$ with normalized orientation vectors $s, s^{'} \in \mathrm{E}^3$ of two closed conductors $l$ and $l^{'}$ carrying currents $I \in \mathrm{R}$ and $I^{'} \in \mathrm{R},$ respectively and the unit vector $n := (r - r^{'}) / |r - r^{'}|,$ fixing the spatial orientations of these infinitesimal elements, and the function $\alpha : (\mathrm{S}^2)^2 \times \mathrm{S}^2 \to \mathrm{R}$ being some real-valued smooth mapping. Taking further into account the mutual symmetry between the infinitesimal elements of currents $dl$ and $dl^{'}$, belonging respectively to these two electric conductors, the infinitesimal force (79) was assumed by A.M. Ampere *to satisfy locally the third Newton's law*:

$$df(r, r^{'}) = -df(r^{'}, r) \tag{80}$$

with the mapping

$$\alpha(s, s^{'}; n) = \frac{\mu_0}{4\pi} (3k_1 < s, n >< s^{'}, n > + k_2 < s, s^{'} >),. \tag{81}$$

where $< \cdot, \cdot >$ is the natural scalar product in $\mathrm{E}^3$ and $k_1, k_2 \in \mathrm{R}$ are some still undetermined real and dimensionless parameters. The assumption (80) is evidently looking very restrictive and can be considered as reasonable only subject to a stationary system of conductors under regard, when the mutual action at a distance principle [1] [5] can be applied. Owing to himself J.C. Maxwell [67]: "*... we may draw the conclusions, first, that action and reaction are not always equal and opposite, and second, that apparatus may be constructed to generate any amount of work from its own resources. For let two oppositely electrified bodies $A$ and $B$ travel along the line joining them with equal velocities in the direction $AB$, then if either the potential or the attraction of the bodies at a given time is that due to their position at some former time (as these authors suppose), $B$, the foremost body, will attract $A$ forwards more than $B$ attracts $A$ backwards. Now let $A$ and $B$ be kept asunder by a rigid rod. The combined system, if set in motion in the direction $AB$, will pull in that direction with a force which may either continually augment the velocity, or may be used as an inexhaustible source of energy.*"

Based on the fact that there is no possibility to measure the force between two infinitesimal current elements, A.M. Ampere took into account (80), (81) and calculated the corresponding force exerted by the whole conductor $l'$ on an infinitesimal current element of other conductor under regard:

$$
\begin{aligned}
dF(r) &:= \oint_{l'} df(r,r') = \\
&= \frac{II'\mu_0}{4\pi}\oint_{l'}\frac{(r-r')}{|r-r'|^2}\left(3k_1 < dr,\frac{r-r'}{|r-r'|} >< dr',\frac{r-r'}{|r-r'|} > +k_2\frac{r-r'}{|r-r'|} < dr,dr' >\right) = \\
&= \frac{II'\mu_0}{4\pi}\oint_{l'}\nabla_{r'}\left(\frac{1}{|r-r'|}\right)(3k_1 < dr,r-r' >< dr',r-r' > +k_2 < dr,dr' >),
\end{aligned}
\tag{82}
$$

which can be equivalently transformed as

$$
\begin{aligned}
dF(r) &= \frac{II'\mu_0}{4\pi}\oint_{l'}\nabla_{r'}\left(\frac{1}{|r-r'|}\right)(3k_1 < dr,r-r' >< dr',r-r' > +k_2 < dr,dr' >) = \\
&= \frac{II'\mu_0}{4\pi}\oint_{l'}\nabla_{r'}\left(\frac{1}{|r-r'|}\right)[k_1(3 < dr,r-r' >< dr',r-r' > - \\
&\quad - < dr,dr' >) + (k_1+k_2) < dr,dr' >] = \\
&= -k_1\frac{\mu_0 I}{4\pi} < dr,\nabla\oint_{l'}\left(\frac{I'dr'}{|r-r'|}\right) > -(k_1+k_2) < \nabla,\oint_{l'} < dr,\frac{I'dr'}{|r-r'|} >,
\end{aligned}
\tag{83}
$$

owing to the integral identity

$$\oint_{l'} \nabla_{r'}\left(\frac{1}{|r-r'|}\right)(3<dr,r-r'><dr',r-r'>-<dr,dr'>)=<dr,\nabla>\oint_{l'}\frac{dr'}{|r-r'|}, \tag{84}$$

which can be easily checked by means of integration by parts. If to introduce the vector potential

$$A(r):=\frac{\mu_0 I'}{4\pi}\oint_{l'}\frac{dr'}{|r-r'|}, \tag{85}$$

generated by the conductor $l'$ at point $r\in \mathrm{E}^3$, belonging to the infinitesimal element $dl$ of the conductor $l$, the resulting infinitesimal force (83) gives rise to the following expression:

$$\begin{aligned} dF(r) &= k_1(-I<dr,\nabla)A(r)+I\nabla<dr,A(r)>)-(2k_1+k_2)I\nabla<dr,A(r)>= \\ &= k_1 I dr\times(\nabla\times A(r))-(2k_1+k_2)I\nabla<dr,A(r)>= \\ &= k_1 J(r)d^3r\times B(r)-(2k_1+k_2)\nabla<Jd^3r,A(r)>, \end{aligned} \tag{86}$$

where we have taken into account the standard magnetic field definition

$$B(r):=\nabla\times A(r) \tag{87}$$

and the corresponding current density relationship

$$J(r)d^3r:=Idr. \tag{88}$$

There are, evidently, many different possibilities to choose the dimensionless parameters $k_1,k_2\in \mathrm{R}$. In his analysis A.M. Ampere had chosen the case when $k_1=1, k_2=-2$ and obtained the well known nowadays *magnetic force* expression

$$dF(r)=J(r)d^3r\times B(r), \tag{89}$$

which easily reduces to the *classical Lorentz expression*

$$df_L(r)=\xi u\times B(r) \tag{90}$$

for a force exerted by an external magnetic field on a moving with a velocity $u\in T(\mathrm{R}^3)$ point particle with an electric charge $\xi\in \mathrm{R}$.

If to take *an alternative choice* and put $k_1=1, k_2=-1,$ the expression (86) yields *a modified magnetic Lorentz type force*, exerted by an external magnetic field generated by a moving charged particle with a velocity $u'\in T(\mathrm{R}^3)$ on a point particle, endowed with the electric charge $\xi\in \mathrm{R}$ and moving with a velocity $u\in T(\mathrm{R}^3)$:

$$dF_L(r)=J(r)d^3r\times B(r)-\nabla<J(r)d^3r,A(r)>, \tag{91}$$

which was before occasionally discussed in different works [9] [10] [11] [69] [100] and recently enough strongly obtained and analyzed in detail from the Lagrangian point of view in works [18] [19] [50] [51] in the following equivalent to (70) infinitesimal form:

$$\delta f_L(r) = \xi u \times (\nabla \times \xi \delta A(r)) - \xi \nabla < u - u_f, \delta A(r) >, \tag{92}$$

where $\delta A(r) \in T^*(\mathrm{R}^3)$ denotes the magnetic potential generated by an external charged point particle moving with velocity $u_f \in T(\mathrm{R}^3)$ and exerting the magnetic force $\delta f_L(r)$ on the charged particle located at point $r \in \mathrm{R}^3$ and moving with velocity $u \in T(\mathrm{R}^3)$ with respect to a common reference system $\mathrm{K}_t$. We also need to mention here that the modified Lorentz force expression (91) does not take naturally into account the resulting pure every weak electric force, as the conductors $l$ and $l'$ are considered to be electrically neutral. Simultaneously, we see that the magnetic potential has a physical significance in its own right [6] [9] [11] [50] [69] and has meaning in a way that extends beyond the calculation of force fields.

Really, to obtain the Lorentz type force (91) exerted by the external magnetic field generated by *the whole conductor* $l'$ on an infinitesimal current element $dl$ of the conductor $l$, it is necessary to integrate the expression (92) along this conductor loop $l'$ :

$$\begin{aligned}
dF_L(r) &:= \oint_{l'} \delta f_L(r) = J(r)dr \times (\nabla \times \oint_{l'} \delta A(r)) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+ \nabla \oint_{l'} < u', \xi \delta A(r) >= J(r)dr \times (\nabla \times A(r)) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+ \nabla \oint_{l'} < dr', \xi \delta A(r)/dt >= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+ \nabla \int_{S(l')} < dS(l'), \nabla \times \xi \delta A(r)/dt >= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+ \nabla \oint_{l'} < dS(l'), \xi \delta B(r)/dt >= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \\
&+ \xi \nabla (d\Phi(r)/dt) = J(r)dr \times B(r) - \nabla < J(r)dr, A(r) > - \rho(r) d^3 r \nabla \overline{W} = \\
&= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \rho(r) d^3 r(-\nabla \overline{W} - \partial A(r)/\partial t) = \\
&= J(r)dr \times B(r) - \nabla < J(r)dr, \oint_{l'} \delta A(r) > + \rho(r) d^3 r E(r),
\end{aligned} \tag{93}$$

that is the equality

$$dF(r) = \rho(r) d^3 r E(r) + J(r) d^3 r \times B(r) - \nabla < J(r) d^3 r, A(r) >, \tag{94}$$

where, by definition, the electric field $E(r) := -\nabla \overline{W} - \partial A(r)/\partial t$. Now one can easily derive from (94) the searched for *Lorentz type force* expression (91), if to take into account that the whole electric field $E(r)\,;\ \ 0$ owing to the neutrality of the conductors.

The presented above analysis of the A.M. Ampere's derivation of the magnetic force expression (86), as well as its consequences (91) and (92) make it possible to suppose that the missed modified Lorentz type force expression (91) could also be embedded into the classical relativistic Lagrangian and related Hamiltonian formalisms, giving rise to eventually new aspects and interpretations of many observed during the past centuries looking "strange" experimental phenomena.

### 1.5. The vacuum field theory electrodynamics equations: Hamiltonian analysis

Any Lagrangian theory has an equivalent canonical Hamiltonian representation via the classical Legendre transformation [64] [66] [96] [101] [102]. As we have already formulated our vacuum field theory of a moving charged particle $\xi$ in Lagrangian form, we proceed now to its Hamiltonian analysis making use of the action functionals (50), (62) and (71).

Take, first, the Lagrangian function (52) and the momentum expression (51) for defining the corresponding Hamiltonian function with respect to the moving reference frame $\mathrm{K}_\tau$ :

$$
\begin{aligned}
H \quad &:=< p,\dot{r} > -\mathrm{L} = \\
&= -< p,p > \bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} + \bar{W}(1-|p|^2/\bar{W}^2)^{-1/2} = \\
&= -|p|^2\,\bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} + \bar{W}^2\bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} = \\
&= -(\bar{W}^2-|p|^2)(\bar{W}^2-|p|^2)^{-1/2} = -(\bar{W}^2-|p|^2)^{1/2}.
\end{aligned} \tag{95}
$$

Consequently, it is easy to show [64] [96] [102] [66] that the Hamiltonian function (95) is a conservation law of the dynamical field equation (49), that is for all $\tau, t \in \mathrm{R}$

$$dH/d\tau = dH/dt = 0, \tag{96}$$

which naturally leads to an energy interpretation of $H$. Thus, we can represent the particle energy as

$$\mathrm{E} = (\bar{W}^2-|p|^2)^{1/2}. \tag{97}$$

Accordingly the Hamiltonian equivalent to the vacuum field equation (49) can be written as

$$
\begin{aligned}
\dot{r} \quad &:= dr/d\tau = \partial H/\partial p = p(\bar{W}^2-|p|^2)^{-1/2} \\
\dot{p} \quad &:= dp/d\tau = -\partial H/\partial r = \bar{W}\nabla\bar{W}(\bar{W}^2-|p|^2)^{-1/2},
\end{aligned} \tag{98}
$$

and we have the following result.

**Proposition 6.** *The alternative freely moving point particle electrodynamic model (49) allows the canonical Hamiltonian formulation (98) with respect to the "rest" reference frame variables, where the Hamiltonian function is given by expression (95). Its electrodynamics is completely* *equivalent to the classical relativistic freely moving point particle electrodynamics described in Subsection 1.2.1.*

In the analogous manner, one can now use the Lagrangian (62) to construct the Hamiltonian function for the dynamical field equation (58), describing the motion of charged particle $\xi$ in an external electromagnetic field in the canonical Hamiltonian form:

$$\dot{r} := dr/d\tau = \partial H/\partial P, \;\; \dot{P} := dP/d\tau = -\partial H/\partial r, \tag{99}$$

where

$$\begin{aligned} H \; &:= < P, \dot{r} > - \mathrm{L} = \\ &= < P, \dot{r}_f - P\bar{W}^{\prime,-1}(1 - |P|^2/\bar{W}^{\prime,2})^{-1/2} > + \bar{W}^{\prime}[\bar{W}^{\prime,2}(\bar{W}^{\prime,2} - |P|^2)^{-1}]^{1/2} = \\ &= < P, \dot{r}_f > + |P|^2 (\bar{W}^{\prime,2} - |P|^2)^{-1/2} - \bar{W}^{\prime,2}(\bar{W}^{\prime,2} - |P|^2)^{-1/2} = \\ &= -(\bar{W}^{\prime,2} - |P|^2)(\bar{W}^{\prime,2} - |P|^2)^{-1/2} + < P, \dot{r}_f > = \\ &= -(\bar{W}^{\prime,2} - |P|^2)^{1/2} - \xi < A^{\prime}, P > (\bar{W}^{\prime,2} - |P|^2)^{-1/2} = \\ &= -(\bar{W}^2 - |\xi A|^2 - |P|^2)^{1/2} - \xi < A, P > (\bar{W}^2 - |\xi A|^2 - |P|^2)^{-1/2}, \end{aligned} \tag{100}$$

being written with respect to the laboratory reference frame $\mathrm{K}_t$. Here we took into account that, owing to definitions (56), (57) and (63),

$$\begin{aligned} \xi A^{\prime} \; &:= \bar{W}^{\prime} u_f^{\prime} = \bar{W}^{\prime} dr_f / dt^{\prime} = \xi A = \\ &= \bar{W}^{\prime} \frac{dr_f}{d\tau} \cdot \frac{d\tau}{dt^{\prime}} = \bar{W}^{\prime} \dot{r}_f (1 - |u - u_f|)^{1/2} = \\ &= \bar{W}^{\prime} \dot{r}_f (1 + |\dot{r} - \dot{r}_f|^2)^{-1/2} = \\ &= -\bar{W}^{\prime} \dot{r}_f (\bar{W}^{\prime,2} - |P|^2)^{1/2} \bar{W}^{\prime,-1} = -\dot{r}_f (\bar{W}^{\prime,2} - |P|^2)^{1/2}, \end{aligned} \tag{101}$$

and, in particular,

$$\dot{r}_f = -\xi A (\bar{W}^{\prime,2} - |P|^2)^{-1/2}, \bar{W} = \bar{W}^{\prime} (1 - |u_f|^2)^{-1/2}, \tag{102}$$

where $A: M^4 \to \mathbb{R}^3$ is the related magnetic vector potential generated by the moving external charged particle $\xi_f$. Equations (99) can be rewritten with respect to the laboratory reference frame $\mathrm{K}_t$ in the form

$$dr/dt = u, \, dp/dt = \xi E + \xi u \times B - \xi \nabla < A, u - u_f >, \tag{103}$$

which coincides with the result (70).

