Coupling of Laplace Transform and Differential Transform for Wave Equations

Abstract

In this paper, we apply Differential Transform Method (DTM) coupled with Laplace Transform Method to solve wave equations and wave-like equations which arise very frequently in physical problems related to engineering and applied sciences. It is observed that the proposed technique gives excellent accuracy for the finding and is very user-friendly. Several examples are given to re-conrm the efciency of the suggested algorithm. The graphs were performed by using Mathematica-8.

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Keywords: DTM, Laplace transformation, Wave equations, Mathematica-8.

1 Introduction.

In recent years, the rapid development of nonlinear sciences [1-9] witnesses a wide range of analytical and numerical techniques by various engineering and scientific applications. Most of the developed schemes have focused on their limited convergence, divergent results, linearization, discretization, unrealistic assumptions and non-compatibility with the versatility of physical problems[1-9].
Different type of schemes and analytical tool have been proposed for solving wave-like equations with variable co-efficents [1], linear and non-linear wave equations [2] and wave systems [3] etc. The Differential Transform Method (DTM) is one of them techniques. It is an iterative procedure for obtaining analytic Taylor's series solution of differential equations was first proposed by J.K. Zhou [4] in 1986, its main application concern with linear and non-linear initial value problems in electric circuit analysis. The Laplace-differential transform method (LDTM) is an approximate analytical technique for solving differential equations that recently introduced by Marwan Alquran *et* al.[6] and it is has been successfully applied to solve different types of physical problems by Kiranta *et* al.[7-8].
The main goal of this work is the coupling of differential transform method and Laplace transformation to obtain exact solutions of wave-like equations. The suggested algorithm is tested on linear, nonlinear wave equations and wave-like equations in bounded and unbounded domains. To the best of our knowledge no such try has been made to combine LTM and DTM for solving wave equations. Four problems for wave equations and wave-like equation are solved to make clear the application of the transform and the numerical results are very encouraging.

2 Differential Transformation Method

The one variable differential transform [7-8] of a function $u(x,t)$, is defined as:

$$U_k(t) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial x^k}\right]_{x=x_0} ; k \geq 0 \tag{2.1}$$

where $u(x,t)$ is the original function and $U_k(t)$ is the transformed function. The inverse differential transform of $U_k(t)$ is defined as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)(x-x_0)^k, \tag{2.2}$$

where x_0 is the initial point for the given initial condition. Then the function $u(x,t)$ can be written as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)x^k. \tag{2.3}$$

Some basic formulas of Differential Transformation are listed in the following [18].

Original Function	Transformed Function
$u(x,t) = f(x,t) \pm g(x,t)$	$U_k(t) = F_k(t) \pm G_k(t)$
$u(x,t) = \alpha f(x,t)$	$U_k(t) = \alpha F_k(t)$
$u(x,t) = \frac{\partial f(x,t)}{\partial x}$	$U_k(t) = (k+1)F_{k+1}(t)$
$u(x,t) = \frac{\partial^r f(x,t)}{\partial x^r}$	$U_k(t) = (k+1)(k+2)...(k+r)F_{k+r}(t)$
$u(x,t) = (x-x_0)^r(t-t_0)^s$	$U_k(t) = (t-t_0)^s\delta(k-r)$ where, $\delta(k-r)$
$u(x,t) = f(x,t)g(x,t)$	$U_k(t) = \sum_{r=0}^{k} F_r(t)G_{k-r}(t)$
$u(x,t) = f(x,t)g(x,t)h(x,t)$	$U_k(t) = \sum_{r=0}^{k}\sum_{q=0}^{r} F_q(t)G_{r-q}(t)H_{k-r}(t)$
$u(x,t) = sin(ax+\alpha)$	$U_k(t) = \frac{a^k}{k!}\left[sin(\frac{k\pi}{2}+\alpha)\right]$
$u(x,t) = cos(ax+\alpha)$	$U_k(t) = \frac{a^k}{k!}\left[cos(\frac{k\pi}{2}+\alpha)\right]$

3 Differential Transform Method Coupled with Laplace Transform

Consider the general nonlinear, inhomogeneous partial differential equation [6]

$$\mathcal{L}[u(x,t)] + \mathcal{R}[u(x,t)] = f(x,t); \quad x \in \mathbb{R},\ t \in \mathbb{R}^+, \tag{3.1}$$

subject to the initial conditions

$$u(x,0) = g_1(x), \quad u_t(x,0) = g_2(x), \tag{3.2}$$

and the spatial conditions

$$u(0,t) = h_1(t), \quad u_x(0,t) = h_2(t), \tag{3.3}$$

where $\mathcal{L}[.]$ is linear operator w.r.to 't', $\mathcal{R}[.]$ is remaining operator and f is a known analytical function.
Taking the Laplace Transform to the both sides of the given equation (3.1), w.r.to 't', and we get

$$L\big[\mathcal{L}\big[u(x,t)\big]\big] + L\big[\mathcal{R}\big[u(x,t)\big]\big] = L\big[f(x,t)\big]. \tag{3.4}$$

