Original Research Article

APROXIMATIONS IN DIVISIBLE GROUPS: PART I

ABSTRACT

We prove a general Dirichlet-type approximation theorem in the setting of Cauchy sequences in normed divisible groups. Essentially, we demonstrate thatthe concept of approximation exponents are extendable to elements belonging to the completion of a normed uniquely-divisible group, where the approximation is given in terms of quasi-order functions on the pre-complete group.

KEYWORDS: Divisible Groups, Cauchy Sequences, Group Norms, Proximity Functions

1.0 INTRODUCTION

The study of group theory naturally leads to problem of finding elements of a group that belongs to cyclic subgroups of the group. It is easy to see that there are groups for which some elements do not belong to any cyclic subgroup other than those generated by the elements; for instance,a prime number does not belong to any cyclic subgroup of the multiplicative group of rational numbers other than that generated by the prime itself.Tostudythegroups for which every element belongs a cyclic group generated by some other element, the notion of *divisible groups* are important. To be precise, a divisiblegroup isa group $(G,\cdot)$ such that for every $g \in G$ and natural number *n* there isan$h \in G$ such that $g = h^n := h \cdot h^{n-1}$—we shall informally say that *G* has *n*-th roots for all *n*. Classically, divisible groups appeared in the theory of Abelian groups; in particular, every Abelian group can be naturally embedded in an Abelian divisible group and an Abelian group is divisible if and only if it is an injective object in the category of Abelian groups (Griffith (1970), Feigelstock (2006), Lang (1984)); moreover in the Abelian,or generally locally nilpotent, torsion-free case Malcev (1949), every divisible group is a*uniquely divisible group*: that is, $g^n = h^n$ implies $g = h$. In any case, non-trivialAbelian divisible groups are not finitely generated, which is easily demonstrable via the Fundamental Theorem of Finitely-generated Abelian groups,anduniquely divisible groupsarenecessarily torsion free. A foremost example is the group of rational numbers $\mathbb{Q}$ under addition. In another but similar vein, given a prime number*p*, a *p*-*divisible* group is a group with *p*-th roots. We extend this further to a subset ϖ of the prime numbersby definingϖ-*divisible* groups as groups with *p*-throots for all *p*inϖ (this is not standard, for instance Baumslag (1958) calls these E_{ϖ}-groups);when ϖ is the whole of the primes, then we get the divisible groups. The archetypal examplesare the additive subgroups of $\mathbb{Q}$ given by$\mathbb{Q}\{\varpi\} = \{q \in \mathbb{Q}: p|\mathrm{D}(q) \Rightarrow p \in$

$\varpi\}$where $\mathrm{D}(q)$ is the denominator of q.We say a group is *uniquely ϖ-divisible* if it is aϖ-divisible group with unique roots. As a further example, if ϖ is all of the prime numbers, then a vector space over a field of characteristic $k$isa well-defined uniquely $\varpi\backslash\{k\}$-divisible group; this latter example shows thatuniquelyϖ-divisible groups can be cyclic groups, torsion groupsor finitely generated groups, in contradistinction to uniquely divisible Abelian groups (the finite fields, being or prime characteristics, are such examples).

2.0 SALIENT EXPOSITION ON NOTATIONANDMOTIVATION

Now given a ϖ-divisiblegroup$(G,\cdot)$, henceforward the notation g^r, where $r \in \mathbb{Q}\{\varpi\}$ and $g \in G$, shall denote (one of possibly many elements)$h \in G$ such that $g^n = h^d$ where $r = n/d$ with$\gcd(n,d) = 1$;in particular,g^rrepresents a unique element in G if G is a uniquely ϖ-divisible group. Now if we denote by $|\cdot|: \mathbb{Q}\{\varpi\} \to \mathbb{R}$ an absolute value function from $\mathbb{Q}\{\varpi\}$ to the real numbers $\mathbb{R}$, then Ostrowski (1916)showed that $|\cdot|$ is, up to equivalence, the usual absolute value $|\cdot|_\infty$ on the real numbers or the usual absolute value $|\cdot|_p$ on the p-adic numbers for a prime p. When$|\cdot| \coloneqq |\cdot|_\infty$, we have the followingclassical elementary but important result:

THEOREM 2.1:*Let $\alpha \in \mathbb{R}$. Then for some $\mu > 1$, there is an infinite sequence $\{r_n\}_{n=1}^{\infty} \in \mathbb{Q}$so that$\alpha \in \mathbb{R}\backslash\mathbb{Q}$ if and only if*$0 < |\alpha - r_n| = O\left(\left(\mathrm{ord}(r_n \; mod \; \mathbb{Z})\right)^{-\mu}\right)$.