Whence, we see that the Hamiltonian function (100) satisfies the energy conservation conditions

$$dH / d\tau = dH / dt^{'} = dH / dt = 0, \tag{104}$$

for all $\tau, t^{'}$ and $t \in \mathrm{R},$ and that the suitable energy expression is

$$\mathrm{E} = (\bar{W}^2 - \xi^2 | A|^2 - | P|^2)^{1/2} + \xi < A, P > (\bar{W}^2 - \xi^2 | A|^2 - | P|^2)^{-1/2}, \tag{105}$$

where the generalized momentum $P = p + \xi A.$ The result (105) differs essentially from that obtained in [5], which makes use of the Einstein's Lagrangian for a moving charged point particle $\xi$ in an external electromagnetic field. Thus, we obtain the following proposition.

**Proposition 7.** *The alternative classical relativistic electrodynamic model (103), which is intrinsically compatible with the classical Maxwell equations (6), allows the Hamiltonian formulation (99) with respect to the proper reference frame variables, where the Hamiltonian function is given by expression (100).*

The inference above is a natural candidate for experimental validation of our theory. It is strongly motivated by the following remark.

**Remark 2.** *It is necessary to mention here that the Lorentz force expression (103) uses the particle momentum* $p = mu,$ *where the dynamical "mass"* $m := -\bar{W}$ *satisfies condition (105). The latter gives rise to the following crucial relationship between the particle energy* $\mathrm{E}_0$ *and its rest mass* $m_0 = -\bar{W}_0$ *(for the velocity* $u = 0$ *at the initial time moment* $t = 0$):

$$\mathrm{E}_0 = m_0 \frac{(1 - |\xi A_0 / m_0|^2)}{(1 - 2|\xi A_0 / m_0|^2)^{1/2}}, \tag{106}$$

*or, equivalently, at the condition* $|\xi A_0 / m_0|^2 < 1/2$

$$m_0 = \mathrm{E}_0 \left( \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4|\xi A_0 / \mathrm{E}_0|^2} + |\xi A_0 / \mathrm{E}_0|^2 \right)^{1/2}, \tag{107}$$

*where* $A_0 := A|_{t=0} \in \mathrm{E}^3,$ *which strongly differs from the classical expression* $m_0 = \mathrm{E}_0 - \xi\varphi_0,$ *following from (44) and is not depending a priori on the external potential energy* $\xi\varphi_0.$ *As the quantity* $|\xi A_0 / \mathrm{E}_0| \to 0,$ *the following asymptotical mass values follow from (107):*

$$m_0 ; \ \mathrm{E}_0 - \frac{|\xi A_0|^4}{2|\mathrm{E}_0|^3 \mathrm{E}_0}, \quad m_0^{(\pm)} ; \ \pm\sqrt{2}|\xi A_0|. \tag{108}$$

*The first mass value* $m_0 ; \ \mathrm{E}_0 - \frac{|\xi A_0|^4}{2|\mathrm{E}_0|^3 \mathrm{E}_0}$ *is physically reasonable from the classic relativistic point of view, giving rise at week enough magnetic potential to the charged particle energy* $\mathrm{E}_0,$ *yet the second mass values* $m_0^{(\pm)} ; \ \pm\sqrt{2}|\xi A_0|$ *still need their physical interpretation, as they may describe both matter and anti-matter states, consisting, at a very huge energy modulus* $|\mathrm{E}_0| \to \infty,$ *of some charged particle excitations of the vacuum. It is also worth of mentioning that the sign of the mass* $m_0$ *coincides with that of the energy* $\mathrm{E}_0$ *if only the* *inequality* $1-|\xi A_0/m_0|^2 \geq 0$ *holds.*

To make this difference more clear, we now analyze the Lorentz force (76) from the Hamiltonian point of view based on the Lagrangian function (74). Thus, we obtain that the corresponding Hamiltonian function

$$\begin{aligned} H \quad &:= <P,\dot{r}> - \mathrm{L} = <P,\dot{r}> + \bar{W}(1+|\dot{r}|^2)^{1/2} - \xi <A,\dot{r}> = \\ &= <P-\xi A,\dot{r}> + \bar{W}(1+|\dot{r}|^2)^{1/2} = \\ &= -<p,p>\bar{W}^{-1}(1-|p|^2/\bar{W}^2)^{-1/2} + \bar{W}(1-|p|^2/\bar{W}^2)^{-1/2} = \\ &= -(\bar{W}^2-|p|^2)(\bar{W}^2-|p|^2)^{-1/2} = -(\bar{W}^2-|p|^2)^{1/2}. \end{aligned} \tag{109}$$

Since $p = P - \xi A,$ expression (109) assumes the final "*no interaction*" [5] [65] [103] [104] form

$$H = -(\bar{W}^2 - |P-\xi A|^2)^{1/2}, \tag{110}$$

which is conserved with respect to the evolution equations (72) and (73), that is

$$dH/d\tau = dH/dt = 0 \tag{111}$$

for all $\tau, t \in \mathbb{R}.$ These equations latter are equivalent to the following Hamiltonian system

$$\begin{aligned} \dot{r} \quad &= \partial H/\partial P = (P-\xi A)(\bar{W}^2 - |P-\xi A|^2)^{-1/2}, \\ \dot{P} \quad &= -\partial H/\partial r = (\bar{W}\nabla\bar{W} - \nabla <\xi A,(P-\xi A)>)(\bar{W}^2 - |P-\xi A|^2)^{-1/2}, \end{aligned} \tag{112}$$

as one can readily check by direct calculations. Actually, the first equation

$$\begin{aligned} \dot{r} \quad &= (P-\xi A)(\bar{W}^2 - |P-\xi A|^2)^{-1/2} = p(\bar{W}^2-|p|^2)^{-1/2} = \\ &= mu(\bar{W}^2-|p|^2)^{-1/2} = -\bar{W}u(\bar{W}^2-|p|^2)^{-1/2} = u(1-|u|^2)^{-1/2}, \end{aligned} \tag{113}$$

holds, owing to the condition $d\tau = dt(1-|u|^2)^{1/2}$ and definitions $p := mu,\ m = -\bar{W},$ postulated from the very beginning. Similarly we obtain that

$$\begin{aligned} \dot{P} \quad &= -\nabla\bar{W}(1-|p|^2/\bar{W}^2)^{-1/2} + \nabla <\xi A,u>(1-|p|^2/\bar{W}^2)^{-1/2} = \\ &= -\nabla\bar{W}(1-|u|^2)^{-1/2} + \nabla <\xi A,u>(1-|u|^2)^{-1/2}, \end{aligned} \tag{114}$$

coincides with equation (75) in the evolution parameter $t \in \mathbb{R}.$ This can be formulated as the next result.

**Proposition 8.** *The dual to the classical relativistic electrodynamic model (76) allows the canonical Hamiltonian formulation (112) with respect to the proper reference frame variables, where the Hamiltonian function is given by expression (110). Moreover, this formulation circumvents the "mass-potential energy" controversy attached to the classical electrodynamic model (42).*

The modified Lorentz force expression (76) and the related rest energy relationship are characterized by the following remark.

**Remark 3.** *If we make use of the modified relativistic Lorentz force expression (76) as an alternative to the classical one of (46), the corresponding charged particle* $\xi$ *energy expression (110) also gives rise to a true physically reasonable energy expression (at the velocity* $u := 0 \in \mathrm{E}^3$ *at the initial time moment* $t = 0$*); namely,* $\mathrm{E}_0 = m_0$ *instead of the physically controversial classical expression* $\mathrm{E}_0 = m_0 + \xi\varphi_0$, *where* $\varphi_0 := \varphi|_{t=0}$, *corresponding to the case (44).*

## 1.6. Conclusions

All of dynamical field equations discussed above are canonical Hamiltonian systems with respect to the corresponding proper reference frames $\mathrm{K}_\tau$, parameterized by suitable time parameters $\tau \in \mathrm{R}$. Upon passing to the basic laboratory reference frame $\mathrm{K}_t$ with the time parameter $t \in \mathrm{R}$, naturally the related Hamiltonian structure is lost, giving rise to a new interpretation of the real particle motion. Namely, one that has an absolute sense only with respect to the proper reference system, and otherwise being completely relative with respect to all other reference frames. As for the Hamiltonian expressions (95), (100) and (110), one observes that they all depend strongly on the vacuum potential energy field function $\bar{W} : M^4 \to \mathrm{R}$, thereby avoiding the mass problem of the classical energy expression pointed out by L. Brillouin **Error! Reference source not found.**. It should be noted that the canonical Dirac quantization procedure can be applied only to the corresponding dynamical field systems considered with respect to their proper reference frames.

**Remark 4.** *Some comments are in order concerning the classical relativity principle. We have obtained our results relying only on the natural notion of the proper reference frame and its suitable Lorentz parametrization with respect to any other moving reference frames. It seems reasonable then that the true state changes of a moving charged* particle $\xi$ *are exactly realized only with respect to its proper reference system. Then the only remaining question would be about the physical justification of the corresponding relationship between time parameters of moving and proper reference frames.*

The relationship between reference frames that we have used through is expressed as

$$d\tau = dt(1 - |u|^2)^{1/2}, \tag{115}$$

where $u := dr/dt \in \mathrm{E}^3$ is the velocity vector with which the proper reference frame $\mathrm{K}_\tau$ moves with respect to another arbitrarily chosen reference frame $\mathrm{K}_t$. Expression (115) implies, in particular, that

$$dt^2 - |dr|^2 = d\tau^2, \tag{116}$$

which is identical to the classical infinitesimal Lorentz invariant. This is not a coincidence, since all our dynamical vacuum field equations were derived in turn [18][19] from the governing equations of the vacuum potential field function $W: M^4 \to \mathbb{R}$ in the form

$$\partial^2 W/\partial t^2 - \nabla^2 W = \xi\rho, \partial W/\partial t + \nabla(vW) = 0, \partial\rho/\partial t + \nabla(v\rho) = 0, \tag{117}$$

which is *a priori* Lorentz invariant. Here $\rho \in \mathbb{R}$ is the charge density and $v := dr/dt$ the associated local velocity of the vacuum field potential evolution. Consequently, the dynamical infinitesimal Lorentz invariant (116) reflects this intrinsic structure of equations (117). If it is rewritten in the following nonstandard Euclidean form:

$$dt^2 = d\tau^2 + |dr|^2 \tag{118}$$

it gives rise to a completely different relationship between the reference frames $K_t$ and $K_\tau$, namely

$$dt = d\tau(1 + |\dot{r}|^2)^{1/2}, \tag{119}$$

where $\dot{r} := dr/d\tau$ is the related particle velocity with respect to the proper reference system. Thus, we observe that all our Lagrangian analysis in this Section is based on the corresponding functional expressions written in these "Euclidean" space-time coordinates and with respect to which the least action principle was applied. So we see that there are two alternatives - the first is to apply the least action principle to the corresponding Lagrangian functions expressed in the Minkowski space-time variables with respect to an arbitrarily chosen reference frame $K_t$, and the second is to apply the least action principle to the corresponding Lagrangian functions expressed in Euclidean space-time variables with respect to the proper reference frame $K_\tau$.

This leads us to a slightly amusing but thought-provoking observation: It follows from our analysis that all of the results of classical special relativity related with the electrodynamics of charged point particles can be obtained (in a one-to-one correspondence) using of our new definitions of the dynamical particle mass and the least action principle with respect to the associated Euclidean space-time variables in the proper reference system.

An additional remark concerning the quantization procedure of the proposed electrodynamics models is in order: If the dynamical vacuum field equations are expressed in canonical Hamiltonian form, as we have done in this paper, only straightforward technical details are required to quantize the equations and obtain the corresponding Schrödinger evolution equations in suitable Hilbert spaces of quantum states. There is another striking implication from our approach: the Einstein equivalence principle [1] [5] [65] [89] is rendered superfluous for our vacuum field theory of electromagnetism and gravity.

Using the canonical Hamiltonian formalism devised here for the alternative charged point particle electrodynamics models, we found it rather easy to treat the Dirac quantization. The results obtained compared favorably with classical quantization, but it must be admitted that we still have not given a compelling physical motivation for our new models. This is something that we plan to revisit in future investigations. Another important aspect of our vacuum field theory no-geometry (geometry-free) approach to combining the electrodynamics with the gravity, is the manner in which it singles out the decisive role of the proper reference frame $K_\tau$. More precisely, all of our electrodynamics models allow both the Lagrangian and Hamiltonian formulations with respect to the proper reference system evolution parameter $\tau \in \mathbb{R}$, which are well suited the to canonical quantization. The physical nature of this fact remains is as yet not quite clear. In fact, as far as we know [5] [65] [75] [76] [89], there is no physically reasonable explanation of this decisive role of the proper reference system, except for that given by R. Feynman who argued in [1] that the relativistic expression for the classical Lorentz force (46) has physical sense only with respect to the proper reference frame variables $(\tau, r) \in \mathrm{R} \times \mathrm{E}^3$. In future research we plan to analyze the quantization scheme in more detail and begin work on formulating a vacuum quantum field theory of infinitely many particle systems.

# 2. The Lorentz type force analysis within the Feynman proper time paradigm and the radiation theory

## 2.1. Introductory setting

The elementary point charged particle, like electron, mass problem was inspiring many physicists [20] from the past as J. J. Thompson, G.G. Stokes, H.A. Lorentz, E. Mach, M. Abraham, P.A. M. Dirac, G.A. Schott and others. Nonetheless, their studies have not given rise to a clear explanation of this phenomenon that stimulated new researchers to tackle it from different approaches based on new ideas stemming both from the classical Maxwell-Lorentz electromagnetic theory, as in [1] [12] [21] [22] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35] [36] [37] [39] [74] [105] [106] [107], and modern quantum field theories of Yang-Mills and Higgs type, as in [40] [41] [43] [108] and others, whose recent and extensive review is done in [44].

In the present work I will mostly concentrate on detail analysis and consequences of the Feynman proper time paradigm [1] [22] [45] [46] subject to deriving the electromagnetic Maxwell equations and the related Lorentz like force expression considered from the vacuum field theory approach, developed in works [49] [50] [51], and further, on its applications to the electromagnetic mass origin problem. Our treatment of this and related problems, based on the least action principle within the Feynman proper time paradigm [1], has allowed to construct the respectively modified Lorentz type equation for a moving in space and radiating energy charged point particle. Our analysis also elucidates, in particular, the computations of the self-interacting electron mass term in [29], where there was proposed a not proper solution to the well known classical Abraham-Lorentz [52] [53] [54] [55] and Dirac [56] electron electromagnetic "4/3-electron mass" problem. As a result of our scrutinized studying the classical electromagnetic mass problem we have stated that it can be satisfactory solved within the classical H. Lorentz and M. Abraham reasonings augmented with the additional electron stability condition, which was not taken before into account yet appeared to be very important for balancing the related electromagnetic field and mechanical electron momenta. The latter, following recent enough works [31] [35], devoted to analyzing the electron charged shell model, can be realized within there suggested *pressure-energy compensation principle,* suitably applied to the ambient electromagnetic energy fluctuations and the own electrostatic Coulomb electron energy.

## 2.2. Feynman proper time paradigm geometric analysis

In this section, we will develop further the vacuum field theory approach within the Feynman proper time paradigm, devised before in [49] [51], to the electromagnetic J.C. Maxwell and H. Lorentz electron theories and show that they should be suitably modified: namely, the basic Lorentz force equations should be generalized following the Landau-Lifschitz least action recipe [5], taking also into account the pure electromagnetic field impact. When applied the devised vacuum field theory approach to the classical electron shell model, the resulting Lorentz force expression appears to satisfactorily explain the electron inertial mass term exactly coinciding with the electron relativistic mass, thus confirming the well known assumption [2] [109] by M. Abraham and H. Lorentz.