By using initial conditions from equation (3.2), we get

$$\bar{u}(x,s) + L\big[\mathcal{R}\big[u(x,t)\big]\big] = \big[\bar{f}(x,s)\big], \tag{3.5}$$

where $\bar{u}(x,s)$ and $\bar{f}(x,s)$ are the Laplace transform on $u(x,t)$ and $f(x,t)$ respectively.
Afterwards, we apply differential transform method on the equation (3.5) with respect to 'x', and we get

$$\bar{U_k}(s) + L\big[\mathcal{R}\big[U_k(t)\big]\big] = \big[\bar{F_k}(s)\big], \tag{3.6}$$

where $\bar{U_k}(s)$ and $\bar{F_k}(s)$ are the differential transform of $\bar{u}(x,s)$ and $\bar{f}(x,s)$ respectively.
Taking the inverse Laplace transform on both sides of the equation (3.6) with respect to 's', and then we get

$$L^{-1}\big[\bar{U_k}(s)\big] + L^{-1}L\big[\mathcal{R}\big[U_k(t)\big]\big] = L^{-1}\big[\bar{F_k}(s)\big],$$

or

$$U_k(t) + \big[\mathcal{R}\big[U_k(t)\big]\big] = \big[F_k(t)\big]. \tag{3.7}$$

Now, apply the differential transform method on the given spatial conditions (3.3), we get

$$U_0(t) = h_1(t), \quad U_1(t) = h_2(t) \tag{3.8}$$

Now by the equations (3.7) and (3.8), the solution in the series form is given by

$$u(x,t) = \sum_{k=0}^{\infty} U(k,t)x^k.$$

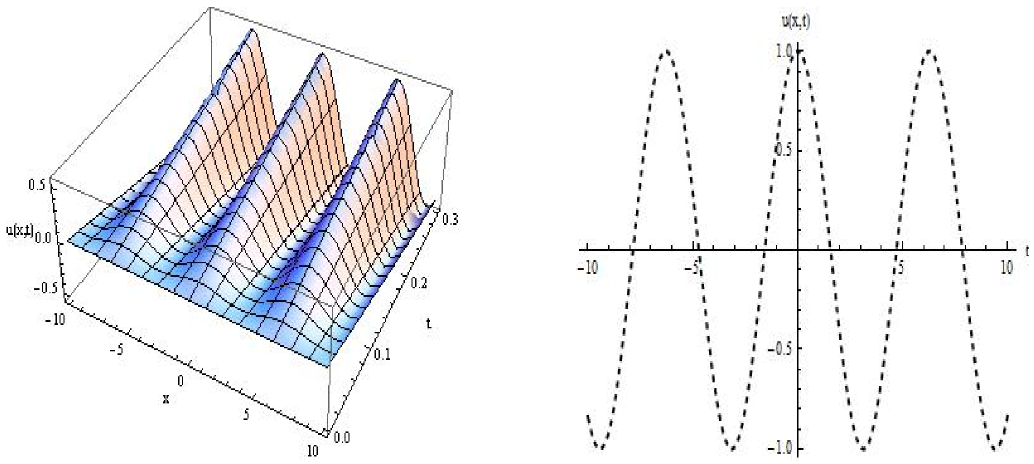
4 Numerical Applications

In this section, we apply Differential Transform Method (DTM) coupled with Laplace transform to solve linear & nonlinear Wave Equations. Numerical results are very encouraging.
Example 4.1: Wave-like equation (see Fig. 4.1)
Consider the following wave like equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}, \tag{4.1}$$

with the initial conditions,

$$u(x,0) = 0, \quad u_t(x,0) = x^2, \tag{4.2}$$



(a) (b)

Figure 4.1: (a): 3D plot of u(x, t) and (b): 2D plot of u(x, t) for Example 4.1.

and

$$u(0,t)=0, \quad u_x(0,t)=0. \tag{4.3}$$

In this technique, first we apply the Laplace transformation on equation (4.1) with respect to t, therefore, we get

$$s^2L\big[u(x,t)\big]-su(x,0)-u_t(x,0)=L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right].$$

