

Original Research Article

Control gain and cost of pinning control for BA scale-free networks

Abstract: This paper investigates the pinning control strategy of BA scale-free network through numerical analysis. A new cost function is defined to measure the cost of pinning control. Compared the control gain and cost of pinning “smallest” nodes and “biggest” nodes, an interesting result is that a smaller control gain and a lower cost are achieved by using the control scheme of pinning nodes with smallest degrees. Moreover, there is a minimal control cost by pinning “smallest” nodes and “biggest” nodes, respectively. The number of controllers by pinning the “smallest” nodes is considered finally.

Key words: BA scale-free network; pinning control; control gain; control cost

1. Introduction

There are many large-scale systems in nature and human societies which can be described by networks where nodes represent individuals in the system and edges represent the connection or interplay among the nodes. More researchers crossing many fields of science, including physics, chemistry, biology, mathematics, are engaged in complex networks in recent years. There are more researches on controlling the dynamics of a network and guiding it to a desired state such as an equilibrium point or a periodic orbit of the network. A special control strategy[1-8], called pinning control by which only a fraction of nodes or even a single node is controlled over the whole network, is an efficient method for dealing with complex network. There are two main problems when pinning the network, how to choose the controlled nodes and how to design the controllers. There are many studies focused on the first problems[4-8] and get a conclusion that pinning the smallest degree nodes is more efficient than that of biggest node(s).

In this paper, the control strategy of pinning the smallest degree nodes and the biggest degree node(s) is analysed from the eigenvalue of controlled network. From this analysis, for a fixed network, the same conclusion is achieved that the smallest degree nodes should be chose to control in the pinning control strategy. Further more, a new control cost function is defined to measure the cost of pinning control. Comparing the control cost between pinning smallest degree nodes and biggest node(s), it is also more effectively by pinning control the smallest degree nodes. Moreover, there is a minimal control cost not only by pinning “smallest” nodes but also by pinning “biggest” node(s). That is, there is an optimal control gains when pinning “smallest” nodes or “biggest” node(s). The number of controllers by pinning the “smallest” nodes is considered finally.

37 2. Pinning control

38 Suppose that a complex network consists of N identical linearly and diffusively coupled
39 nodes with each node being an n -dimensional dynamical system. The state equations of this
40 dynamical network are given by

$$41 \quad \dot{x}_i = f(x_i) + c \sum_{j=1}^N a_{ij} \Gamma x_j, i = 1, 2, \dots, N, \quad (1)$$

42 where $x_i = (x_{i1}, x_{i2}, \dots, x_{in})^T \in R^n$ are the state variables of node i , $f(\cdot)$ is the dynamical
43 function of an isolated node, c is the positive constant representing the coupling strength,
44 $\Gamma = \text{diag}(r_1, r_2, \dots, r_n)$ is the inner coupling matrix if $r_i = 1$ means that two coupled nodes are
45 linked through their i th state variables; otherwise $r_i = 0$.

46 The coupling matrix $A = (a_{ij}) \in R^{N \times N}$ represents the coupling configuration of the
47 network: if there is a connection between node i and node j , then $a_{ij} = a_{ji} = 1$; otherwise,
48 $a_{ij} = a_{ji} = 0$; and the diagonal elements of matrix A are defined by

$$49 \quad a_{ii} = - \sum_{\substack{j=1 \\ j \neq i}}^N a_{ij}, i = 1, 2, \dots, N, \quad (2)$$

50 which ensures the diffusion that $\sum_{j=1}^N a_{ij} = 0$ for all rows.

51 Suppose that we want to stabilize network (1) onto a homogeneous stationary state defined
52 by

$$53 \quad x_1 = x_2 = \dots = x_N = \bar{x}, f(\bar{x}) = 0. \quad (3)$$

54 To achieve the goal (3), we apply the pinning control strategy on a small fraction of the nodes in
55 network (1). Without loss of generality, rearrange the order of the nodes in the network, and let the
56 first l nodes be controlled. Thus, the pinning controlled network can be described by

$$57 \quad \begin{aligned} \dot{x}_i &= f(x_i) + c \sum_{j=1}^N a_{ij} \Gamma x_j + u_i, i = 1, 2, \dots, l, \\ \dot{x}_i &= f(x_i) + c \sum_{j=1}^N a_{ij} \Gamma x_j, i = l+1, l+2, \dots, N, \end{aligned} \quad (4)$$

58 where

$$59 \quad u_i = -cd_i \Gamma(x_i - \bar{x}), i = 1, 2, \dots, l, \quad (5)$$

60 are n -dimensional linear feedback controllers with all the control gains $d_i > 0$.

61 **Lemma:** Consider the controlled network (4) with (5). Suppose that there exists a constant
 62 $\rho < 0$ such that $[Df(\bar{x}) + \rho\Gamma]$ is a Hurwitz matrix. If

$$63 \quad c \geq \frac{|\rho|}{\min \lambda(-A + D)}, \quad (6)$$

64 then the homogeneous stationary state $\bar{x}$ of the controlled network (4) is locally stable, where
 65 $D = \text{diag}(d_1, \dots, d_l, 0, \dots, 0) \in R^{N \times N}$ and $\lambda(-A + D)$ represents the eigenvalue of matrix
 66 $-A + D$.