As was reported by F. Dyson [45] [46], the original Feynman approach derivation of the electromagnetic Maxwell equations was based on an *a priori* general form of the classical Newton type force, acting on a charged point particle moving in three-dimensional space $\mathrm{R}^3$ endowed with the canonical Poisson brackets on the phase variables, defined on the associated tangent space $T(\mathrm{R}^3)$. As a result of this approach there only the first part of the Maxwell equations were derived, as the second part, owing to F. Dyson [45], is related with the charged matter nature, which appeared to be hidden. Trying to complete this Feynman approach to the derivation of Maxwell's equations more systematically we have observed [49] that the original Feynman's calculations, based on Poisson brackets analysis, were performed on the *tangent space* $T(\mathrm{R}^3)$. which is, subject to the problem posed, not physically proper. The true Poisson brackets can be correctly defined only on the *coadjoint phase space* $T^*(\mathrm{R}^3)$. as seen from the classical Lagrangian equations and the related Legendre transformation [47] [64] [96] [110] from $T(\mathrm{R}^3)$. to $T^*(\mathrm{R}^3)$. Moreover, within this observation, the corresponding dynamical Lorentz type equation for a charged point particle should be written for the particle momentum, not for the particle velocity, whose value is well defined only with respect to the proper relativistic reference frame, associated with the charged point particle owing to the fact that the Maxwell equations are Lorentz invariant.

Thus, from the very beginning, we shall reanalyze the structure of the Lorentz force exerted on a moving charged point particle with a charge $\xi \in \mathrm{R}$ by another point charged particle with a charge $\xi_f \in \mathrm{R}$, making use of the classical Lagrangian approach, and rederive the corresponding electromagnetic Maxwell equations. The latter appears to be strongly related to the charged point mass structure of the electromagnetic origin as was suggested by R. Feynman and F. Dyson.

Consider a charged point particle moving in an electromagnetic field. For its description, it is convenient to introduce a trivial fiber bundle structure $\pi : \mathrm{M} \to \mathrm{R}^3, \mathrm{M} = \mathrm{R}^3 \times G$, with the abelian structure group $G := \mathrm{R} \setminus \{0\}$, equivariantly acting on the canonically symplectic coadjoint space $T^*(\mathrm{M})$ endowed both with the canonical symplectic structure

$$\begin{aligned} \omega^{(2)}(p,y;r,g) \ &:= dpr^*\alpha^{(1)}(r,g) = < dp, \wedge dr > + \\ &+ \ < dy, \wedge g^{-1}dg >_{\mathrm{G}} + < ydg^{-1}, \wedge dg >_{\mathrm{G}} \end{aligned} \qquad (120)$$

for all $(p,y;r,g) \in T^*(\mathrm{M})$, where $\alpha^{(1)}(r,g) := < p, dr > + < y, g^{-1}dg >_{\mathrm{G}} \in T^*(\mathrm{M})$ is the corresponding Liouville form on $\mathrm{M}$, and with a connection one-form $\mathrm{A} : M \to T^*(M) \times \mathrm{G}$ as

$$\mathrm{A}(r,g) := g^{-1} < \xi A(r), dr > g + g^{-1}dg, \tag{121}$$

with $\xi \in \mathrm{G}^*, (r,g) \in \mathrm{R}^3 \times G,$ and $< \cdot, \cdot >$ being the scalar product in $\mathrm{E}^3$. The corresponding curvature 2-form $\Sigma^{(2)} \in \Lambda^2(\mathrm{R}^3) \otimes \mathrm{G}$ is

$$\Sigma^{(2)}(r) := d\mathrm{A}(r,g) + \mathrm{A}(r,g) \wedge \mathrm{A}(r,g) = \xi \sum_{i,j=1}^{3} F_{ij}(r) dr^i \wedge dr^j, \tag{122}$$

where

$$F_{ij}(r) := \frac{\partial A_j}{\partial r^i} - \frac{\partial A_i}{\partial r^j} \tag{123}$$

for $i,j = \overline{1,3}$ is the electromagnetic tensor with respect to the reference frame $\mathrm{K}_t$, characterized by the phase space coordinates $(r,p) \in T^*(\mathrm{R}^3)$. As an element $\xi \in \mathrm{G}^*$ is still not fixed, it is natural to apply the standard [47] [64] [96] [110] invariant Marsden-Weinstein-Meyer reduction to the orbit factor space $\tilde{P}_\xi := P_\xi / G_\xi$ subject to the related momentum mapping $l : T^*(\mathrm{M}) \to \mathrm{G}^*$, constructed with respect to the canonical symplectic structure (120) on $T^*(\mathrm{M})$, where, by definition, $\xi \in \mathrm{G}^*$ is constant, $P_\xi := l^{-1}(\xi) \subset T^*(\mathrm{M})$ and $G_\xi = \{ g \in G : Ad_G^* \xi \}$ is the isotropy group of the element $\xi \in \mathrm{G}^*$.

As a result of the Marsden-Weinstein-Meyer reduction, one finds that $G_\xi \; ; \; G$, the factor-space $\tilde{P}_\xi \; ; \; T^*(\mathrm{R}^3)$ is endowed with a suitably reduced symplectic structure $\bar{\omega}_\xi^{(2)} \in T^*(\tilde{P}_\xi)$ and the corresponding Poisson brackets on the reduced manifold $\tilde{P}_\xi$ are

$$\begin{aligned} \{r^i, r^j\}_\xi &= 0, \{p_j, r^i\}_\xi = \delta_j^i, \\ \{p_i, p_j\}_\xi &= \xi F_{ij}(r) \end{aligned} \tag{124}$$

for $i,j = \overline{1,3}$, considered with respect to the reference frame $\mathrm{K}_t$. Introducing a new momentum variable

$$\tilde{\pi} := p + \xi A(r) \tag{125}$$

on $\tilde{P}_\xi$, it is easy to verify that $\bar{\omega}_\xi^{(2)} \to \tilde{\omega}_\xi^{(2)} :=< d\tilde{\pi}, \wedge dr >$, giving rise to the following *"minimal interaction"* canonical Poisson brackets:

$$\{r^i, r^j\}_{\tilde{\omega}_\xi^{(2)}} = 0, \{\tilde{\pi}_j, r^i\}_{\tilde{\omega}_\xi^{(2)}} = \delta_j^i, \{\tilde{\pi}_i, \tilde{\pi}_j\}_{\tilde{\omega}_\xi^{(2)}} = 0 \tag{126}$$

for $i,j = \overline{1,3}$ with respect to some new reference frame $\tilde{\mathrm{K}}_{t'}$, characterized by the phase space coordinates $(r, \tilde{\pi}) \in \tilde{P}_\xi$ and a new evolution parameter $t' \in \mathrm{R}$ if and only if the Maxwell field compatibility equations

$$\partial F_{ij} / \partial r_k + \partial F_{jk} / \partial r_i + \partial F_{ki} / \partial r_j = 0 \tag{127}$$

are satisfied on $\mathrm{R}^3$ for all $i,j,k = \overline{1,3}$ with the curvature tensor (123).

Now we proceed to a dynamic description of the interaction between two moving charged point particles $\xi$ and $\xi_f$, moving respectively, with the velocities $u := dr/dt$ and $u_f := dr_f / dt$ subject to the reference frame $\mathrm{K}_t$. Unfortunately, there is a fundamental problem in correctly formulating a physically suitable action functional and the related least action condition. There are clearly possibilities such as

$$S_p^{(t)} := \int_{t_1}^{t_2} dt \mathrm{L}_p^{(t)}[r; dr/dt] \tag{128}$$

on a temporal interval $[t_1, t_2] \subset \mathbb{R}$ with respect to the laboratory reference frame $\mathrm{K}_t$,

$$S_p^{(t')} := \int_{t_1'}^{t_2'} dt' \mathrm{L}_p^{(t')}[r; dr/dt'] \tag{129}$$

on a temporal interval $[t_1', t_2'] \subset \mathbb{R}$ with respect to the moving reference frame $\mathrm{K}_{t'}$ and

$$S_p^{(\tau)} := \int_{\tau_1}^{\tau_2} d\tau \mathrm{L}_p^{(\tau)}[r; dr/d\tau] \tag{130}$$

on a temporal interval $[\tau_1, \tau_2] \subset \mathbb{R}$ with respect to the proper time reference frame $\mathrm{K}_\tau$, naturally related to the moving charged point particle $\xi$.

It was first observed by Poincaré and Minkowski [65] that the temporal differential $d\tau$ is not a closed differential one-form, which physically means that a particle can traverse many different paths in space $\mathbb{R}^3$ with respect to the reference frame $\mathrm{K}_t$ during any given proper time interval $d\tau$, *naturally* related to its motion. This fact was stressed [65] [111] [112] [113] [114] by Einstein, Minkowski and Poincaré, and later exhaustively analyzed by R. Feynman, who argued [1] that the dynamical equation of a moving point charged particle is physically sensible only with respect to its proper time reference frame. This is Feynman's proper time reference frame paradigm, which was recently further elaborated and applied both to the electromagnetic Maxwell equations in [23] [24] [74] and to the Lorentz type equation for a moving charged point particle under external electromagnetic field in [47] [49] [50] [51]. As it was there argued from a physical point of view, the least action principle should be applied only to the expression (130) written with respect to the proper time reference frame $\mathrm{K}_\tau$, whose temporal parameter $\tau \in \mathbb{R}$ is independent of an observer and is a closed differential one-form. Consequently, this action functional is also mathematically sensible, which in part reflects the Poincaré's and Minkowski's observation that the infinitesimal quadratic interval

$$d\tau^2 = (dt')^2 - |dr - dr_f|^2, \tag{131}$$

relating the reference frames $\mathrm{K}_{t'}$ and $\mathrm{K}_\tau$, can be invariantly used for the four-dimensional relativistic geometry. The most natural way to contend with this problem is to first consider the quasi-relativistic dynamics of the charged point particle $\xi$ with respect to the moving reference frame $\mathrm{K}_{t'}$ subject to which the charged point particle $\xi_f$ is at rest. Therefore, it possible to write down a suitable action functional (129), up to $O(1/c^4)$, as the light velocity $c \to \infty$, where the quasi-classical Lagrangian function $\mathrm{L}_p^{(t')}[r; dr/dt']$ can be naturally chosen as

$$\mathrm{L}_p^{(t')}[r; dr/dt'] := m'(r)\left|dr/dt' - dr_f/dt'\right|^2/2 - \xi\varphi'(r). \tag{132}$$

where $m'(r) \in \mathbb{R}_+$ is the charged particle $\xi$ inertial mass parameter and $\varphi'(r)$ is the potential function generated by the charged particle $\xi_f$ at a point $r \in \mathbb{R}^3$ with respect to the reference frame $\mathrm{K}_{t'}$. Since the standard temporal relationships between reference frames $\mathrm{K}_t$ and $\mathrm{K}_{t'}$:

$$dt' = dt(1 - |dr_f / dt'|^2)^{1/2}, \tag{133}$$

as well as between the reference frames $\mathrm{K}_{t'}$ and $\mathrm{K}_\tau$:

$$d\tau = dt'(1 - |dr / dt' - dr_f / dt'|^2)^{1/2}, \tag{134}$$

give rise, up to $O(1/c^2)$, as $c \to \infty$, to $dt'$; $dt$ and $d\tau$; $dt'$, respectively, it is easy to verify that the least action condition $\delta S_p^{(t')} = 0$ is equivalent to the dynamical equation

$$d\pi / dt = \nabla \mathrm{L}_p^{(t')}[r; dr/dt] = (\frac{1}{2}|dr/dt - dr_f / dt|^2)\nabla m - \xi \nabla \varphi(r), \tag{135}$$

where we have defined the generalized canonical momentum as

$$\pi := \partial \mathrm{L}_p^{(t')}[r; dr/dt] / \partial(dr/dt) = m(dr/dt - dr_f / dt), \tag{136}$$

with the dash signs dropped and denoted by " $\nabla$" the usual gradient operator in $\mathrm{E}^3$. Equating the canonical momentum expression (136) with respect to the reference frame $\mathrm{K}_{t'}$ to that of (125) with respect to the canonical reference frame $\tilde{\mathrm{K}}_{t'}$, and identifying the reference frame $\tilde{\mathrm{K}}_{t'}$ with $\mathrm{K}_{t'}$, one obtains that

$$m(dr/dt - dr_f / dt) = m\,dr/dt - \xi A(r), \tag{137}$$

giving rise to the important inertial particle mass determining expression

$$m = -\xi \varphi(r), \tag{138}$$

which right away follows from the relationship

$$\varphi(r) dr_f / dt = A(r). \tag{139}$$

The latter is well known in the classical electromagnetic theory [2] [5] for potentials $(\varphi, A) \in T^*(M^4)$ satisfying the Lorentz condition

$$\partial \varphi(r) / \partial t + < \nabla, A(r) >= 0, \tag{140}$$

yet the expression (138) looks very nontrivial in relating the " *inertial*" mass of the charged point particle $\xi$ to the electric potential, being both generated by the ambient charged point particles $\xi_f$. As was argued in articles [49] [50], the above mass phenomenon is closely related and from a physical perspective shows its deep relationship to the classical electromagnetic mass problem.

Before further analysis of the completely relativistic the charge $\xi$ motion under consideration, we substitute the mass expression (138) into the quasi-relativistic action functional (129) with the Lagrangian (132). As a result, we obtain two possible action functional expressions, taking into account two main temporal parameters choices:

$$S_p^{(t')} = -\int_{t_1'}^{t_2'} \xi \varphi'(r)(1 + \frac{1}{2}|dr/dt' - dr_f / dt'|^2) dt' \tag{141}$$

on an interval $[t_1', t_2'] \subset \mathrm{R}$, or

$$S_p^{(\tau)} = -\int_{\tau_1}^{\tau_2} \xi \varphi'(r)(1 + \frac{1}{2}|dr/d\tau - dr_f / d\tau|^2) d\tau \tag{142}$$

on an $[\tau_1, \tau_2] \subset \mathrm{R}$ . The direct relativistic transformations of (142) entail that

$$\begin{aligned} S_p^{(\tau)} &= -\int_{\tau_1}^{\tau_2} \xi \varphi^{'}(r)(1+\frac{1}{2}\left|dr/d\tau - dr_f/d\tau\right|^2)d\tau \,; \\ &; \; -\int_{\tau_1}^{\tau_2} \xi \varphi^{'}(r)(1+\left|dr/d\tau - dr_f/d\tau\right|^2)^{1/2} d\tau = \qquad (143) \\ &= -\int_{\tau_1}^{\tau_2} \xi \varphi^{'}(r)(1-|\,dr/dt^{'} - dr_f/dt^{'}\,|)^{-1/2} d\tau = -\int_{t_1^{'}}^{t_2^{'}} \xi \varphi^{'}(r) dt^{'}, \end{aligned}$$

giving rise to the correct, from the physical point of view, relativistic action functional form (129), suitably transformed to the proper time reference frame representation (130) via the Feynman proper time paradigm. Thus, we have shown that the true action functional procedure consists in a physically motivated choice of either the action functional expression form (128) or (129). Then, it is transformed to the proper time action functional representation form (130) within the Feynman paradigm, and the least action principle is applied.