By using initial conditions from equation (4.2), we get

$$L\big[u(x,t)\big]=\frac{x^2}{s^2}+\frac{1}{s^2}L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right].$$

Now, we applying the Inverse Laplace transformation w.r.t.'s' on both sides:

$$u(x,t)=x^2t+L^{-1}\left[\frac{1}{s^2}L\left[\frac{x^2}{2}\frac{\partial^2 u}{\partial x^2}\right]\right]. \tag{4.4}$$

The next step is applying the Differential transformation method on equations (4.3) and (4.4) with respect to space variable x, we get

$$U_k(t)=t\delta(k-2,t)+L^{-1}\left[\frac{1}{2s^2}L\left[\sum_{r=0}^{k}(r+2)(r+1)U_{r+2}(t)\delta(k-r-2,t)\right]\right]; \tag{4.5}$$

$$where \quad \delta(k-1,t)=\begin{cases}1 & k=1\\ 0 & otherwise\end{cases}$$

and

$$U_0(t)=0, \quad U_1(t)=0. \tag{4.6}$$

Substituting (4.6) into (4.5) and the following approximations are obtained successively

$$U_2(t) = sinh(t), \quad U_3(t) = 0, \quad U_4(t) = 0, \quad U_5(t) = 0...$$

Finally, the closed form solution is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)x^k = x^2 sinh(t).$$

which is the exact solution.
Example 4.2: Inhomogeneous wave equation (see Fig.4. 2)
Consider the following inhomogeneous nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + u + u^2 - xt - x^2t^2, \tag{4.7}$$

with the initial conditions,

$$u(x,0) = 0, \quad u_t(x,0) = x, \tag{4.8}$$

and

$$u(0,t) = 0, \quad u_x(0,t) = t. \tag{4.9}$$

In this technique, first we apply the Laplace transformation on equation (4.7) with respect to t, therefore, we get

$$s^2 L\big[u(x,t)\big] - su(x,0) - u_t(x,0) = -\frac{x}{s^2} - \frac{2x^2}{s^3} + L\left[\frac{\partial^2 u}{\partial x^2} + u + u^2\right].$$

By using initial conditions from equation (4.8), we get

$$L\big[u(x,t)\big] = \frac{x}{s^2} - \frac{x}{s^4} - \frac{2x^2}{s^5} + \frac{1}{s^2}L\left[\frac{\partial^2 u}{\partial x^2} + u + u^2\right].$$

Now, we applying the Inverse Laplace transformation w.r.t.'s on both sides:

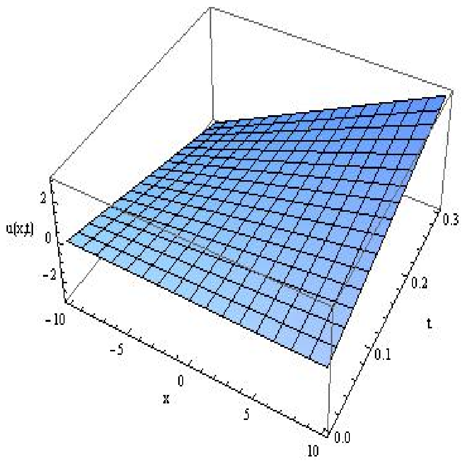
$$u(x,t) = xt - \frac{xt^3}{3!} - \frac{2x^2t^4}{4!} + L^{-1}\left[\frac{1}{s^2}L\left[\frac{\partial^2 u}{\partial x^2} + u + u^2\right]\right]. \tag{4.10}$$

The next step is applying the Differential transformation method on equations (4.9) and (4.10) with respect to space variable x, we get

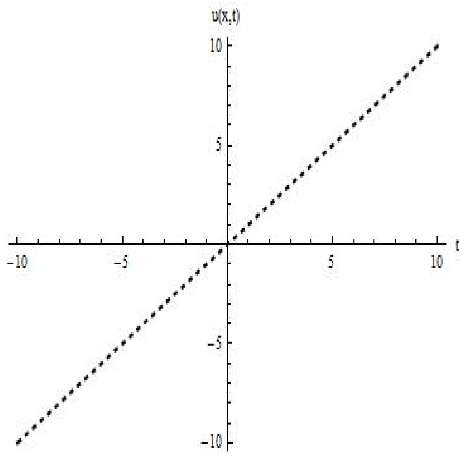
$$U_k(t) = t\delta(k-1,t) - \frac{t^3}{3!}\delta(k-1,t) - \frac{2t^4}{4!}\delta(k-2,t) + L^{-1}\left[\frac{1}{s^2}L\Big[U_k(t)\Big]\right]$$
$$+L^{-1}\left[\frac{1}{s^2}L\Big[(k+2)(k+1)U_{k+2}(t) + \sum_{r=0}^{k} U_r(t)U_{k-r}(t)\Big]\right]; \tag{4.11}$$

where

$$\delta(k-1,t) = \begin{cases} 1 & k=1 \\ 0 & otherwise \end{cases}$$



(a)