67 **Remark1:** As shown in Ref.[2], if Γ is a positive definite matrix, then $|\rho| = L_f$
 68 represents the globally stable about the homogenous state $\bar{x}$ and $|\rho| = h_{\max}$ represents the
 69 locally stable, where $L_f > 0$ is the Lipshitz constant of $f(\cdot)$, $h_{\max} > 0$ is the maximum
 70 positive Lyapunov Exponent of chaotic system $\dot{x} = f(x)$.

71 **Remark2:** As shown in Ref.[3], a single controller can pin a coupled complex network to a
 72 homogenous solution if the coupling strength c is large enough.

74 3. Pinning strategy

75 Two well-known strategy of pinning control schemes are randomly and specifically pinning.
 76 As discussed in ref[], controllers are generally preferred to be added to nodes with larger degrees.
 77 However, it is also known that under such pinning schemes, the feedback gains d usually have
 78 to be relatively large. From the viewpoint of realistic applications, using large control gains is not
 79 expected and sometimes cannot be realized. Practically, a designed control strategy should be
 80 effective and also easily implementable. So, we introduce a new concept of cost function to
 81 evaluate the efficiency of our designed control schemes.

82 **Definition**(Control Cost): If under the controller (5), network (4) is stable. The Control Cost
 83 is defined as

$$84 \quad CC = c \sum_{i=1}^l d_i \int_0^{+\infty} \|\Gamma(x_i - \bar{x})\| dt, \quad (7)$$

85 where $\|\cdot\|$ represents the Euclid norm.

86 According to Lemma, for a fixed network or a fixed coupling strength c , our aim is to select
 87 node(s) to control and makes

$$88 \quad \min \lambda(-A + D) \geq \frac{|\rho|}{c}, \quad (8)$$

89 for proper control gains d_i .

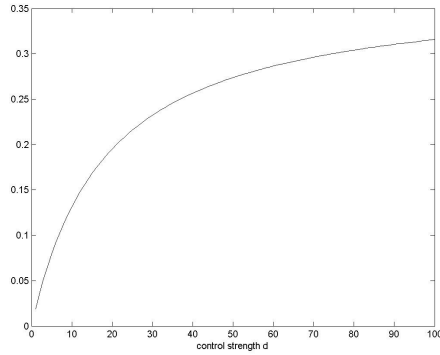
90 In the following simulations, we consider a 50-node BA scale-free network composed of Chen

oscillators with $\Gamma = \text{diag}(1,1,1)$, $|\rho| = h_{\max} = 2.0184$, $d_i = d$. There are only one node with biggest degree 22 and 19 nodes with smallest degree 3.

Fig.1 and Fig.2 show $\min \lambda(-A+D)$ versus the control gains when pinning the “biggest” node and all of the “smallest” nodes, respectively. We can see that for the same control gains d , $\min \lambda(-A+D)$ of controlling smallest degree nodes is bigger than that of controlling the biggest degree nodes. That is, for a fixed coupling strength, we should control the smallest degree nodes.

Fig.3 and Fig.4 show the process of controlling “biggest” node of degree 22 with $d = 30, c = 20$ and all “smallest” nodes of 3 with $d = 1, c = 20$, respectively. The control cost are 4.8079×10^6 and 1.5577×10^6 , respectively. It can be seen controlling nodes with smaller degrees is more efficiency than controlling nodes with larger degrees.

Remark3: When controlling the “biggest” nodes with $c = 20$, the control gain $d = 7$ satisfies Eq.(8), but for apparently efficiency control in 10s, we set $d = 30$.



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105 Fig.1 $\min \lambda(-A+D)$ versus the control gains

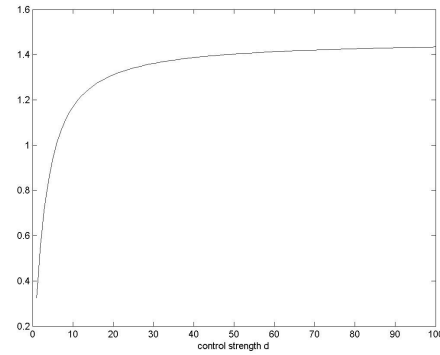
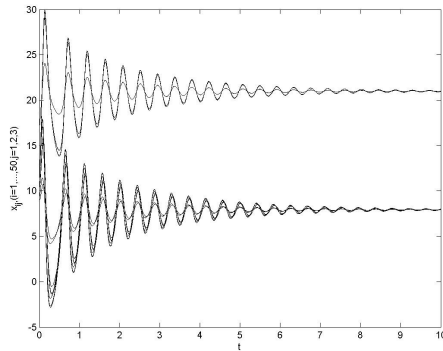


Fig.2 $\min \lambda(-A+D)$ versus the control gains

when pinning the “biggest” node

when pinning the “smallest” nodes



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108 Fig.3 Convergence of the network when controlling