Concerning the above discussed problem of describing the motion of a charged point particle $\xi$ in the electromagnetic field generated by another moving charged point particle $\xi_f$, it must be mentioned that we have chosen the quasi-relativistic functional expression (132) in the form (129) with respect to the moving reference frame $\mathrm{K}_{t^{'}}$, because its form is physically reasonable and acceptable, since the charged point particle $\xi_f$ is then at rest, generating no magnetic field.

Based on the above relativistic action functional expression

$$S_p^{(\tau)} := -\int_{\tau_1}^{\tau_2} \xi \varphi^{'}(r)(1+\left|dr/d\tau - dr_f/d\tau\right|^2)^{1/2} d\tau \qquad (144)$$

written with respect to the proper reference from $\mathrm{K}_\tau$, one finds the following evolution equation:

$$d\pi_p/d\tau = -\xi \nabla \varphi^{'}(r)(1+\left|dr/d\tau - dr_f/d\tau\right|^2)^{1/2}, \qquad (145)$$

where the generalized momentum is given exactly by the relationship (136):

$$\pi_p = m(dr/dt - dr_f/dt). \qquad (146)$$

Making use of the relativistic transformation (133) and the next one (134), the equation (145) is easily transformed to

$$\frac{d}{dt}(p+\xi A) = -\nabla \varphi(r)(1-\left|u_f\right|^2), \qquad (147)$$

where we took into account the related definitions: (138) for the charged particle $\xi$ mass, (139) for the magnetic vector potential and $\varphi(r) = \varphi^{'}(r)/(1-\left|u_f\right|^2)^{1/2}$ for the scalar electric potential with respect to the laboratory reference frame $\mathrm{K}_t$. Equation (147) can be further transformed, using elementary vector algebra, to the classical Lorentz type form:

$$dp/dt = \xi E + \xi u \times B - \xi \nabla < u - u_f, A >, \qquad (148)$$

where

$$E := -\partial A/\partial t - \nabla \varphi \qquad (149)$$

is the related electric field and

$$B := \nabla \times A \qquad (150)$$

is the related magnetic field, exerted by the moving charged point particle $\xi_f$ on the charged point particle $\xi$ with respect to the laboratory reference frame $\mathrm{K}_t$. The Lorentz type force equation (148) was obtained in [49] [50] in terms of the moving reference frame $\mathrm{K}_{t'}$, and recently reanalyzed in [34] [50]. The obtained results follow in part [16] [17] from Ampère's classical works on constructing the magnetic force between two neutral conductors with stationary currents.

## 3. The self-interaction problem: historical preliminaries

The elementary point charged particle, like electron, mass problem was inspiring many physicists [20] from the past as J. J. Thompson, G.G. Stokes, H.A. Lorentz, E. Mach, M. Abraham, P.A. M. Dirac, G.A. Schott, J. Schwinger and many others. Nonetheless, their studies have not given rise to a clear explanation of this phenomenon that stimulated new researchers to tackle it from different approaches based on new ideas stemming both from the classical Maxwell-Lorentz electromagnetic theory, as in [1] [21] [22] [24] [25] [26] [34] [74] [109], and modern quantum field theories of Yang-Mills and Higgs type, as in [40] [41] [43] [108] and others, whose recent and extensive review is done in [44].

In the present work we mostly concentrate on detailed quantum and classical analysis of the self-interacting shell model charged particle within the Fock many-temporal approach [115] [116] and the Feynman proper time paradigm [1] [22] [45] [46] subject to deriving the electromagnetic Maxwell equations and the related Lorentz like force expression within the vacuum field theory approach, devised in works [24] [49] [50] [51] [74] [117], and further, we elaborate the obtained results to treating the classical H. Lorentz and M. Abraham [12] [27] [28] [29] [30] [31] [32] [33] [35] [36] [37] [39] [52] [53] [54] [107] [118] electromagnetic mass origin problem. For the first time the proper time approach to classical electrodynamics and quantum mechanics was possibly suggested still in 1937 by V. Fock [119], in which, in particular, there was constructed an alternative proper time based Lagrangian description of a point charged particle under external electromagnetic field. A more detailed motivation of using the proper time approach was later presented by R. Feynman in his Lectures [1]. Concerning the alternative and much later investigations of the *a priori* given quantum electromagnetic Maxwell equations in the Fock space one can mention the Gupta-Bleiler [120] [121] [122] and [61] [71] [88] approaches. The first one, as it is well known [71] [121], contradicts a one of the most important field theoretical principles - the positive definiteness of the quantum event probability and is strongly based on making nonphysical use of an indefinite metric on quantum states. The second one is completely non-relativistic and based on the canonical quantization scheme [71] in the case of the Coulomb gauge condition. Inspired by these and related classical results, we have stated that the self-interacting quantum mechanism of the charged particle with its self-generated electromagnetic field consists of two physically different phenomena, whose influence on the structure of the resulting Hamilton interaction operator appeared to be crucial and gave rise to a modified analysis of the related classical shell model charged particle within the Lagrangian formalism. As a result of our scrutinized studying the classical electromagnetic mass problem there was demonstrated that it can be satisfactory solved within the classical H. Lorentz and M. Abraham reasonings augmented with the additional electron stability condition, which was not taken before into account yet appeared to be very important for balancing the related electromagnetic field and mechanical electron momenta. The latter, following the recent enough works [31] [35] [118] devoted to analyzing the electron charged shell model, was realized within there suggested *pressure-energy compensation principle,* suitably applied to the ambient electromagnetic energy fluctuations and the self-generated electrostatic Coulomb electron energy. In the case of a point charged particle the alternative relativistic invariant approach to studying the radiation reaction force was before suggested by Teitelbom [37], which was based on a formal relativistic invariant splitting of the electromagnetic energy-momentum tensor and deriving the related suitably renormalized charged particle equations of motion.

## 4. The charged particle self-interaction quantum origin

Consider a free relativistic quantum fermionic *a priori* massless particle field described [121] [123] by means of the secondly-quantized self-adjoint Dirac-Weil type Hamiltonian

$$H_f = \int_{\mathbb{R}^3} d^3x \psi^+ < c\alpha, \frac{\hbar}{i}\nabla > \psi, \tag{151}$$

where $\alpha \in \mathbb{E}^3 \otimes End \ \ M^4$ denotes the standard Dirac spin matrix vector representation in the Minkowski space $M^4$, $c \in \mathbb{R}_+$ is the light velocity, $<\cdot,\cdot>$ denotes the usual scalar product in the Euclidean space $\mathbb{E}^3$, $\psi : \mathbb{R}^3 \to (End \ \ \Phi)^4$ - a spinor of the quantum annihilation operators, acting in a suitable Fock space $\Phi$ endowed with the usual scalar product $(\cdot,\cdot)$ and $\psi^+ : \mathbb{R}^3 \to (End \ \ \Phi)^4$ - the respectively adjoint co-spinor of creation operators in the Fock space $\Phi$. The following anticommuting [121] [123] operator relationships

$$\begin{aligned} \psi_j(x)\psi_l^+(y) + \psi_l^+(y)\psi_j(x) &= \delta_{jl}\delta(x-y), \\ \psi_j(x)\psi_l(y) + \psi_l(y)\psi_j(x) &= 0, \\ \psi_j^+(x)\psi_l^+(y) + \psi_l^+(y)\psi_j^+(x) &= 0 \end{aligned} \tag{152}$$

hold for any $x, y \in \mathbb{R}^3$ and $j,l \in \overline{1,4}$, being compatible with the related Heisenberg operator dynamics, generated by the fermionic Hamiltonian operator (151):

$$\partial\psi / \partial\overline{t} := \frac{i}{\hbar}[H_f, \psi], \ \partial\psi^+ / \partial\overline{t} := \frac{i}{\hbar}[H_f, \psi^+] \tag{153}$$

with respect to its own laboratory reference frame $\mathrm{K}_{\overline{t}}$, parameterized by the evolution parameter $\overline{t} \in \mathbb{R}$.

It is clear that the Hamiltonian (151) possesses no information of such an important characteristic as the electric charge $\xi \in \mathbb{R}$, which generates the own electromagnetic field interacting both with it and with other ambient charged particles. As it is usually accepted, we will model a free electromagnetic field by its bosonic self-adjoint operator four-potential $(\varphi, A) : \mathbb{R}^3 \to Hom \ \ (\Phi, \Phi^4)$, whose evolution is generated by the self-adjoint Hamiltonian

$$H_b = 2\int_{\mathrm{R}^3} d^3k\,|k|^2\,[< A^+(k), A(k) > - \varphi(k)\varphi^+(k)], \tag{154}$$

acting in the before introduced common Fock space $\Phi$ and represented by means of the field operators expanded into the Fourier integrals

$$\begin{aligned} \varphi(x) \quad : &= \frac{1}{(2\pi)^{3/2}}\int_{\mathrm{R}^3} d^3k\varphi(k)\exp(i < k, x >) + \frac{1}{(2\pi)^{3/2}}\int_{\mathrm{R}^3} d^3k\varphi^+(k)\exp(-i < k, x >), \\ A(x) \quad : &= \frac{1}{(2\pi)^{3/2}}\int_{\mathrm{R}^3} d^3kA(k)\exp(i < k, x >) + \frac{1}{(2\pi)^{3/2}}\int_{\mathrm{R}^3} d^3kA^+(k)\exp(-i < k, x >). \end{aligned} \tag{155}$$

The coefficients of the expansions (155) satisfy the following [115] [116] [121] commutation operator relationships:

$$\begin{aligned} \varphi(k), \varphi^+(s)] \quad &= \quad -\frac{c\hbar}{2|k|}\delta(k-s), \\ \varphi(k), A_j(s)] \quad &= \quad 0, \\ [\varphi(k), \varphi(s)] \quad &= \quad 0 = [\varphi^+(k), \varphi^+(s)], \\ [A_j(k), A_l^+(s)] \quad &= \quad \frac{c\hbar}{2|k|}\delta_{jl}\delta(k-s), \\ [A_j(k), A_l(s)] \quad &= \quad 0 = [A_j^+(k), A_l^+(s)] \end{aligned} \tag{156}$$

for all $k, s \in \mathrm{E}^3$ and $j, l \in \overline{1,3}$, compatible with the related Heisenberg operator dynamics [121] generated by the electromagnetic field Hamiltonian (154):

$$\frac{\partial A}{\partial \tilde{t}} := \frac{i}{\hbar}[H_b, A], \frac{\partial \varphi}{\partial \tilde{t}} := \frac{i}{\hbar}[H_b, \varphi], \tag{157}$$

with respect to its own laboratory reference frame $\mathrm{K}_{\tilde{t}}$, parameterized by the temporal parameter $\tilde{t} \in \mathrm{R}$. In particular, based on the commutation relationships (156), one can check that the electric

$$E := -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial \tilde{t}} \tag{158}$$

and magnetic

$$B := \nabla \times A \tag{159}$$

fields satisfy the operator Maxwell equations in vacuum, and the following weak Lorenz type constraints

$$\begin{aligned} C_0(k)\Phi \quad : &= i[< k, A(k) > - |k|\varphi(k)]\Phi = 0, \\ C_0^+(k)\Phi \quad : &= -i[< k, A^+(k) > - |k|\varphi^+(k)]\Phi = 0 \end{aligned} \tag{160}$$

hold in the Fock space $\Phi$ for all $k \in \mathrm{E}^3$. As the operators $C_0(k) : \Phi \to \Phi$ and $C_0^+(k) : \Phi \to \Phi$

are commuting both to each other for all $k \in \mathrm{E}^3$ and with the Hamiltonian (154), that is

$$\begin{aligned} C_0(k), C_0(l)] &= 0 = [C_0(k), C_0^+(l)], \\ C_0(k), H_b] &= 0 = [C_0^+(k), H_b] \end{aligned} \tag{161}$$

for any $k, l \in \mathrm{E}^3$, the constraints (160) are compatible with the evolution operator equations (157). Moreover, concerning the Hamiltonian operator (154), whose equivalent operator expression is

$$H_b = \frac{1}{2}\int_{\mathrm{R}^3} (|E|^2 + |B|^2), \tag{162}$$

the following proposition holds.

**Proposition 9.** *The Hamiltonian operator (154) on the reduced by means of constraints (160) Fock subspace* $\Phi$ *is Hermitian and non-negative definite.*

*Proof.* Really, if to define the operator

$$B(k) := A(k) - \frac{k}{|k|^2} < k, A(k) >, \tag{163}$$

the Hamiltonian operator (154) can be rewritten equivalently as

$$\begin{aligned} H_b &= 2\int_{\mathrm{R}^3} d^3k\, |k|^2 \{< \frac{k}{|k|} \times B^+(k), \frac{k}{|k|} \times B(k) > + \\ &+ \frac{i}{|k|}\varphi(k) C_0^+(k) + \frac{1}{|k|^2}[{}_0^+(k) - i|k|\varphi^+(k)] C_0(k)\}. \end{aligned} \tag{164}$$

The latter, owing to the weak Lorenz type constraints (160), gives rise to the inequality

$$\begin{aligned} (f, H_b f) &= 2\int_{\mathrm{R}^3} d^3k\, |k|^2 |(< \frac{k}{|k|} \times B(k) f, \frac{k}{|k|} \times B(k) f >) = \\ &= 2\int_{\mathrm{R}^3} d^3k\, \| k \times B(k) f \|^2 \geq 0 \end{aligned} \tag{165}$$

for any vector $f \in \Phi$, proving the proposition.

**Remark 5.** *The Hamiltonian operator expression (154) easily follows* [116] [121] [123] *from the well known relativistic invariant classical Fock-Podolsky electromagnetic Lagrangian*

$$\begin{aligned} \mathrm{L}_b &:= \frac{1}{2}\int_{\mathrm{R}^3} d^3x [< \nabla\varphi + \frac{1}{c}\frac{\partial A}{\partial \tilde{t}}, \nabla\varphi + \frac{1}{c}\frac{\partial A}{\partial \tilde{t}} > - \\ &- < \nabla \times A, \nabla \times A > - (\frac{1}{c}\frac{\partial \varphi}{\partial \tilde{t}} + < \nabla, A >)^2] \end{aligned} \tag{166}$$

*Based on the corresponding to (166) Euler-Lagrange equations one finds that*

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial \tilde{t}^2} - \Delta A = 0, \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial \tilde{t}^2} - \Delta \varphi = 0, \tag{167}$$

*whose wave solutions allow to determine the electromagnetic fields (158) and (159) and to check that the related Maxwell field equations in vacuum are satisfied if the Lorenz condition*

$$C_0(\bar{t}, x) := \frac{1}{c}\frac{\partial \varphi}{\partial \tilde{t}} + < \nabla, A >= 0 \tag{168}$$

*holds for all* $(\tilde{t}, x) \in M^4$. *Moreover, from the Lagrangian expression (166) one easily obtains by means of the corresponding Legendre transformation* [64] [96] [121] *the Hamiltonian operator*

$$H_b = \frac{1}{2}\int_{\mathrm{R}^3} d^3x(|E|^2 + |B|^2 - C_0^2) + \int_{\mathrm{R}^3} d^3x(< \nabla, A >^2 - < \nabla\varphi, \nabla\varphi >), \tag{169}$$

*being equivalent in the Fock space* $\Phi$*, modulo the solutions (155) of the wave equations (167), to the written above operator expression (154).*

Taking into account the operator equations (157), one easily obtains that

$$\begin{array}{lcl} C_0(k) & = & i[< k, A(k) > - |k| \varphi(k)] \neq 0, \\ C_0^+(k) & = & -i[< k, A^+(k) > - |k| \varphi^+(k)] \neq 0, \end{array} \tag{170}$$

contradicting the imposed above Lorenz constraint *(168)*. As the latter should be vanishing in the Fock space, it was suggested in [115] to reduce the Fock space $\Phi$ to a subspace, on which there are satisfied only the weak Lorenz type operator constraints (160). Concerning these constraints, imposed on the Fock space $\Phi$, it is necessary to mention that a corresponding vacuum vector $|0> \in \Phi$ does not, evidently, annihilate the operators $\varphi(k): \Phi \to \Phi$ and $A^+(k): \Phi \to \Phi^3$, as they do not form computing pairs with operators $C_0^+(k)$ and $C_0(k)$, respectively.

