Original Research Article

THE HAMILTONIAN OPERATOR AND EULER POLYNOMIALS

Abstract. In this paper we obtain some identities related to the Hamiltonian operator and Euler polynomials and confirm these properties through examples.

1. Introduction

Various functions appear in many areas of theoretical physics, for example, Euler polynomials is shown in the field of non-commutative operators in quantum physics. Let us define the commutator of two operators p and q as

$$[p, q] = pq - qp$$

and their anti-commutator as

$$\{p, q\} = pq + qp.$$

Generally we define the iterated anti-commutators as

$$\{p, q\}_2 = \{\{p, q\}, q\}, \qquad \{p, q\}_3 = \{\{\{p, q\}, q\}, q\} = \{\{p, q\}_2, q\}$$

and moreover for all positive integer n, we have

$$\{p, q\}_n = \{\{p, q\}_{n-1}, q\}.$$

We introduce the Hamiltonian operator H as

$$H = \frac{1}{2}\left(p^2 + q^2\right).$$

C. Bender and L. Bettencourt [2] suggest the following result

$$\frac{1}{2^n}\{q, H\}_n = \frac{1}{2}\left\{q, E_n(H + \frac{1}{2})\right\} \tag{1}$$

where we can find the Euler polynomials $E_n(x)$ $(n \in \mathbb{N})$ are given by the power series

$$\sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!} = \frac{2e^{xz}}{e^z + 1}. \tag{2}$$

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The integers $E_n = 2^n E_n(1/2)$ are called Euler numbers. The first few Euler polynomials are

$$E_0(x) = 1, \qquad E_1(x) = x - \frac{1}{2}, \qquad E_2(x) = x^2 - x,$$
$$E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \qquad E_4(x) = x^4 - 2x^3 + x,$$
$$E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}.$$

It is well-known [3] that

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_{n-k}(0) x^k \tag{3}$$

and

$$E_n(x) + E_n(x+1) = 2x^n \qquad \text{for all } n \in \mathbb{N}. \tag{4}$$

In this article we start from the paper [1] and we try to generalize some identities shown on it thus we obtain the following relations of the Hamiltonian operator involving Euler polynomials :

Theorem 1.1. *Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Then we have*

$$\sum_{k=0}^{n} \binom{n}{k} E_{n-k}(0) \frac{1}{2^k} \left(\left\{ q, H - \frac{a}{2} \right\}_k + \left\{ q, H - \frac{a}{2} + 1 \right\}_k \right)$$
$$= \frac{1}{2^{n-1}} \left\{ q, H - \frac{a}{2} \right\}_n .$$

Corollary 1.2. *Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Then we have*

$$\sum_{k=0}^{n} \binom{n}{k} E_{n-k}(0) \frac{1}{2^k} \left(\left\{ q, H - \frac{a}{2} \right\}_k - \left\{ q, H - \frac{a}{2} + 2 \right\}_k \right)$$
$$= \frac{1}{2^{n-1}} \left\{ q, H - \frac{a}{2} \right\}_n - \frac{1}{2^{n-1}} \left\{ q, H - \frac{a}{2} + 1 \right\}_n .$$

2. Some identities for the Hamiltonian operator

Let $\mathbb{N}$ and $\mathbb{R}$ denote the sets of all positive integers and real numbers, respectively. We introduce the symbolic notation, with $a \in \mathbb{R}$,

$$(\{q, H\} + a)_n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \{q, H\}_k \tag{5}$$

and the convention $\{q, H\}_0 = q$.