(b)

Figure 4.2: (a): 3D plot of u(x, t) and (b): 2D plot of u(x, t) for Example 4.2.

and

$$U_0(t) = 0, \quad U_1(t) = t. \tag{4.12}$$

Substituting (4.12) into (4.11) and the following approximations are obtained successively

$$U_2(t) = 0, \;\; U_3(t) = 0, \;\; U_4(t) = 0, \;\; U_5(t) = 0...$$

Finally, the closed form solution is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)x^k = xt.$$

which is the exact solution.
Example 4.3: Homogeneous wave equation (see Fig. 4.3)
Consider the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - 3u, \tag{4.13}$$

with the initial conditions,

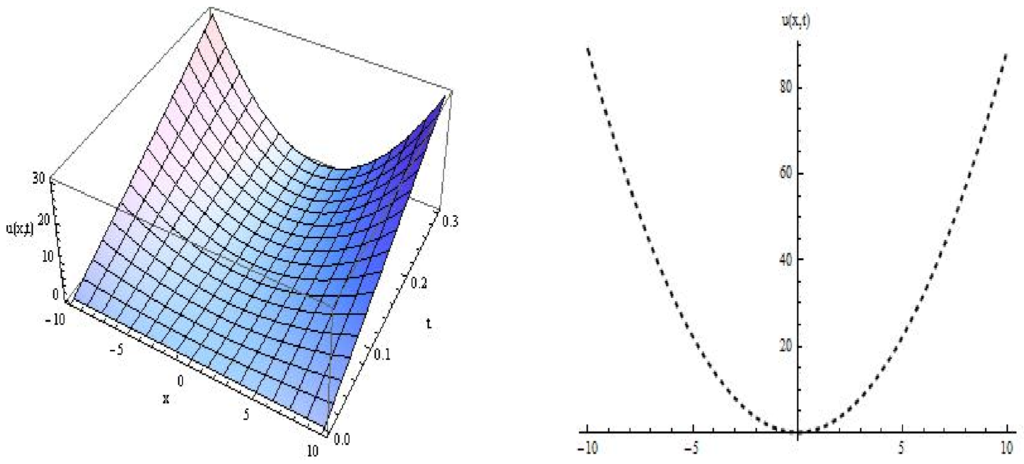
$$u(x,0) = 0, \quad u_t(x,0) = 2cos(x), \tag{4.14}$$

and

$$u(0,t) = sin(2t), \quad u_x(0,t) = 0. \tag{4.15}$$

In this technique, first we apply the Laplace transformation on equation (4.13) with respect to t, therefore, we get

$$s^2 L\big[u(x,t)\big] - su(x,0) - u_t(x,0) = L\left[\frac{\partial^2 u}{\partial x^2} - 3u\right].$$



(a) (b)

Figure 4.3: (a): 3D plot of u(x, t) and (b): 2D plot of u(x, t) for Example 4.3.

By using initial conditions from equation (4.14), we get

$$L\big[u(x,t)\big] = \frac{2cos(x)}{s^2} + \frac{1}{s^2}L\left[\frac{\partial^2 u}{\partial x^2} - 3u\right].$$

Now, we applying the Inverse Laplace transformation w.r.t.'s on both sides:

$$u(x,t) = 2tcos(x) + L^{-1}\left[\frac{1}{s^2}L\left[\frac{\partial^2 u}{\partial x^2} - 3u\right]\right]. \tag{4.16}$$

The next step is applying the Differential transformation method on equations (4.15) and (4.16) with respect to space variable x, we get

$$U_k(t) = \frac{2t}{k!}cos(\frac{k\pi}{2}) + L^{-1}\left[\frac{1}{s^2}L\Big[(k+2)(k+1)U_{k+2}(t) - 3U_k(t)\Big]\right], \tag{4.17}$$

and

$$U_0(t) = sin(2t), \quad U_1(t) = 0. \tag{4.18}$$

Substituting (4.18) into (4.17) and the following approximations are obtained successively

$$U_2(t) = -\frac{1}{2}sin(2t), \quad U_3(t) = 0, \quad U_4(t) = \frac{1}{4!}sin(2t), \quad U_5(t) = 0...$$

Finally, the solution in series form is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t)x^k = sin(2t)\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right].$$

the closed form solution is given as

$$u(x,t) = sin(2t)cos(x).$$

which is the exact solution.