## 5. The transformed Fock space, its Lorenz type reduction and the Quantum Maxwell equations

As we are interested in describing the self-interaction of the fermionic quantum particle field $\psi: \Phi \to \Phi^4$ with the related self-generated bosonic electromagnetic potentials field $(\varphi, A): \Phi \to \Phi^4$, we need, within the Fock many-temporal description approach [115] [116], first to consider the fermionic particle and bosonic electromagnetic fields with respect to the common reference frame $\mathrm{K}_t$ specified by the temporal parameter $t \in \mathrm{R}$. Secondly, we need to make use of the classical "*minimum interaction*" principle [47] [117], (whose sketched backgrounds are presented in Supplement, Section 9. and to apply to the Hamiltonian operator expression (151):

$$H_f^{(int)} = \int_{\mathrm{R}^3} d^3x\psi^+ < c\alpha, \frac{\hbar}{i}\nabla > \psi + \int_{\mathrm{R}^3} d^3x(\xi\psi^+\psi\varphi - \xi\psi^+ < c\alpha, A > \psi), \tag{171}$$

in which the fermionic $\psi: \Phi \to \Phi^4$ and bosonic $(\varphi, A): \Phi \to \Phi^4$ operators are commuting *a priori* to each other as quantum fields of different nature. Since the whole quantum field system consists of the fermionic particle and bosonic self-generated electromagnetic fields, its evolution is described by means of the joint Hamiltonian operator

$$H_{f-b} := H_f^{(int)} + H_b \tag{172}$$

via the Heisenberg equations

$$\frac{\partial \psi}{\partial t} := \frac{i}{\hbar}[H_{f-b}, \psi], \frac{\partial \psi^+}{\partial t} := \frac{i}{\hbar}[H_{f-b}, \psi^+],$$
$$\frac{\partial A}{\partial t} := \frac{i}{\hbar}[H_{f-b}, A], \frac{\partial \varphi}{\partial t} := \frac{i}{\hbar}[H_{f-b}, \varphi] \tag{173}$$

with respect to the common temporal parameter $t \in \mathrm{R}$, as in this case there is assumed that the corresponding temporal parameters $\tilde{t} \in \mathrm{R}$ and $\bar{t} \in \mathrm{R}$ coincide, that is $\tilde{t} = \bar{t} = t \in \mathrm{R}$ and, by definition, the operator spinor $\psi(t,x) := \psi(\bar{t}, \tilde{t})|_{\tilde{t}=\bar{t}=t}$. Simultaneously, there should be, evidently, satisfied the before derived both the electromagnetic field evolution equations (157) with respect to the own reference frame $\mathrm{K}_{\bar{t}}$ and the modified fermionic charged particle field equations

$$\frac{\partial \psi}{\partial \bar{t}} := \frac{i}{\hbar}[H_f^{(int)}, \psi], \frac{\partial \psi^+}{\partial \bar{t}} := \frac{i}{\hbar}[H_f^{(int)}, \psi^+] \tag{174}$$

with respect to the own reference frame $\mathrm{K}_{\bar{t}}$.

Being mostly interested in the evolution of the quantum particle fermionic field $\psi : \Phi \to \Phi,$ we can get rid of the bosonic Hamiltonian impact into (174) having applied to the Fock space $\Phi$ the unitary canonical transformation

$$\Phi \to \tilde{\Phi} := U(t)\Phi, \tag{175}$$

where we denoted by $U(t) : \Phi \to \Phi$ the unitary operator satisfying the determining equation

$$dU(t)/dt = \frac{i}{\hbar} H_b U(t) \tag{176}$$

subject to the bosonic Hamiltonian operator (154) and the temporal parameter $t \in \mathrm{R}$. As a consequence of the transformation (175) we obtain the effective fermionic particle field interaction Hamiltonian operator

$$\tilde{H}_f^{(int)} := U(t) H_f^{(int)} U^*(t) =$$
$$= \int_{\mathrm{R}^3} d^3x \psi^+ < c\alpha, \frac{\hbar}{i}\nabla > \psi + \int_{\mathrm{R}^3} d^3x (\xi \psi^+ \psi \tilde{\varphi} - \xi \psi^+ < c\alpha, \tilde{A} > \psi), \tag{177}$$

where, by definition,

$$\tilde{A} := U(t) A U^*(t), \tilde{\varphi} := U(t) \varphi U^*(t)], \tag{178}$$

subject to which the evolution in the transformed Fock space $\tilde{\Phi},$ induced by the Hamiltonian operator (154)

$$\tilde{H}_b := U(t) H_b U^*(t) = 2\int_{\mathrm{R}^3} d^3k\, |k|^2 \,[< \tilde{A}^+(k), \tilde{A}(l) > - \tilde{\varphi}(k)\tilde{\varphi}^+(k)], \tag{179}$$

became completely eliminated. Concerning the Hamiltonian operator (179) here we need to mention that the related commutation relationships for the operators $(\tilde{\varphi}(k), \tilde{A}(k)) : \tilde{\Phi} \to \tilde{\Phi}^4$

and $(\tilde{\varphi}^{+}(k),\tilde{A}^{+}(k)):\tilde{\Phi}\rightarrow\tilde{\Phi}^{4}$ remain the same as (156), that is

$$\begin{aligned}
\tilde{\varphi}(k),\tilde{\varphi}^{+}(s)] &= -\frac{c\hbar}{2|k|}\delta(k-s),[\tilde{\varphi}(k),\tilde{A}_{j}(s)]=0,\\
\tilde{A}_{j}(k),\tilde{A}_{l}^{+}(s)] &= \frac{c\hbar}{2|k|}\delta_{jl}\delta(k-s),\\
[\tilde{\varphi}(k),\tilde{\varphi}(s)] &= 0=[\tilde{\varphi}^{+}(k),\tilde{\varphi}^{+}(s)],\\
[\tilde{A}_{j}(k),\tilde{A}_{l}(s)] &= 0=[\tilde{A}_{j}^{+}(k),\tilde{A}_{l}^{+}(s)],
\end{aligned} \tag{180}$$

for all $k,s\in\mathrm{E}^{3}$ and $j,l\in\overline{1,3}.$

Now, concerning the Hamiltonian operators (177) and (179), the following Heisenberg evolutions equations

$$\frac{\partial\psi}{\partial\bar{t}}:=\frac{i}{\hbar}[\tilde{H}_{f}^{(int)},\psi],\quad \frac{\partial\psi^{+}}{\partial\bar{t}}:=\frac{i}{\hbar}[\tilde{H}_{f}^{(int)},\psi^{+}] \tag{181}$$

with respect to the own reference frame $\mathrm{K}_{\bar{t}}$ and the Heisenberg evolution equations

$$\frac{\partial\tilde{\varphi}}{\partial\tilde{t}}:=\frac{i}{\hbar}[\tilde{H}_{b},\tilde{\varphi}],\frac{\partial\tilde{A}}{\partial\tilde{t}}:=\frac{i}{\hbar}[\tilde{H}_{b},\tilde{A}] \tag{182}$$

with respect to the own reference frame $\mathrm{K}_{\tilde{t}}$ hold. Being further interested in the evolution equations (173), suitably rewritten in the transformed Fock space $\tilde{\Phi}$ with respect to the common temporal parameter $t\in\mathrm{R},$ we need to take into account [116] that the following functional relationships

$$\psi(t):=\psi(\bar{t},\tilde{t})|_{\bar{t}=\tilde{t}=t},\ \tilde{A}(t):=\tilde{A}(\bar{t},\tilde{t})|_{\bar{t}=\tilde{t}=t} \tag{183}$$

hold. In particular, from (183) the evolution expressions follow

$$\begin{aligned}
\partial\psi(t)/\partial t &= \partial\psi(\bar{t},\tilde{t})/\partial\bar{t}\,|_{\bar{t}=\tilde{t}=t}+\partial\psi(\bar{t},\tilde{t})/\partial\tilde{t}\,|_{\bar{t}=\tilde{t}=t},\\
\partial\tilde{A}(t)/\partial t &= \partial\tilde{A}(\bar{t},\tilde{t})/\partial\bar{t}\,|_{\bar{t}=\tilde{t}=t}+\partial\tilde{A}(\bar{t},\tilde{t})/\partial\tilde{t}\,|_{\bar{t}=\tilde{t}=t},
\end{aligned} \tag{184}$$

for all $t\in\mathrm{R}.$ The latter will be useful when deriving the resulting quantum Maxwell electromagnetic equations.

Before doing this, we need to take into account that the weak operator Lorenz constraints (160), rewritten in the transformed Fock space $\tilde{\Phi},$ is compatible with the evolution equations (182):

$$[\tilde{C}_{0}(k),\tilde{H}_{b}]=0=[\tilde{C}_{0}^{+}(k),\tilde{H}_{b}], \tag{185}$$

yet they fail to be compatible with the evolution equations (181), that is

$$[\tilde{C}_{0}(k),\tilde{H}_{f}^{(int)}]\neq 0\neq[\tilde{C}_{0}^{+}(k),\tilde{H}_{f}^{(int)}].$$

This means that we can not impose on the transformed Fock pace $\tilde{\Phi}$ the constraints

$$\tilde{C}_0(k)\tilde{\Phi} \quad : \quad = i(<k,\tilde{A}(k)> - |k|\tilde{\varphi}(k))\tilde{\Phi} \neq 0,$$

$$\tilde{C}_0^+(k)\tilde{\Phi} \quad : \quad = -i(<k,\tilde{A}^+(k)> - |k|\tilde{\varphi}^+(k))\tilde{\Phi} \neq 0 \tag{186}$$

invariantly for all $k \in \mathrm{E}^3$. Notwithstanding, it is easy enough to check that the following slightly perturbed operators

$$\tilde{C}(k) \quad : \quad = \tilde{C}_0(k) + \frac{i\xi \exp(-ic|k|\bar{t})}{2|k|(2\pi)^{3/2}} \int_{\mathrm{R}^3} \exp(-i<k,y>)\psi^+(y)\psi(y)d^3y,$$

$$\tilde{C}^+(k) \quad : \quad = \tilde{C}_0^+(k) - \frac{i\xi \exp(ic|k|\bar{t})}{2|k|(2\pi)^{3/2}} \int_{\mathrm{R}^3} \exp(i<k,y>)\psi^+(y)\psi(y)d^3y, \tag{187}$$

are commuting both to each other and with the Hamiltonian operators (177) and (179):

$$[\tilde{C}(k),\tilde{C}(s)] \quad = \quad 0 = [\tilde{C}^+(k),\tilde{C}(s)]$$

$$[\tilde{C}(k),\tilde{H}_f^{(int)}] \quad = \quad 0 = [\tilde{C}^+(k),\tilde{H}_f^{(int)}], \tag{188}$$

$$[\tilde{C}(k),\tilde{H}_b] \quad = \quad 0 = [\tilde{C}^+(k),\tilde{H}_b]$$

for all $k,s \in \mathrm{E}^3$. Thus, the related evolution flows (181) and (182) in the transformed Fock space $\tilde{\Phi}$ should be considered under the modified constraints

$$\tilde{C}(k)\tilde{\Phi} = 0 = \tilde{C}^+(k)\tilde{\Phi} \tag{189}$$

for all $k \in \mathrm{E}^3$. Taking into account the exact expressions (187), the constraints (189) can be equivalently rewritten as

$$\tilde{C}(\bar{t};\tilde{t},x)\tilde{\Phi} = 0, \tag{190}$$

where for all $x \in \mathrm{R}$ and the corresponding temporal parameters $\bar{t}$ and $\tilde{t} \in \mathrm{R}$

$$\tilde{C}(\bar{t};\tilde{t},x) := \frac{1}{(2\pi)^{3/2}} \int_{\mathrm{R}^3} d^3k \tilde{C}(k) \exp(i<k,x> - i|k|\tilde{t}) +$$

$$+ \frac{1}{(2\pi)^{3/2}} \int_{\mathrm{R}^3} d^3k \tilde{C}^+(k) \exp(-i<k,x> + i|k|\tilde{t}) = \tag{191}$$

$$= <\nabla,\tilde{A}> + \frac{1}{c}\frac{\partial\tilde{\varphi}}{\partial\tilde{t}} - \frac{\xi}{2\pi} \int_{\mathrm{R}^3} d^3y \Theta(c(\bar{t}-\tilde{t}),|x-y|)\psi^+(y)\psi(y)$$

in which we put, by definition, the relativistic generalized function

$$\Theta(c(t-t),|x-y|) := \frac{\delta(|x-y| + c(\bar{t}-\tilde{t})) - \delta(|x-y| - c(\bar{t}-\tilde{t}))}{2|x-y|}, \tag{192}$$

dual to the well known generalized solution [123] [124]

$$\delta(|x-y|^2 - c^2(\bar{t}-\tilde{t})^2) = \frac{\delta(|x-y| + c(\bar{t}-\tilde{t})) + \delta(|x-y| - c(\bar{t}-\tilde{t}))}{2|x-y|}$$

to the relativistic wave equation.