Proposition 2.1. *(See [1]) For $a \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$\left\{ q, H + \frac{a}{2} \right\}_n = (\{q, H\} + a)_n .$$

Corollary 2.2. *Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Then*

(a)

$$\sum_{k=0}^{n} \binom{2n}{2k} \{q, H\}_{2k} a^{2n-2k} = \frac{1}{2} \left(\left\{q, H + \frac{a}{2}\right\}_{2n} + \left\{q, H - \frac{a}{2}\right\}_{2n} \right),$$

(b)

$$\sum_{k=0}^{n} \binom{2n+1}{2k+1} \{q, H\}_{2k+1} a^{2n-2k} = \frac{1}{2} \left(\left\{q, H + \frac{a}{2}\right\}_{2n+1} + \left\{q, H - \frac{a}{2}\right\}_{2n+1} \right).$$

Proof. By (5) and Proposition 2.1 we observe that

$$\left\{q, H + \frac{a}{2}\right\}_n = (\{q, H\} + a)_n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} \{q, H\}_k \tag{6}$$

and

$$\left\{q, H - \frac{a}{2}\right\}_n = (\{q, H\} - a)_n = \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} \{q, H\}_k. \tag{7}$$

(a) After putting $n = 2N$ in Eq. (6) and (7), adding them we obtain

$$\begin{aligned} &2 \sum_{k=0}^{N} \binom{2N}{2k} \{q, H\}_{2k} a^{2N-2k} \\ &= \sum_{k=0}^{2N} \binom{2N}{k} a^{2N-k} \{q, H\}_k + \sum_{k=0}^{2N} \binom{2N}{k} (-a)^{2N-k} \{q, H\}_k \\ &= \left\{q, H + \frac{a}{2}\right\}_{2N} + \left\{q, H - \frac{a}{2}\right\}_{2N}. \end{aligned}$$

(b) Let $n = 2N + 1$ in (6) and (7). Then adding them we have

$$\begin{aligned} &2 \sum_{k=0}^{N} \binom{2N+1}{2k+1} \{q, H\}_{2k+1} a^{2N-2k} \\ &= \sum_{k=0}^{2N+1} \binom{2N+1}{k} a^{2N+1-k} \{q, H\}_k \\ &\quad + \sum_{k=0}^{2N+1} \binom{2N+1}{k} (-a)^{2N+1-k} \{q, H\}_k \\ &= \left\{q, H + \frac{a}{2}\right\}_{2N+1} + \left\{q, H - \frac{a}{2}\right\}_{2N+1}. \end{aligned}$$

□

Proposition 2.3. *(See [1]) An equivalent form of identity (1) is*

$$\frac{1}{2^n}\left\{q, H-\frac{1}{2}\right\}_n + \frac{1}{2^n}\left\{q, H+\frac{1}{2}\right\}_n = \{q, H^n\}.$$

From the above proposition we consider the following lemma and we can see that Proposition 2.3 is the special case $a = 1$.

Lemma 2.4. *Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Then we have*

$$\frac{1}{2^n}\left\{q, H-\frac{a}{2}\right\}_n + \frac{1}{2^n}\left\{q, H-\frac{a}{2}+1\right\}_n = \left\{q, \left(H-\frac{a-1}{2}\right)^n\right\}.$$

Proof. From (1) we can easily know that

$$\frac{1}{2^n}\left\{q, H-\frac{1}{2}\right\}_n = \frac{1}{2}\left\{q, E_n(H)\right\},$$

which deduces that by (4)

$$\begin{aligned}
&\frac{1}{2^n}\left\{q, H-\frac{a}{2}\right\}_n + \frac{1}{2^n}\left\{q, H-\frac{a}{2}+1\right\}_n \\
&= \frac{1}{2}\left\{q, E_n\left(H-\frac{a-1}{2}\right)\right\} + \frac{1}{2}\left\{q, E_n\left(H-\frac{a-1}{2}+1\right)\right\} \\
&= \left\{q, \frac{1}{2}E_n\left(H-\frac{a-1}{2}\right)\right\} + \left\{q, \frac{1}{2}E_n\left(H-\frac{a-1}{2}+1\right)\right\} \\
&= \left\{q, \frac{1}{2}\left(E_n\left(H-\frac{a-1}{2}\right) + E_n\left(H-\frac{a-1}{2}+1\right)\right)\right\} \\
&= \left\{q, \frac{1}{2}\cdot 2\left(H-\frac{a-1}{2}\right)^n\right\} \\
&= \left\{q, \left(H-\frac{a-1}{2}\right)^n\right\}.
\end{aligned}$$