Here$\mathbb{R}\backslash\mathbb{Q}$ is the complement of $\mathbb{Q}$ in $\mathbb{R}$—that is, the irrational numbers; the notation$x = O(y)$ implies $|x| \leq My$ for some absolute constant $M > 0$; also,$\mathrm{ord}(h \in H)$ denotes the order (or period) of an element h in the group $H$and $\mathbb{Z}$ denotes the set ofintegers(thus, $\mathrm{ord}(r_n \; mod \; \mathbb{Z})$ gives denominator of r_n). Dirichletproved that in fact with the implied constant being $M = 1$, the theorem holds with $\mu \geq 2$; the optimal situation occurs when $M = 1/\sqrt{5}$ (see Hurwitz (1891)) still with $\mu \geq 2$. An important remark is that the sequence $\{r_n\}_{n=1}^{\infty}$ in the Theorem above is a Cauchy sequence, therefore Theorem 2.1 equally states that there are no Cauchy sequences converging inside $\mathbb{Q}$ withthe given estimate. The object of this paper is to extend the "if" part of the above theorem to uniquely ϖ-divisible groups$G$ and their completionsvia norms,with the estimatesmeasured in termsofquasi-order functions on G.We address the "only if" part in a sequel to this paper. First, we define our main functions:

DEFINITION 2.2(Norm on ϖ-Divisible Groups): For a set of primes ϖ, let $(G,\cdot)$ be a ϖ-divisible group with identity element$e$ and let $|\cdot|: \mathbb{Q}\{\varpi\} \to \mathbb{R}$ be an absolute value function. Then a function $\|\cdot\|: G \to \mathbb{R}$ is a *norm*on G if it satisfies:

i. $\|g\| = 0$only if$g = e$
ii. $\|gh\| \leq \|g\| + \|h\|$
iii. $\|g^r\| = |r|\|g\|, r \in \mathbb{Q}\{\varpi\}$

We denote by $(G,\cdot,\|\cdot\|)$ a ϖ-divisible group with a norm $\|\cdot\|$. If G is Abelian, then it is just a normed linear space but over the integral domain $\mathbb{Q}\{\varpi\}$. Indeed if $(G,+,\|\cdot\|)$ is a normed vector space over a field $\mathbb{F}$, then $|\cdot|$ is the well-defined absolute value function induced by the absolute value function on $\mathbb{F}$ over the vector space.

DEFINITION 2.3 (Proximity Function on Groups): Let G be a group with identity e. Then a function $\varrho: G\backslash\{e\} \to \mathbb{R}$ is a *proximity function* on G iff for all $g \neq h$:

i. $\varrho(g \neq e) = \varrho(g^{-1}) > 0$
ii. $\varrho(gh^{-1}) \leq C\varrho(g)\varrho(h)$
iii. $\varrho(gh^{-1}) \leq C\varrho(g)$ if $\varrho(g) = \varrho(h)$

where $C > 0$ is an absolute constant. If in (ii) we have the stronger bound $\varrho(gh^{-1}) \leq C\max\{\varrho(g), \varrho(h)\}$, then we say ϱ is an *ultra-metric proximity function*. Especially, if ϱ is integer-valued with $C = 1$ and that (ii) and (iii) read $\varrho(gh^{-1}) \mid \operatorname{lcm}\big(\varrho(g), \varrho(h)\big)$ and $\varrho(gh^{-1}) \mid \varrho(g)$ if $\varrho(g) = \varrho(h)$ respectively, then we say ϱ is an *order function*.

We shall typify a proximity function by ϱ with the constant C understood. Obviously the product of two proximity functions is a proximity function; and also if ϱ is a proximity function, then so is ϱ^{μ} for any real number $\mu > 0$; thus we say two proximity functions ϱ_1, ϱ_2 are *equivalent* if $\varrho_1 = \varrho_2^{\mu}$ for some $\mu > 0$.

Examples 2.4:

- For Abelian torsion groups G, the function $\varrho(\cdot) \coloneqq \operatorname{ord}(\cdot)$ is an order function with $C = 1$.

- For groups with ultra-metric norms $\|\cdot\|$, the functions $\varrho(\cdot) \coloneqq \|\cdot\|$ and $\varrho(\cdot) \coloneqq \alpha^{\|\cdot\|}$, where $\alpha \geq 1$ is real, are ultra-metric proximity functions with $C = 1$.