**Remark 5.** *It is here worthy to mention that the above defined operator* $\tilde{C}(\bar{t}):\tilde{\Phi}\to\tilde{\Phi}$, *depending parameterically on the bosonic temporal parameter* $\bar{t}\in\mathrm{R}$, *satisfies the relativistic wave equation*

$$\frac{1}{c^2}\frac{\partial^2\tilde{C}}{\partial\tilde{t}^2}-\Delta\tilde{C}=0, \qquad (193)$$

*that can be easily checked, if to make use of the rewritten in the Fock space wave equations (167):*

$$\frac{1}{c^2}\frac{\partial^2\tilde{A}}{\partial\tilde{t}^2}-\Delta\tilde{A}=0,\frac{1}{c^2}\frac{\partial^2\tilde{\varphi}}{\partial\tilde{t}^2}-\Delta\tilde{\varphi}=0. \qquad (194)$$

*Moreover, as it can be shown by means of direct calculations, the transformed bosonic Hamiltonian operator (179) on the reduced via the modified Lorenz type constraints (190) Fock space* $\tilde{\Phi}$ *persists to be, as before, non-negative definite.*

Now we can proceed to deriving the quantum Maxwell equations starting from the operator equations *(194)* and the suitably transformed to the Fock space $\tilde{\Phi}$ electromagnetic fields definitions (158) and (159):

$$(\nabla\times\tilde{B}-\frac{1}{c}\frac{\partial\tilde{E}}{\partial\tilde{t}})\tilde{\Phi}=\frac{\xi}{2\pi}\nabla\int_{\mathrm{R}^3}{}^{3}y\Theta(c(\bar{t}-\tilde{t}),|x-y|)\psi^{+}(y)\psi(y)\tilde{\Phi} \qquad (195)$$

and

$$<\nabla,\tilde{E}>\tilde{\Phi}=-\frac{\xi}{2\pi}\frac{\partial}{\partial\tilde{t}}\int_{\mathrm{R}^3}d^3y\Theta(c(\bar{t}-\tilde{t}),|x-y|)\psi^{+}(y)\psi(y)\tilde{\Phi}, \qquad (196)$$

which are considered in the weak operator sense. Taking now into account the relationships (182) and (184), one can obtain strong operator relationships for the electrical and magnetic fields

$$\tilde{E}=-\frac{1}{c}\frac{\partial\tilde{A}}{\partial\tilde{t}}-\nabla\tilde{\varphi}=-\frac{1}{c}\frac{\partial\tilde{A}}{\partial\tilde{t}}-\nabla\tilde{\varphi},\ \tilde{B}=\nabla\times\tilde{A}. \qquad (197)$$

with respect to the common reference frame $\mathrm{K}_t$. Similarly one can easily calculate the weak operator relationship

$$(\frac{1}{c}\frac{\partial\tilde{\varphi}}{\partial t}+<\nabla,\tilde{A}>)\tilde{\Phi}=0, \qquad (198)$$

which holds for the common temporal parameter $t\in\mathrm{R}$. Now we will calculate the weak Maxwell type operator relationships (195) and (196) with respect to the common reference frame $\mathrm{K}_t$ :

$$(\nabla\times\tilde{B}-\frac{1}{c}\frac{\partial\tilde{E}}{\partial\tilde{t}})|_{\bar{t}=\tilde{t}=t}\ \tilde{\Phi}=\frac{\xi}{2\pi}\nabla\int_{\mathrm{R}^3}d^3y\Theta(c(\bar{t}-\tilde{t}),|x-y|)\psi^{+}(y)\psi(y)|_{\bar{t}=\tilde{t}=t}\ \tilde{\Phi}=0 \qquad (199)$$

and

$$<\nabla,\tilde{E}>\tilde{\Phi}=-\frac{\xi}{2\pi}\frac{\partial}{\partial\tilde{t}}\int_{\mathrm{R}^3}d^3y\Theta(c(\bar{t}-\tilde{t}),|x-y|)\psi^{+}(y)\psi(y)|_{\bar{t}=\tilde{t}=t}\ \tilde{\Phi}=\xi\psi^{+}\psi\tilde{\Phi}, \qquad (200)$$

where there was used the known [121] [124] generalized function relationship

$$\frac{1}{c}\frac{\partial}{\partial s}\Theta(cs,|z|)|_{s=0}=-2\pi\delta(z) \tag{201}$$

for all $z\in\mathrm{R}^3$. To calculate further the expression (199), we need to make use of the strong operator relationships (184) and find that

$$\frac{\partial\tilde{E}}{\partial\tilde{t}}|_{\tilde{t}=\bar{t}=t}=\frac{\partial\tilde{E}}{\partial t}-\frac{\partial\tilde{E}}{\partial\bar{t}}|_{\bar{t}=\tilde{t}=t}=\frac{\partial\tilde{E}}{\partial t}-\frac{i}{\hbar}[\tilde{H}_f^{(int)},\tilde{E}]=\frac{\partial\tilde{E}}{\partial t}+\xi\psi^+\alpha\psi. \tag{202}$$

Thus, from (202) and (199) one can obtain that

$$(\nabla\times\tilde{B}-\frac{1}{c}\frac{\partial\tilde{E}}{\partial t})\tilde{\Phi}=\xi\psi^+\alpha\psi\tilde{\Phi} \tag{203}$$

with respect to the common reference frame $\mathrm{K}_t$. The combined together weak operator relationships (200) and (203)

$$(\nabla\times\tilde{B}-\frac{1}{c}\frac{\partial\tilde{E}}{\partial t})\tilde{\Phi}=\xi\psi^+\alpha\psi\tilde{\Phi},<\nabla,\tilde{E}>\tilde{\Phi}=\xi\psi^+\psi\tilde{\Phi} \tag{204}$$

in the reduced by the weak constraint (198) Fock space $\tilde{\Phi}$ jointly with the evident strong operator relationships

$$\nabla\times\tilde{E}+\frac{1}{c}\frac{\partial\tilde{B}}{\partial\tilde{t}}=0,\nabla\times\tilde{B}=0 \tag{205}$$

compile the complete system of quantum Maxwell equations with respect to the common reference frame $\mathrm{K}_t$.

Really, from the Heisenberg evolution equations (181) one easily obtains the strong operator charge conservative flow relationship

$$\frac{\partial}{\partial t}(\xi\psi^+\psi)+<\nabla,\xi\psi^+\alpha\psi>=0, \tag{206}$$

in which the quantity

$$\rho:=\xi\psi^+\psi \tag{207}$$

is interpreted as the operator charge density and the quantity

$$J:=\xi\psi^+c\alpha\psi \tag{208}$$

is naturally interpreted as the operator current density in the space $\mathrm{R}^3$. Whence the weak operator equations (204) can be rewritten, taking into account the definitions (207) and (208), in the standard Maxwell equations weak form:

$$(\nabla\times\tilde{B}-\frac{1}{c}\frac{\partial\tilde{E}}{\partial t})\tilde{\Phi}=\frac{J}{c}\tilde{\Phi},<\nabla,\tilde{E}>\tilde{\Phi}=\rho\tilde{\Phi} \tag{209}$$

under the Fock space $\tilde{\Phi}$ constraint (198). Moreover, based on the weak operator Maxwell equations (209) and the Lorenz constraint (198), one can derive easily the following weak operator linear wave equations

$$(\frac{1}{c^2}\frac{\partial^2\tilde{\varphi}}{\partial t^2}-\Delta\tilde{\varphi})\tilde{\Phi}=\rho\tilde{\Phi},(\frac{1}{c^2}\frac{\partial^2\tilde{A}}{\partial t^2}-\Delta\tilde{A})\tilde{\Phi}=\frac{J}{c}\tilde{\Phi} \tag{210}$$

in respect to the common laboratory reference frame $\mathrm{K}_t$, allowing to calculate the induced by the charged fermionic field causal quantum bosonic potentials $(\tilde{\varphi}_\xi,\tilde{A}_\xi):\tilde{\Phi}\to\tilde{\Phi}^4$ in the analytical form:

$$\tilde{\varphi}_\xi = \frac{1}{4\pi}\int_{\mathrm{R}^3}\frac{\rho(t',y)d^3y}{|x-y|},\ \tilde{A}_\xi = \frac{1}{4\pi c}\int_{\mathrm{R}^3}\frac{J(t',y)d^3y}{|x-y|}, \tag{211}$$

where the "retarded" temporal parameter $t' := t - |x-y|/c \in \mathrm{R},$ making the equations (210) exactly satisfied modulo the solutions to their uniform forms. Moreover, owing to (206), the expressions (211) satisfy exactly the strong operator Lorenz constraint

$$\frac{1}{c}\frac{\partial \tilde{\varphi}_\xi}{\partial t} + < \nabla, \tilde{A}_\xi >= 0 \tag{212}$$

with respect to the laboratory reference frame $\mathrm{K}_t$.

From the analysis of the quantum charged particle fermionic field model, interacting with the self-generated quantum bosonic electromagnetic field, one can infer the following important consequences:

- the physical effective evolution of the fermionic-bosonic system with respect to the common reference frame $\mathrm{K}_t$ is governed by the reduced fermionic Hamiltonian operator (177), acting on the canonically transformed Fock space $\tilde{\Phi},$ reduced by means of the weak Lorenz type operator constraint (198);

- the compatibility of evolutions of the quantum fermionic and bosonic fields with respect to the common temporal reference frame $\mathrm{K}_t$ entails the reciprocal influence of the fermionic field on the bosonic one and vice versa, being clearly demonstrated both by the weak field potentials operator equations (210) and the Lorentz type weak constraint (198) imposed on the Fock space $\tilde{\Phi}$;

- subject to the basic self-interacting fermionic-bosonic system described by the joint Hamiltonian operator (172) in the transformed Fock space $\tilde{\Phi},$ one can claim that the bosonic electromagnetic impact into the quantum charged particle dynamics is decisive, as owing to it the fermionic system can realize its charge interaction property through the physical vacuum deformation, caused by the related deformation of the weak Lorenz type operator constraint (190), and resulting into the weak operator potential equations (211).

The consequences formulated above subject to the quantum fermionic-bosonic self-interacting phenomenon, as it was shown in **Error! Reference source not found.**, appeared to be very important from classical point of view, especially for physical understanding the inertial properties of a charged particle under action of the self-generated electromagnetic field.

## 6. Classical reduction of the quantum charged particle and electromagnetic field evolutions

Let's consider the vector position operator $\hat{x}:\tilde{\Phi}\to\tilde{\Phi}^3$ and its weak in the reduced Fock space $\tilde{\Phi}$ evolution with respect to the complete and suitably renormalized charged particle

Hamiltonian operator (177). Taking into account that the Hamiltonian operator $\tilde{H}_f^{(int)}:\tilde{\Phi}\to\tilde{\Phi}$ can be represented as

$$\tilde{H}_f^{(int)}=\int_{\mathrm{R}^3}d^3x\psi^+ < c\alpha,\hat{p}_x > \psi+\int_{\mathrm{R}^3}d^3x(\xi\psi^+\psi\tilde{\varphi}_\xi-\xi\psi^+ < c\alpha,\tilde{A}_\xi > \psi), \tag{213}$$

within which the operators $(\tilde{\varphi}_\xi,\tilde{A}_\xi):\tilde{\Phi}\to\tilde{\Phi}^3$ are given by the nonlocal integral expressions (211) and $\hat{p}_x:\tilde{\Phi}\to\tilde{\Phi}^3$ is the locally defined charged particle $\xi$ momentum operator $\hat{p}_x:=\frac{\hbar}{i}\nabla_x$, canonically conjugated [71] to the position operator $\hat{x}:\tilde{\Phi}\to\tilde{\Phi}^3$, that is

$$[\hat{p}_y,\hat{x}]=\frac{\hbar}{i}\delta(x-y) \tag{214}$$

for any $x,y\in\mathrm{R}^3$. This also, in particular, means that the position operator $\hat{x}:\tilde{\Phi}\to\tilde{\Phi}^3$ is *a priori* given in the diagonal representation: $\hat{x}\tilde{f}:=x\tilde{f}$ for any vector $\tilde{f}\in\tilde{\Phi}$.

As a result of a simple calculation one finds the expression

$$d\hat{x}/dt=\psi^+c\alpha\psi, \tag{215}$$

which can be used for obtaining the classical charged particle $\xi$ velocity $u(t,x)\in T(\mathrm{R}^3)$ as

$$u(t,x):=(\Omega,d\hat{x}/dt\Omega)=(\Omega,\psi^+c\alpha\psi\Omega), \tag{216}$$

where the vector $\Omega\in\tilde{\Phi}$ is the ground state of the Hamiltonian operator (213) acting in the Lorenz type reduced and suitably renormalized [71] [88] [121] [123] Fock space $\tilde{\Phi}$. Substituting (215) and (207) into the Hamiltonian expression (213) one obtains the expression

$$\tilde{H}_f^{(int)}=\int_{\mathrm{R}^3}d^3x < d\hat{x}/dt,\hat{p}_x > +\int_{\mathrm{R}^3}d^3x(\rho\tilde{\varphi}_\xi-\frac{1}{c}J,\tilde{A}_\xi >), \tag{217}$$

whose classical counterpart looks as

$$\bar{H}_f^{(int)}=\int_{\mathrm{R}^3}d^3x(\rho\tilde{\varphi}_\xi-<\frac{1}{c}J,\tilde{A}_\xi >), \tag{218}$$

within which there was taken into account the previously assumed quantum massless charged particle $\xi$ fermionic field. The expression (218) jointly with the renormalized bosonic field Hamiltonian (162) gives rise to the complete classical Hamiltonian function

$$\bar{H}_{f-b}^{(int)}=\int_{\mathrm{R}^3}d^3x[\frac{1}{2}(|\tilde{E}|^2+|\tilde{B}|^2)+\rho\tilde{\varphi}_\xi-<\frac{1}{c}J,\tilde{A}_\xi >], \tag{219}$$

governing the temporal evolution both of the charged particle $\xi$ and of the electromagnetic fields with respect to the laboratory reference frame $\mathrm{K}_t$. The obtained above Hamiltonian function and its corresponding Lagrangian form *(166)* have been effectively used before in [125] for describing the classical self-interacting charged particle dynamics and its inertial properties.

Being experienced with the analysis of a self-interacting charged quantum particle fermionic field with the self-generated quantum bosonic electromagnetic field, we understand well that the influence of the electromagnetic field on the charged particle should be considered as crucial, strongly modifying the related fermionic Hamiltonian operator, describing the charged particle dynamics. As the simultaneously modified bosonic electromagnetic operator depends, owing to the self-interaction, on the charge and current particle field densities, the joint impact on the charged particle dynamics can be effectively classically modeled by means of its inertial mass parameter. In the quantum operator case the physical charged particle mass parameter $m_{ph} \in \mathrm{R}_+$ can be naturally defined by means of the least quantum renormalized Hamiltonian (172) eigenvalue

$$m_{ph} := c^{-2} \inf_{\tilde{f} \in \tilde{\Phi}, \|\tilde{f}\|=1} (\tilde{f}, \tilde{H}^{(int)}_{f-b} \tilde{f}), \; \tilde{H}^{(int)}_{f-b} := \tilde{H}^{(int)}_{f} + \tilde{H}_b, \tag{220}$$

in the suitably transformed and reduced by means of the operator Lorenz type constraint (198) Fock space $\tilde{\Phi}$ with respect to the common reference frame $\mathrm{K}_t$. As the quantum spectral problem (220) is very complicated, new tools are needed to be developed for its successful analysis.

# 7. Classical self-interacting charged particle dynamics and its inertial properties

Being experienced with the analysis of a self-interacting charged quantum particle fermionic field with the self-generated quantum bosonic electromagnetic field, we understand well that the influence of the electromagnetic field on the charged particle should be considered as crucial, strongly modifying the related fermionic Hamiltonian operator, describing the charged particle dynamics. As the simultaneously modified bosonic electromagnetic operator depends, owing to the self-interaction, on the charge and current particle field densities, the joint impact on the charged particle dynamics can be effectively classically modeled by means of its inertial mass parameter. In the quantum operator case the physical charged particle mass parameter $m_{ph} \in \mathrm{R}_+$ can be naturally defined by means of the least quantum renormalized Hamiltonian (172) eigenvalue (220) in the suitably transformed and reduced by means of the operator Lorenz type constraint (198) Fock space $\tilde{\Phi}$ with respect to the common reference frame $\mathrm{K}_t$. As the quantum spectral problem (220) is very complicated, we will try below to analyze it from the classical point of view.