□

Example 2.5. *In Lemma 2.4 the case $n = 1$ implies that*

$$\begin{aligned}
&\frac{1}{2}\left(\left\{q, H-\frac{a}{2}\right\}_1 + \left\{q, H-\frac{a}{2}+1\right\}_1\right) \\
&= \frac{1}{2}\left(q\left(H-\frac{a}{2}\right) + \left(H-\frac{a}{2}\right)q + q\left(H-\frac{a}{2}+1\right) + \left(H-\frac{a}{2}+1\right)q\right) \\
&= qH + Hq - aq + q \\
&= \left\{q, H-\frac{a-1}{2}\right\}.
\end{aligned}$$

But since

$$[p, H] = -iq \qquad \text{and} \qquad [q, H] = ip$$

we have

$$\begin{aligned}qH^2-2HqH+H^2q=[q,H]H-H[q,H]=[[q,H],H]=[ip,H]&=i[p,H]\\&=i(-iq)\\&=q\end{aligned}$$

and

$$HqH=\frac{qH^2+H^2q-q}{2}.$$

This leads for the case $n=2$ that

$$\begin{aligned}&\frac{1}{4}\left(\left\{q,H-\frac{a}{2}\right\}_2+\left\{q,H-\frac{a}{2}+1\right\}_2\right)\\&=\frac{1}{4}\left(\left\{\left\{q,H-\frac{a}{2}\right\},H-\frac{a}{2}\right\}+\left\{\left\{q,H-\frac{a}{2}+1\right\},H-\frac{a}{2}+1\right\}\right)\\&=\frac{1}{4}\left(q\left(H-\frac{a}{2}\right)^2+2\left(H-\frac{a}{2}\right)q\left(H-\frac{a}{2}\right)+\left(H-\frac{a}{2}\right)^2q\right.\\&\qquad\left.+q\left(H-\frac{a}{2}+1\right)^2+2\left(H-\frac{a}{2}+1\right)q\left(H-\frac{a}{2}+1\right)+\left(H-\frac{a}{2}+1\right)^2q\right)\\&=\frac{H^2q}{2}+\frac{qH^2}{2}+HqH-(a-1)Hq-(a-1)qH+\left(\frac{a^2}{2}-a+1\right)q\\&=H^2q+qH^2-(a-1)Hq-(a-1)qH+\left(\frac{a^2}{2}-a+\frac{1}{2}\right)q\\&=\left\{q,\left(H-\frac{a-1}{2}\right)^2\right\}.\end{aligned}$$

Proof of Theorem 1.1. By (5) and Proposition 2.1 we note that

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left(\left\{q,H-\frac{a}{2}\right\}_k+\left\{q,H-\frac{a}{2}+1\right\}_k\right)\\&=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left\{q,H-\frac{a}{2}\right\}_k+\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left\{q,H-\frac{a-2}{2}\right\}_k\\&=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left(\{q,H\}-a\right)_k+\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left(\{q,H\}-a+2\right)_k\\&=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\cdot\sum_{l=0}^{k}\binom{k}{l}(-a)^{k-l}\{q,H\}_l\\&\qquad+\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\cdot\sum_{l=0}^{k}\binom{k}{l}(-a+2)^{k-l}\{q,H\}_l.\end{aligned}$$

Then by replacing $k-l$ with p and using

$$\binom{n}{k}\binom{k}{l} = \frac{n!}{k!(n-k)!}\cdot\frac{k!}{l!(k-l)!} = \frac{n!}{l!}\cdot\frac{1}{(n-k)!(k-l)!}$$
$$= \frac{n!}{l!(n-l)!}\cdot\frac{(n-l)!}{(n-k)!(k-l)!} = \binom{n}{l}\binom{n-l}{k-l} = \binom{n}{l}\binom{n-l}{p},$$