109 “biggest” with $d = 30, c = 20$

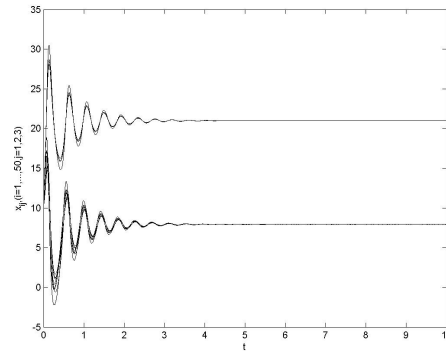


Fig.4 Convergence of the network when controlling

“smallest” nodes with $d = 1, c = 20$

110 4. Control cost and controller number

111 The control cost versus control gains is simulated in Fig.5 and Fig.6 with controlling the
 112 “biggest” node and all “smallest” nodes, respectively. From the two figures we can see that the
 113 control cost decreases rapidly at first, then reaches a minimal value and increases slowly with the
 114 increase of control gains at last. The critical control gains with the minimal control cost can be
 115 chosen as the optimal control gains in the sense of consumed energy. The optimal control gains
 116 and minimal control cost are 210 and 1.5385×10^6 in Fig.5, 11 and 0.6414×10^6 in Fig.6,
 117 respectively.

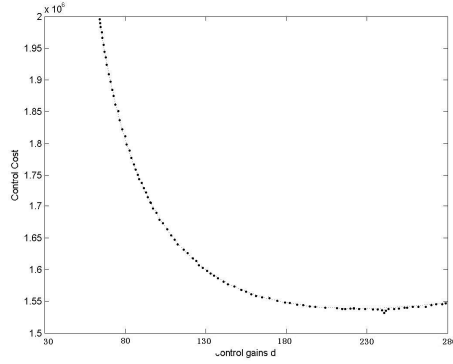


Fig.5 Control cost of pinning the “biggest” node

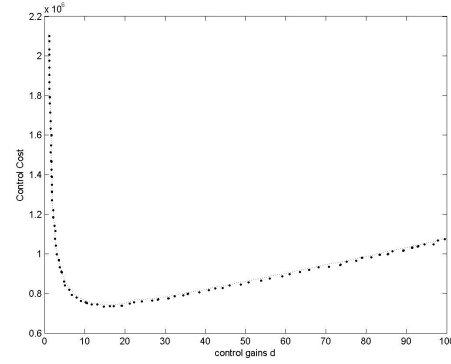


Fig.6 Control cost of pinning the “smallest” node

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121 As shown above, the control scheme of pinning nodes with smaller degrees can be much more
 122 efficient than that of pinning nodes with larger degrees. However, because the number of the
 123 nodes with smaller degrees in a network is often large, the number of controllers to be applied will
 124 also be large if all these nodes are controlled. Thus, the number of controllers by pinning smaller
 125 nodes required to achieve a certain control effect when the coupling strength and control gains are
 126 fixed is a worthy question.

127 Fig.7 shows $\min \lambda(-A + D)$ versus the number of controllers by pinning smaller nodes
 128 when $d = 5, c = 20$. We can see that $\min \lambda(-A + D)$ increases rapidly with the increase of
 129 control number. We also see when pinning 5 “smallest” nodes, Eq.(8) is satisfies. Fig.8 shows the
 130 process of controlling 5 “smallest” nodes of degree 3.

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132 5. Conclusions

133 In this paper, pinning control for BA scale-free networks has been further investigated.
 134 According to the eigenvalue analysis, it is said that controlling the smallest degree nodes is more
 135 effectively than controlling the biggest one; the same conclusion is achieved by analyzing the
 136 control cost based on the new control cost function. The controller number of pinning smallest
 137 degree nodes is investigated in the end.

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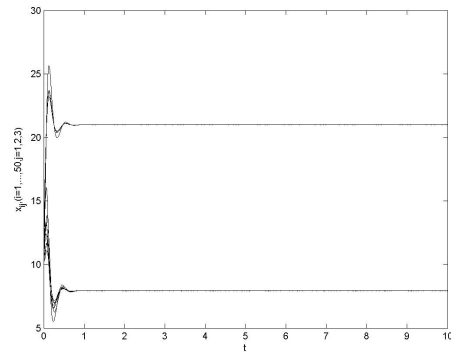
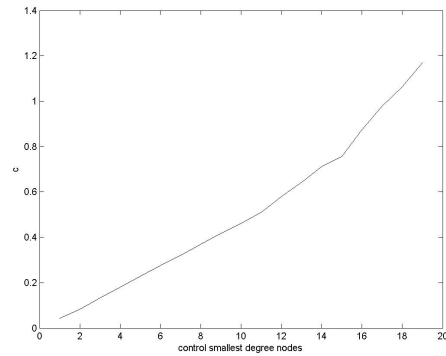


Fig.7 $\min \lambda(-A+D)$ versus the number of controllers Fig.8 Convergence of pinning 5 “smallest” nodes

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