The quantum operator Hamiltonian approach of Section 4. makes it possible to treat analytically the charged particle self-interaction mechanism, which can be described by means of the following two steps. The first one consists in producing the charged particle dynamics governed by the gauge type component of the charged particle Hamiltonian operator (177), and the second one - consists in modifying this dynamics by means of the self-generated electromagnetic field, whose influence is governed by the bosonic Hamiltonian (179), perturbed by the dependence of the electromagnetic field potentials on the related charge and current densities through the differential relationships (210). This mechanism can be classically realized analytically by means of the alternative and already before mentioned Lagrangian least action formalism, following the well known slightly modified [5] Landau-Lifschitz scheme. Namely, the Lagrangian function for the classical charged particle $\xi$, interacting with the self-generated electromagnetic field, is easily derived from the corresponding Hamiltonian function (219), giving rise to the classical Lagrangian expressions *(166)* in the following slightly extended form:

$$\tilde{\mathrm{L}}_{(f-b)} = \int_{\mathrm{R}^3} d^3x(<\frac{1}{c}J, \tilde{A}> - \rho\tilde{\varphi}) +$$

$$+\frac{1}{2}\int_{\mathrm{R}^3} d^3x(<\nabla\tilde{\varphi}+\frac{1}{c}\frac{\partial\tilde{A}}{\partial t}, \nabla\tilde{\varphi}+\frac{1}{c}\frac{\partial\tilde{A}}{\partial t}> - \tag{221}$$

$$-<\nabla\times\tilde{A}, \nabla\times\tilde{A}>) - <k, dx/dt>,$$

where vector $k := k(t,x) \in \mathrm{E}^3$ models the related radiation reaction momentum, caused by the accelerated charged particle $\xi$ with respect to the laboratory reference frame $\mathrm{K}_t$, as well as there implied that the classical Lorenz type constraint (198) is satisfied *a priori.* Here we need to mention that the first part of the Lagrangian (221) is responsible for the internal gauge type charged particle self-interaction and the second one is responsible for the external charged particle self-interaction induced by the suitably perturbed electromagnetic field, depending on the particle charge and current densities. The physical difference between these two phenomena proves to be especially important for calculation of an effective Lagrangian function for the related dynamical properties of the self-interacting charged particle.

Before proceeding further we need to make an important comment concerning the least action properties of the classical relativistic self-interacting Lagrangian (221). Namely, taking into account a deep quantum vacuum origin [121] of the electromagnetic field and its effective measuring only with respect to the common laboratory reference frame $\mathrm{K}_t$, we can state that the related Maxwell equations should be naturally derived from the following least action principle: the variation $\delta\tilde{S}^{(t)}_{f-b} = 0,$ where by definition, the action functional

$$\tilde{S}^{(t)}_{f-b} := \int_{t_1}^{t_2} \tilde{\mathrm{L}}_{(f-b)} dt \tag{222}$$

is calculated with respect to the laboratory reference frame $\mathrm{K}_t$ on a fixed temporal interval $[t_1, t_2] \subset \mathrm{R}$. Yet, as it is easy to check, the above action functional (222) fails to derive the corresponding Lorentz type dynamical equations for the self-interacting charged particle $\xi$, if to take into account that the related charged particle is considered to be pointwise, located at point $x(t) \in \mathrm{E}^3$ for $t \in \mathrm{R}$ and endowed with the current density vector $J = \rho dx(t)/dt \in \mathrm{E}^3$ and the charge density $\rho := \xi\delta(x - x(t)), x \in \mathrm{E}^3$. This, evidently, means that the action functional (222) should be suitably modified with respect to the [1] [51] Feynman proper time reference frame paradigm, owing to which the action functional for the charged particle dynamics has a physical sense if and only if it is considered with respect to the proper time reference frame $\mathrm{K}_\tau$:

$$\tilde{S}^{(\tau)}_{f-b} := \int_{\tau_1}^{\tau_2} \tilde{\mathrm{L}}_{(f-b)}) \sqrt{(1+|\dot{x}|^2/c^2)} d\tau \tag{223}$$

on a fixed temporal interval $[\tau_1, \tau_2] \subset \mathrm{R}$, where we took into account, that $dt := \sqrt{(1+|\dot{x}|^2/c^2)} d\tau$ and, by definition, the velocity $\dot{x} := dx/d\tau$ with respect to the proper temporal parameter $\tau \in \mathrm{R}$. Then from the least action condition $\delta\tilde{S}^{(\tau)}_{f-b} = 0$ on the fixed temporal interval $[\tau_1, \tau_2] \subset \mathbb{R}$ one easily obtains the well known classical Lorentz dynamical equation

$$\frac{d}{dt}(mu) = \xi \tilde{E} + \xi u \times \tilde{B}, \tag{224}$$

written with respect to the laboratory reference frame $\mathrm{K}_t$. When deriving (224) there was put, by definition, the inertial mass definition $m := -\tilde{\varphi}/c^2$. The reasonings presented above will be in part employed below when analyzing a suitably reduced Lagrangian function (221).

For the self-interacting charged particle to be physically specified by the mentioned above phenomena in detail, we will consider below a so-called shell model particle, whose charge is uniformly distributed on a sphere of a very small yet fixed radius. Then, following the similar calculations from [5], one can obtain from (221) that

$$\begin{aligned}
\tilde{\mathrm{L}}_{(f-b)} &= \frac{1}{2}\int_{\mathbb{R}^3} d^3x (\tilde{\varphi} < \nabla, \tilde{E} > + \frac{1}{c} < \tilde{A}, \frac{\partial \tilde{E}}{\partial t} > - \frac{1}{c} < \tilde{A}, J + \frac{\partial \tilde{A}}{\partial t} >) - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3x < \tilde{A}, \tilde{E} > + \int_{\mathbb{R}^3} d^3x \, (< \frac{1}{c} J, \tilde{A} > - \rho\tilde{\varphi}) - \\
&- \frac{1}{2}\lim_{r\to\infty}\int_{\mathbb{S}_r^2} < \tilde{\varphi}\tilde{E} + \tilde{A}\times\tilde{B}, dS_r^2 > - < k, dx/dt > = \\
&= -\int_{\Omega_-(\xi)} d^3x \, (< \frac{1}{c} J, \tilde{A} > - \rho\tilde{\varphi}) + \int_{\Omega_-(\xi)\cup\Omega_+(\xi)} d^3x (< \frac{1}{c} J, \tilde{A} > - \rho\tilde{\varphi}) - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3x < \tilde{A}, \tilde{E} > - < k, dx/dt > = \\
&= \frac{1}{2}\int_{\Omega_-(\xi)} d^3x \, (< \frac{1}{c} J, \tilde{A} > - \rho\tilde{\varphi}) + \frac{1}{2}\int_{\Omega_-(\xi)\cup\Omega_+(\xi)} d^3x (< \frac{1}{c} J, \tilde{A} > - \rho\tilde{\varphi}) - \\
&- \frac{1}{2c}\frac{d}{dt}\int_{\mathbb{R}^3} d^3x < \tilde{A}, \tilde{E} > - < k, dx/dt >,
\end{aligned} \tag{225}$$

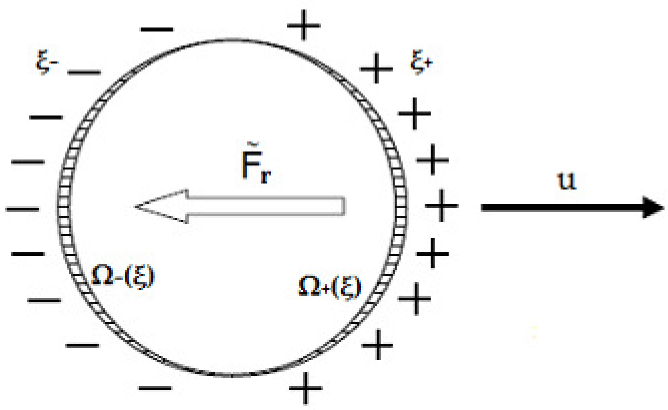


Fig. 1. The courtesy picture from [31]

where we took into account that $\lim_{r\to\infty}\int_{S_r^2} <\tilde{\varphi}\tilde{E}+\tilde{A}\times\tilde{B}, dS_r^2>=0,$ meaning the vanishing of the radiated by the accelerated charged particle energy, as well as we denoted by $\Omega_-(\xi):=$ supp $\xi_-\subset S^2$ and by $\Omega_+(\xi):=$ supp $\xi_+\subset S^2$ the corresponding charge $\xi$ parts supports, located on the electromagnetic field shadowed rear and on the electromagnetic field exerted on semi-spheres of the charged particle spherical shell $\Omega(\xi):=\Omega_-(\xi)\cup\Omega_+(\xi),$ respectively (see Fig 1.) to its motion with velocity $u:=dx/dt\in E^3$ with respect to the laboratory reference frame $K_t$.

The expression (225) demonstrates explicitly that during the charged particle motion the self-generated electromagnetic field interacts effectively only with its frontal part $\Omega_+(\xi)\subset S^2$ of the particle spherical shell $S^2$, as the rear part $\Omega_-(\xi)\subset S^2$ of the particle shell enters during its motion into the shadowed interior region of the sphere, where the net electric field $\tilde{E}\in E^3$ is vanishing owing to the charged particle spherical symmetry. To proceed further we need to calculate the electromagnetic potentials $(\tilde{\varphi},\tilde{A}):M^4\to R\times E^3$, using the determining expressions (211) as $1/c\to 0$:

$$\tilde{\varphi} = \int_{\mathrm{R}^3} d^3y \frac{\rho(t', y)}{|x-y|}\bigg|_{t'=t-|x-y|/c} = \lim_{\varepsilon \downarrow 0} \int_{\mathrm{R}^3} d^3y \frac{\rho(t-\varepsilon, y)}{|x-y|} +$$

$$+\lim_{\varepsilon \downarrow 0} \frac{1}{2c^2} \int_{\mathrm{R}^3} d^3y\, |x-y|\, \partial^2 \rho(t-\varepsilon, y)/\partial t^2 +$$

$$+\lim_{\varepsilon \downarrow 0} \frac{1}{6c^3} \int_{\mathrm{R}^3} d^3y\, |x-y|^2\, \partial \rho(t-\varepsilon, y)/\partial t + O(1/c^4) = \tag{226}$$

$$= \int_{\Omega_+(\xi)} d^3y \frac{\rho(t, y)}{|x-y|} + \frac{1}{2c^2} \int_{\Omega_+(\xi)} d^3y\, |x-y|\, \partial^2 \rho(t, y)/\partial t^2 +$$

$$+\frac{1}{6c^3} \int_{\Omega_+(\xi)} d^3y\, |x-y|^2\, \partial \rho(t, y)/\partial t + O(1/c^4),$$

$$\tilde{A} = \frac{1}{c} \int_{\mathrm{R}^3} d^3y \frac{J(t', y)}{|x-y|}\bigg|_{t'=t-|x-y|/c} = \lim_{\varepsilon \downarrow 0} \frac{1}{c} \int_{\mathrm{R}^3} d^3y \frac{J(t-\varepsilon, y)}{|x-y|} -$$

$$-\lim_{\varepsilon \downarrow 0} \frac{1}{c^2} \int_{\mathrm{R}^3} d^3y\, \partial J(t-\varepsilon, y)/\partial t +$$

$$+\lim_{\varepsilon \downarrow 0} \frac{1}{2c^3} \int_{\mathrm{R}^3} d^3y\, |x-y|\, \partial^2 J(t-\varepsilon, y)/\partial t^2 + O(1/c^4) =$$

$$= \frac{1}{c} \int_{\Omega_+(\xi)} d^3y \frac{J(t, y)}{|x-y|} - \frac{1}{c^2} \int_{\Omega_+(\xi)} d^3y\, \partial J(t, y)/\partial t +$$

$$+\frac{1}{2c^3} \int_{\Omega_+(\xi)} d^3y\, |x-y|\, \partial^2 J(t, y)/\partial t^2 + O(1/c^4),$$

where the limit $lim_{\varepsilon \downarrow 0}(...)$ was treated physically, that is taking into account the assumed above spherical shell model of the charged particle $\xi$ and its corresponding self-interaction during its motion. Now, as a result of simple enough calculations based on the electromagnetic potentials (226), the effective expression for the classical Lagrangian (225) can be equivalently rewritten up to $O(1/c^4)$ accuracy with respect to the laboratory reference frame $\mathrm{K}_t$ as

$$\tilde{\mathrm{L}}^{(t)}_{(f-b)} = \frac{\mathrm{E}_{es}}{2c^2} |u|^2, \tag{227}$$

if to make use of the following integral expressions:

$$\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3x \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3y\rho(t,y)\rho(t,y) := \xi^2,$$

$$\frac{1}{2}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3x \int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3y \frac{\rho(t,y)\rho(t,y)}{|x-y|} := \mathrm{E}_{es},$$

$$\int_{\Omega_+(\xi)} d^3x\rho(t,x) \int_{\Omega_+(\xi)} d^3y \frac{\rho(t;y)}{|y-x|} = \frac{1}{2}\mathrm{E}_{es},$$

$$\int_{\Omega_-(\xi)} d^3x\rho(t,x) \int_{\Omega_-(\xi)} d^3y \frac{\rho(t;y)}{|y-x|} = \frac{1}{2}\mathrm{E}_{es}, \tag{228}$$

$$\int_{\Omega_-(\xi)} d^3x\rho(t,x) \int_{\Omega_+(\xi)} d^3y \frac{\rho(t;y)}{|x-y|} \left|\frac{<y-x,u>}{|y-x|}\right|^2 > := \frac{\mathrm{E}_{es}}{6}|u|^2,$$

$$\int_{\Omega_+(\xi)} d^3x\rho(t,x) \int_{\Omega_+(\xi)} d^3y \frac{\rho(t;y)}{|x-y|} \left|\frac{<y-x,u>}{|y-x|}\right|^2 > := \frac{\mathrm{E}_{es}}{6}|u|^2,$$

obtained owing to the reasonings similar to those in [2] [126]. Now, to derive from the reduced Lagrangian function (227) the corresponding dynamic equation for the charged shell model particle $\xi$, we need within the discussed above the Feynman proper time paradigm to transform this Lagrangian with respect to the charged particle proper time reference frame $\mathrm{K}_\tau$:

$$\tilde{\mathrm{L}}^{(t)}_{(f-b)} \to \tilde{\mathrm{L}}^{(\tau)}_{(f-b)} = \frac{\bar{m}_{es}}{2}|\dot{x}|^2 - <k,\dot{x}>, \tag{229}$$

where we denoted by

$$\bar{m}_{es} := m_{es}\sqrt{1-|u|^2/c^2} \tag{230}$$

the so-called relativistic rest mass of the charged particle with respect to the proper time reference frame $\mathrm{K}_\tau$, and by

$$m_{es} := \mathrm{E}_{es}/c^2 \tag{231}$$

the so-called charged particle electromagnetic mass with respect to the laboratory reference frame $\mathrm{K}_t$. Based on the Lagrangian function (229) one can construct up to $O(1/c^2)$ the generalized charged particle inertial momentum

$$\tilde{\pi}_f := m_{ph}u - k \tag{232}$$

as

$$\tilde{\pi}_f = \partial\tilde{\mathrm{L}}^{(\tau)}_{(f-b)}/\partial\dot{x} = m_{es}u - k, \tag{233}$$

satisfying with respect to the proper time reference frame $\mathrm{K}_\tau$ the evolution equation

$$d\tilde{\pi}_f/d\tau = \partial\tilde{\mathrm{L}}^{(\tau)}_{(f-b)}/\partial x = 0, \tag{234}$$

which is equivalent to Lorentz type equation

$$d(m_{es}u)/dt = dk(t)/dt := \tilde{F}_r \tag{235}$$

with respect to the laboratory reference frame $\mathrm{K}_t$, where the right hand side of (235) means, by definition, the corresponding radiation reaction force $\tilde{F}_r$. Having applied to the Lagrangian function (229) the standard Legendre transformation, one easily finds the quasi-classical conserved Hamiltonian function

$$\mathrm{H}_{f-b}^{(t)} :=< \tilde{\pi}_f, \dot{x} > - \tilde{\mathrm{L}}_{(f-b)}^{(\tau)} = \frac{m_{es}|u|^2}{2}(1+\frac{1}{2}|u|^2/c^2), \tag{236}$$

satisfying with respect to the laboratory reference frame $\mathrm{K}_t$ the condition $d\mathrm{H}_{f-b}^{(t)}/dt = 0$ for all $t \in \mathrm{R}$. Yet, the most interesting and important consequence from (236)and the dynamic equation (235) consists in coinciding the electromagnetic mass parameter $m_{es} \in \mathrm{R}_+$ :

$$m_{phys} := m_{es}, \tag{237}$$

defined by (231), with the naturally related and physically observed inertial mass $m_{phys} \in \mathrm{R}_+$, as it was conceived by H. Lorentz and M. Abraham more than one hundred years ago.