(3), and (4), the above identity becomes

$$\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left(\left\{q, H-\frac{a}{2}\right\}_k + \left\{q, H-\frac{a}{2}+1\right\}_k\right)$$
$$= \sum_{l=0}^{n}\binom{n}{l}\{q,H\}_l\sum_{p=0}^{n-l}\binom{n-l}{p}E_{n-l-p}(0)\frac{(-a)^p}{2^{p+l}}$$
$$+ \sum_{l=0}^{n}\binom{n}{l}\{q,H\}_l\sum_{p=0}^{n-l}\binom{n-l}{p}E_{n-l-p}(0)\frac{(-a+2)^p}{2^{p+l}}$$
$$= \sum_{l=0}^{n}\binom{n}{l}\frac{\{q,H\}_l}{2^l}\sum_{p=0}^{n-l}\binom{n-l}{p}E_{n-l-p}(0)\left(\frac{-a}{2}\right)^p$$
$$+ \sum_{l=0}^{n}\binom{n}{l}\frac{\{q,H\}_l}{2^l}\sum_{p=0}^{n-l}\binom{n-l}{p}E_{n-l-p}(0)\left(\frac{-a+2}{2}\right)^p$$
$$= \sum_{l=0}^{n}\binom{n}{l}\frac{\{q,H\}_l}{2^l}\left(E_{n-l}(-\frac{a}{2}) + E_{n-l}(-\frac{a}{2}+1)\right)$$
$$= \sum_{l=0}^{n}\binom{n}{l}\frac{\{q,H\}_l}{2^l}\cdot 2\left(-\frac{a}{2}\right)^{n-l}.$$

This concludes that by (5) and Proposition 2.1

$$\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left(\left\{q, H-\frac{a}{2}\right\}_k + \left\{q, H-\frac{a}{2}+1\right\}_k\right)$$
$$= \frac{1}{2^{n-1}}\sum_{l=0}^{n}\binom{n}{l}\{q,H\}_l(-a)^{n-l}$$
$$= \frac{1}{2^{n-1}}\left(\{q,H\}-a\right)_n$$
$$= \frac{1}{2^{n-1}}\left\{q, H-\frac{a}{2}\right\}_n.$$

□

Example 2.6. *The case $n = 1$ in Theorem 1.1 shows that*

$$
\begin{aligned}
&\sum_{k=0}^{1}\binom{1}{k}E_{1-k}(0)\frac{1}{2^k}\left(\left\{q,H-\frac{a}{2}\right\}_k+\left\{q,H-\frac{a}{2}+1\right\}_k\right)\\
&=\binom{1}{0}E_1(0)\cdot 1\cdot 2q+\binom{1}{1}E_0(0)\cdot\frac{1}{2}\left(\left\{q,H-\frac{a}{2}\right\}_1+\left\{q,H-\frac{a}{2}+1\right\}_1\right)\\
&=-q+\frac{1}{2}\left(q\left(H-\frac{a}{2}\right)+\left(H-\frac{a}{2}\right)q+q\left(H-\frac{a}{2}+1\right)+\left(H-\frac{a}{2}+1\right)q\right)\\
&=qH+Hq-aq\\
&=\left\{q,H-\frac{a}{2}\right\}_1
\end{aligned}
$$