Example 4.4: Wave-like equation in unbounded domain (see Fig. 4.4)

We finally study the wave equation in an unbounded domain.

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \tag{4.19}$$

with the initial conditions,

$$u(x,0) = sin(x), \quad u_t(x,0) = 0, \tag{4.20}$$

and

$$u(0,t) = 0, \quad u_x(0,t) = cos(t). \tag{4.21}$$

In this technique, first we apply the Laplace transformation on equation (4.19) with respect to t, therefore, we get

$$s^2 L\big[u(x,t)\big] - su(x,0) - u_t(x,0) = L\left[\frac{\partial^2 u}{\partial x^2}\right].$$

By using initial conditions from equation (4.20), we get

$$L\big[u(x,t)\big] = \frac{sin(x)}{s} + \frac{1}{s^2} L\left[\frac{\partial^2 u}{\partial x^2}\right].$$

Now, we applying the Inverse Laplace transformation w.r.t.'s on both sides:

$$u(x,t) = sin(x) + L^{-1}\left[\frac{1}{s^2} L\left[\frac{\partial^2 u}{\partial x^2}\right]\right]. \tag{4.22}$$

The next step is applying the Differential transformation method on equations (4.21) and (4.22) with respect to space variable x, we get

$$U_k(t) = \frac{1}{k!} sin(\frac{k\pi}{2}) + L^{-1}\left[\frac{1}{s^2} L\Big[(k+2)(k+1)U_{k+2}(t)\Big]\right], \tag{4.23}$$

and

$$U_0(t) = 0, \quad U_1(t) = cos(t). \tag{4.24}$$

Substituting (4.24) into (4.23) and the following approximations are obtained successively

$$U_2(t) = 0, \ \ U_3(t) = -\frac{1}{3!} cos(t), \ \ U_4(t) = 0, \ \ U_5(t) = \frac{1}{5!} cos(t)...$$

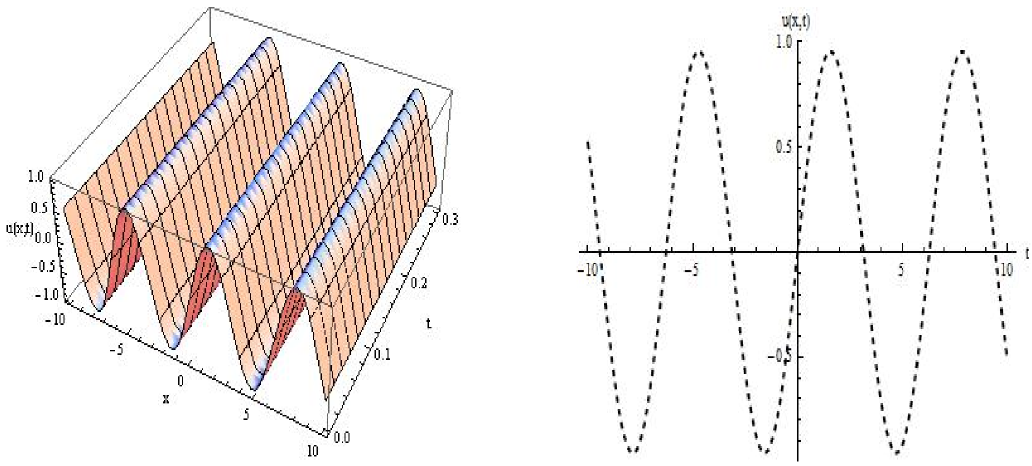
Finally, the solution in series form is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(t) x^k = cos(t)\left[x - \frac{x^3}{3!} + \frac{x^5}{5!}\right].$$

the closed form solution is given as

$$u(x,t) = sin(x)cos(t).$$

which is the exact solution.



(a) (b)

Figure 4.4: (a): 3D plot of u(x, t) and (b): 2D plot of u(x, t) for Example 4.4.

5 Conclusion

In this study, the coupling of Differential Transform Method (DTM) and Laplace Transformation is successfully expanded for the solution of linear & nonlinear Wave equations. The proposed algorithm is suitable for such problems and it gives rapidly converging series solutions. The Laplace differential transform method is more effective and convenient than other methods. The present method reduces the computational work and subsequent results are fully supportive of the reliability, efficiency and accuracy of the suggested scheme. The computations of this paper have been carried out using Mathematica 8.

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