## 8. The radiation reaction force analysis

To calculate the radiation reaction force (235) one can make use of the classical Lorentz type force expression (224) and obtain in the case of the charged particle shell model, similarly to [2] [126], up to $O(1/c^4)$ accuracy, the resulting self-interacting Abraham-Lorentz type force expression with respect to the laboratory reference frame $\mathrm{K}_t$. Owing to the zero net force condition, we have that

$$d\tilde{\pi}_f/dt + \tilde{F}_s = 0, \tag{238}$$

where, by definition, $\tilde{\pi}_f := m_{ph}u$, the Lorentz force can be rewritten in the following form:

$$\begin{aligned} \tilde{F}_s \;=\; & -\frac{1}{2c}\int_{\Omega_-(\xi)} d^3x\rho(t,x)\frac{d}{dt}\tilde{A}(t,x) - \\ & -\frac{1}{2c}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3x\rho(t,x)\frac{d}{dt}\tilde{A}(t,x) - \\ & -\frac{1}{2}\int_{\Omega_-(\xi)} d^3x\rho(t,x)\nabla\tilde{\varphi}(t,x)\,(1-|u/c|^2) - \\ & -\frac{1}{2}\int_{\Omega_+(\xi)\cup\Omega_-(\xi)} d^3x\rho(t,x)\nabla\tilde{\varphi}(t,x)\,(1-|u/c|^2). \end{aligned} \tag{239}$$

Based on calculations similar to those of [2] [126], from (239) and (226) one can obtain, within the charged particle shell model, for small $|u/c| = 1$ and slow enough acceleration that

$$\tilde{F}_s = \sum_{n\in Z_+} \frac{(-1)^{n+1}}{2n!c^n}(1-|u/c|^2)[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+$$

$$+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\int_{\Omega_+(\xi)}d^3y\frac{\partial^n}{\partial t^n}\rho(t,y)\nabla|x-y|^{n-1}+$$

$$+\sum_{n\in Z_+}\frac{(-1)^{n+1}}{2n!c^{n+2}}[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+$$

$$+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\int_{\Omega_+(\xi)}d^3y|x-y|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}J(t,y]=$$

$$= \sum_{n\in Z_+} \frac{(-1)^{n+1}}{2n!c^{n+2}}(1-|u/c|^2)[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+ \tag{240}$$

$$+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\int_{\Omega_+(\xi)}d^3y\frac{\partial^{n=2}}{\partial t^{n+2}}\rho(t,y)\nabla|x-y|^{n+1}+$$

$$+\sum_{n\in Z_+}\frac{(-1)^{n+1}}{2n!c^{n+2}}[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+$$

$$+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\int_{\Omega_+(\xi)}d^3y|x-y|^{n-1}\frac{\partial^{n+1}}{\partial t^{n+1}}J(t,y).$$

The relationship above can be rewritten, owing to the charge continuity equation (206)-(208) and the rotational symmetry property, giving rise to the radiation force differential-integral expression:

$$\tilde{F}_s = \frac{d}{dt}[\sum_{n\in Z_+}\frac{(-1)^{n+1}}{6n!c^{n+2}}[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\times$$

$$\times\int_{\Omega_+(\xi)}d^3y|x-y|^{n-1}\frac{\partial^n}{\partial t^n}J(t,y)-\sum_{n\in Z_+}\frac{(-1)^n\ |u|^2}{6n!c^{n+4}})[\int_{\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)+ \tag{241}$$

$$+\int_{\Omega_+(\xi)\cup\Omega_-(\xi)}\rho(t,x)d^3x(\cdot)]\int_{\Omega_+(\xi)}d^3y|x-y|^{n-1}\frac{\partial^n}{\partial t^n}J(t,y)].$$

From the latter, taking into account the integral expressions (228), one finds from (241) up to the $O(1/c^4)$ accuracy the final radiation reaction force expression

$$\tilde{F}_s \quad = -\frac{d}{dt}\left(\frac{\mathrm{E}_{es}}{c^2}u\right)+\frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}\,) =$$

$$= -\frac{d}{dt}(m_{es}u)+\frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2} = -\frac{d}{dt}(m_{es}u-\frac{2\xi^2}{3c^3}\frac{du}{dt}) \tag{242}$$

holds. We mention here that following the reasonings from [7] [31] [35] [105] [106], in the expressions above there is taken into account an additional hidden and the velocity $u \in T(\mathrm{R}^3)$ directed electrostatic Coulomb surface self-force, acting only on the *front half part* of the spherical electron shell. As a result, from (238), (239) and the relationship (232) one obtains that the generalized charged particle momentum

$$\tilde{\pi}_p := m_{es}u-\frac{2\xi^2}{3c^3}\frac{du}{dt} = m_{es}u-k, \tag{243}$$

thereby defining both the radiation reaction momentum $k(t)=\frac{2\xi^2}{3c^3}\frac{du(t)}{dt}$ for all $t\in \mathrm{R}$ and the corresponding radiation reaction force

$$\tilde{F}_r = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}, \tag{244}$$

coincides exactly with the classical Abraham-Lorentz--Dirac expression. From (243) one easily follows that the observable physical charged particle shell model inertial mass

$$m_{ph} = m_{es} = \mathrm{E}_{es}/c^2 \tag{245}$$

is of the electromagnetic origin, coinciding exactly with the result (237) obtained above. Moreover, (243) ensues the final force expression

$$\frac{d}{dt}(m_{es}u) = \frac{2\xi^2}{3c^3}\frac{d^2u}{dt^2}+O(1/c^4). \tag{246}$$

The latter means, in particular, that the real physically observed " inertial" mass $m_{ph}$ of the charged shell model particle $\xi$ is strongly determined by its electromagnetic self-interaction energy $\mathrm{E}_{es}$ with respect to the laboratory reference frame $\mathrm{K}_t$. A similar statement there was recently discussed in [31] [35], based on the vacuum Casimir effect type considerations. Moreover, the assumed above boundedness of the electrostatic self-energy $\mathrm{E}_{es}$ appears to be completely equivalent both to the presence of the so-called intrinsic Poincaré type " *tensions*" , analyzed in [7] [31] [118], and to the existence of a special compensating Coulomb " *pressure*" , suggested in [35], guaranteeing the assumed electron stability in the works of H. Lorentz and M. Abraham.

### 8.1. Comments

The electromagnetic mass origin problem was reanalyzed in details within the Feynman proper time paradigm and related vacuum field theory approach by means of the fundamental least action principle and the Lagrangian and Hamiltonian formalisms. The resulting electron inertia appeared to coincide in part, in the quasi-relativistic limit, with the momentum expression obtained more than one hundred years ago by M. Abraham and H. Lorentz [53] [54] [55] [64], yet it proved to contain an additional hidden impact owing to the imposed electron stability constraint, which was taken into account in the original action functional as some preliminarily undetermined constant component. As it was demonstrated in [31] [35], this stability constraint can be successfully realized within the charged shell model of electron at rest, if to take into account the existing ambient electromagnetic " dark" energy fluctuations, whose inward directed spatial pressure on the electron shell is compensated by the related outward directed electrostatic Coulomb spatial pressure as the electron shell radius satisfies some limiting compatibility condition. The latter also allows to compensate simultaneously the corresponding electromagnetic energy fluctuations deficit inside the electron shell, thereby forbidding the external energy to flow into the electron. In contrary to the lack of energy flow inside the electron shell, during the electron movement the corresponding internal momentum flow is not vanishing owing to the nonvanishing hidden electron momentum flow caused by the surface pressure flow and compensated by the suitably generated surface electric current flow. As it was shown, this backward directed hidden momentum flow makes it possible to justify the corresponding self-interaction electron mass expression and to state, within the electron shell model, the fully electromagnetic electron mass origin, as it has been conceived by H. Lorentz and M. Abraham and strongly supported by R. Feynman in his Lectures [1]. This consequence is also independently supported by means of the least action approach, based on the Feynman proper time paradigm and the suitably calculated regularized retarded electric potential impact into the charged particle Lagrangian function.

The charged particle radiation problem, revisited in this Section, allowed to conceive the explanation of the charged particle mass as that of a compact and stable object which should be exerted by a vacuum field self-interaction energy. The latter can be satisfied if to impose on the intrinsic charged particle structure [30] some nontrivial geometrical constraints. Moreover, as follows from the physically observed particle mass expressions (245), the electrostatic potential energy being of the self-interaction origin, contributes into the inertial mass as its main relativistic mass component.

There exist different relativistic generalizations of the force expression (246), which suffer the common physical inconsistency related to the no radiation effect of a charged particle in uniform motion.

Another deeply related problem to the radiation reaction force analyzed above is the search for an explanation to the Wheeler and Feynman reaction radiation mechanism, called the absorption radiation theory, strongly based on the Mach type interaction of a charged particle with the ambient vacuum electromagnetic medium. Concerning this problem, one can also observe some of its relationships with the one devised here within the vacuum field theory approach, but this question needs a more detailed and extended analysis.

## 9. Supplement: the "minimum" interaction principle and its geometric backgrounds

In this Section we will sketch analytical backgrounds of the "minimum" interaction principle widely used in modern theoretical and mathematical physics. For description of a moving point charged particle under external electromagnetic field, we will make use of the geometric approach [64]. Namely, let a trivial fiber bundle structure $\pi : \mathrm{M} \to \mathrm{R}^3, \mathrm{M} = \mathrm{R}^3 \times G$, with the abelian structure group $G := \mathrm{R} \setminus \{0\}$, equivariantly act on the canonically symplectic coadjoint space $T^*(\mathrm{M})$. The latter possesses the canonical symplectic structure

$$\begin{aligned} \omega^{(2)}(p,z;x,g) \quad &:= d(pr_*)^* \alpha^{(1)}(x,g) = < dp, \wedge dx > + \\ + \quad &< dz, \wedge g^{-1}dg >_{\mathrm{G}} + < zdg^{-1}, \wedge dg >_{\mathrm{G}} \end{aligned} \tag{247}$$

for all $(p,z;x,g) \in T^*(\mathrm{M})$, where $\alpha^{(1)}(x,g) := < p, dx > + < z, g^{-1}dg >_{\mathrm{G}} \in T^*(\mathrm{M})$ is the corresponding Liouville form on $T^*(\mathrm{M})$ and $< \cdot, \cdot >$ is the usual scalar product in $\mathrm{E}^3$. On the fibered space $\mathrm{M}$ one can define a connection $\Gamma$ by means of an one-form $\mathrm{A} : M \to T^*(\mathrm{M}) \times \mathrm{G}$, determined as

$$\mathrm{A}(x,g) := g^{-1} < \xi A(x), dx > g + g^{-1}dg \tag{248}$$

with $\xi \in \mathrm{G}^*, (x,g) \in \mathrm{R}^3 \times G$. The corresponding curvature 2-form $\Sigma^{(2)} \in \Lambda^2(\mathrm{R}^3) \otimes \mathrm{G}$ is

$$\Sigma^{(2)}(x) := d\mathrm{A}(x,g) + \mathrm{A}(x,g) \wedge \mathrm{A}(x,g) = \xi \sum_{i,j=1}^{3} F_{ij}(x) dx^i \wedge dx^j, \tag{249}$$

where

$$F_{ij}(x) := \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \tag{250}$$

for $i,j = \overline{1,3}$ is the spatial electromagnetic tensor with respect to the reference frame $\mathrm{K}_t$. For an element $\xi \in \mathrm{G}^*$ to be compatibly fixed, we need to construct the related momentum mapping $l : T^*(\mathrm{M}) \to \mathrm{G}^*$ with respect to the canonical symplectic structure (247) on $T^*(\mathrm{M})$, and put, by definition, $l(x,p) := \xi \in \mathrm{G}^*$ to be constant, $P_\xi := l^{-1}(\xi) \subset T^*(\mathrm{M})$ and $G_\xi = \{g \in G : Ad_G^* \xi\}$ to be the corresponding isotropy group of the element $\xi \in \mathrm{G}^*$. Next we can apply the standard [47] [64] [96] invariant Marsden-Weinstein-Meyer reduction scheme to the orbit factor space $\tilde{P}_\xi := P_\xi / G_\xi$ subject to the corresponding group $G$ action. Then, as a result of the Marsden-Weinstein-Meyer reduction, one finds that $G_\xi ; G$, the factor-space $\tilde{P}_\xi ; T^*(\mathrm{R}^3)$ becomes Poisson space with the suitably reduced symplectic structure $\bar{\omega}_\xi^{(2)} \in T^*(\tilde{P}_\xi)$. The corresponding Poisson brackets on the reduced manifold $\tilde{P}_\xi$ equal to

$$\begin{aligned} \{x^i, x^j\}_\xi \quad &= 0, \{p_j, x^i\}_\xi = \delta_j^i, \\ \{p_i, p_j\}_\xi \quad &= \xi F_{ij}(x) \end{aligned} \tag{251}$$

for $i,j = \overline{1,3}$, being considered with respect to the laboratory reference frame $\mathrm{K}_t$. Based on (251) is worth to observe that a new so called "*shifted*" momentum variable

$$\tilde{\pi} := p + \xi A(x) \tag{252}$$

on $\tilde{P}_\xi$ gives rise to the symplectomorphic transformation $\bar{\omega}_\xi^{(2)} \to \tilde{\omega}_\xi^{(2)} := < d\tilde{\pi}, \wedge dx > \in \Lambda^2(T^*(\mathrm{R}^3))$. The latter gives rise to the following important in theoretical physics *"minimal interaction"* canonical Poisson brackets:

$$\{x^i, x^j\}_{\tilde{\omega}^{(2)}_\xi} = 0, \{\tilde{\pi}_j, x^i\}_{\tilde{\omega}^{(2)}_\xi} = \delta^i_j, \{\tilde{\pi}_i, \tilde{\pi}_j\}_{\tilde{\omega}^{(2)}_\xi} = 0 \qquad (253)$$

for $i, j = \overline{1,3}$, represented already with respect to some new reference frame $\tilde{\mathrm{K}}_{t'}$, characterized by the phase space coordinates $(x, \tilde{\pi}) \in \tilde{P}_\xi$ and a new evolution parameter $t' \in \mathrm{R}$, as the spatial Maxwell field compatibility equations

$$\partial F_{ij} / \partial x_k + \partial F_{jk} / \partial x_i + \partial F_{ki} / \partial x_j = 0 \qquad (254)$$

are identically satisfied on $\mathrm{R}^3$ for all $i, j, k = \overline{1,3}$, owing to the electromagnetic curvature tensor (250) definition.


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