thus it is satisfied. Also if $n=2$ *in Theorem 1.1 then we have*

$$
\begin{aligned}
&\sum_{k=0}^{2}\binom{2}{k}E_{2-k}(0)\frac{1}{2^k}\left(\left\{q,H-\frac{a}{2}\right\}_k+\left\{q,H-\frac{a}{2}+1\right\}_k\right)\\
&=\binom{2}{0}E_2(0)\cdot 1\cdot 2q+\binom{2}{1}E_1(0)\cdot\frac{1}{2}\left(\left\{q,H-\frac{a}{2}\right\}_1+\left\{q,H-\frac{a}{2}+1\right\}_1\right)\\
&\quad+\binom{2}{2}E_1(0)\cdot\frac{1}{4}\left(\left\{q,H-\frac{a}{2}\right\}_2+\left\{q,H-\frac{a}{2}+1\right\}_2\right)\\
&=\frac{1}{2}\left(\left\{q,H-\frac{a}{2}\right\}_1+\left\{q,H-\frac{a}{2}+1\right\}_1\right)\\
&\quad+\frac{1}{4}\left(\left\{\left\{q,H-\frac{a}{2}\right\},H-\frac{a}{2}\right\}+\left\{\left\{q,H-\frac{a}{2}+1\right\},H-\frac{a}{2}+1\right\}\right)\\
&=-\frac{1}{2}\left(q\left(H-\frac{a}{2}\right)+\left(H-\frac{a}{2}\right)q+q\left(H-\frac{a}{2}+1\right)+\left(H-\frac{a}{2}+1\right)q\right)\\
&\quad+\frac{1}{4}\left(q\left(H-\frac{a}{2}\right)^2+2\left(H-\frac{a}{2}\right)q\left(H-\frac{a}{2}\right)+\left(H-\frac{a}{2}\right)^2q\right.\\
&\quad\left.+q\left(H-\frac{a}{2}+1\right)^2+2\left(H-\frac{a}{2}+1\right)q\left(H-\frac{a}{2}+1\right)+\left(H-\frac{a}{2}+1\right)^2q\right)\\
&=\frac{H^2q}{2}+\frac{qH^2}{2}-aHq-aqH+HqH+\frac{a^2}{2}q\\
&=\frac{1}{2}\left\{q,H-\frac{a}{2}\right\}_2
\end{aligned}
$$

and so it is satisfied.

Proof of Corollary 1.2. From Theorem 1.1 we deduce that

$$\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left(\left\{q,H-\frac{a}{2}\right\}_k-\left\{q,H-\frac{a}{2}+2\right\}_k\right)$$
$$=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left(\left\{q,H-\frac{a}{2}\right\}_k+\left\{q,H-\frac{a}{2}+1\right\}_k\right)$$
$$-\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\frac{1}{2^k}\left(\left\{q,H-\frac{a}{2}+1\right\}_k+\left\{q,H-\frac{a}{2}+2\right\}_k\right)$$
$$=\frac{1}{2^{n-1}}\left\{q,H-\frac{a}{2}\right\}_n-\frac{1}{2^{n-1}}\left\{q,H-\frac{a}{2}+1\right\}_n.$$

□

Example 2.7. *If $n=1$ in Corollary 1.2 then we obtain*

$$\sum_{k=0}^{1}\binom{1}{k}E_{1-k}(0)\frac{1}{2^k}\left(\left\{q,H-\frac{a}{2}\right\}_k-\left\{q,H-\frac{a}{2}+2\right\}_k\right)$$
$$=-2q$$
$$=\left\{q,H-\frac{a}{2}\right\}_1-\frac{1}{2^{n-1}}\left\{q,H-\frac{a}{2}+1\right\}_1.$$

And if $n=2$ in Corollary 1.2 then

$$\sum_{k=0}^{2}\binom{2}{k}E_{2-k}(0)\frac{1}{2^k}\left(\left\{q,H-\frac{a}{2}\right\}_k-\left\{q,H-\frac{a}{2}+2\right\}_k\right)$$
$$=-2qH-2Hq+2aq-2q$$
$$=\frac{1}{2}\left\{q,H-\frac{a}{2}\right\}_2-\frac{1}{2}\left\{q,H-\frac{a}{2}+1\right\}_2.$$

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