- For groups with bounded norms—that is, $\|\cdot\| \leq M$ with M fixed—the function $\varrho(\cdot) \coloneqq \alpha^{\|\cdot\| - M}$, where $\alpha \geq 1$ is real, is a proximity function with $C = \alpha^{M}$.

- If G is the additive group of an algebraic number field, then the absolute Weil height $h(\cdot) \coloneqq \prod_{v\ place} \max\{1, |\cdot|_v\}$ is a proximity function with $C = 2$.

We shall be interested in those proximity functions ϱ on $(G,\cdot,\|\cdot\|)$ such that for some $\mu_0 > 0$ the function $\varrho(\cdot)^{\mu_0}\|\cdot\|: G\backslash\{e\} \to \mathbb{R}$ is, in essence, discontinuous at the identity e; precisely,

DEFINITION 2.5 (Proximity Function on ϖ-Divisible Groups): Let $(G,\cdot,\|\cdot\|)$ be a normed ϖ-divisible group with identity e and let ϱ be a proximity function on G. Then ϱ is said to be a *close proximity* function on G if there exists a $\mu_0 > 0$ such thatinf$\{\varrho(g_n)^\mu \|g_n\|\} = 0$for some null sequence $\{g_n\}_{n=1}^\infty \subset G\backslash\{e\}$if and only if $\mu < \mu_0$; otherwise, thenϱ is said to be an *openproximity* function on *G*.

REMARKS: Otherwise stated, $\inf\{\varrho(g_n)^\mu \|g_n\|\} > 0$ for all null sequences$\{g_n\}_{n=1}^\infty \subset G\backslash\{e\}$ if and only if$\mu \geq \mu_0$. We typify a close proximity function on *G* by $(\varrho; C, \mu_0)$and in that case we shall say that the elements in *G* are *in close proximit y*(or *in close order*) to each other; else, where necessary, we shall say the elements are*in open proximity*(resp. *in open order*)to each other.

Our interest in close proximity functions on normed ϖ-divisible groups is the following result, which is the main theorem of this paper:

THEOREM 2.6: *Let* $(\varrho; C, \mu_0)$ *be a close proximity function on* $(G,\cdot,\|\cdot\|)$ *and let* $g \in G$. *Then for every* $\mu > \mu_0$ *and Cauchy sequence* $\{g_n\}_{n=1}^\infty \subset G\backslash\{g, e\}$*converging to g, there exists N such that* $\|gg_n^{-1}\| = O(\varrho(g_n)^{-\mu})$ *if and only if* $n \leq N$, *where the implied constant is independent of n or g; moreover, this is also true for* $\mu = \mu_0$ *if*ϱ*is ultra-metric and the implied constant is less than* $\frac{1}{C^{\mu_0}} \inf_{g \neq g_n}\{\varrho(gg_n^{-1})^{\mu_0} \|gg_n^{-1}\|\}$.

In other words, there are only finitely many elements of *G* in close proximity to any element in *G* with respect to the given estimates;or equivalently, Cauchy sequences in *G* do not converge inside *G* with respect to the given estimates.

3.0 PRELIMINARYRESULTS

We establish here some elementary but noteworthy properties of normed ϖ-divisiblegroups endowed with close proximity functions. We also state some close proximity functions on certain ϖ-divisible groups butfirst, we prove the following:

COROLLARY 3.1: *Every normed* ϖ*-divisibleAbelian group is a uniquely* ϖ*-divisible group.*

Proof: Indeed, for some $g \neq h$ suppose $g^n = h^n$ where $n > 1$ is a natural number whose prime divisors belong toϖ. Then $g^n h^{-n} = (gh^{-1})^n = e$, thus

$$|n|\|gh^{-1}\| = \|(gh^{-1})^n\| = \|e\| = 0$$

But$|n| \neq 0$ and so $\|gh^{-1}\| = 0$, implying$gh^{-1} = e$ or $g = h$, a contradiction. **QED**

COROLLARY 3.2:*Any normed* ϖ*-divisible group is non-cyclic and torsion-free*.

Proof: Let $\{g \neq e\}$ generate the group. Then $g^{1/p} = g^n$ for some $p \in \varpi$ and integer $n$and so $g^{pn-1} = e$, implying that g is a torsion element. But if $h \neq e$ is a torsion element with $h^r = e$ for some $r \neq 0$, then $0 = \|e\| = \|h^r\| = |r|\|h\|$. It follows that $\|h\| = 0$or $h = e$, which is a contradiction. Thus there are no torsion elements.**QED**

COROALLRY 3.3: *Let $(G,\cdot,\|\cdot\|)$ be a normed ϖ-divisible group and let $\hat{G}$ be itscompletion with respect to $\|\cdot\|$. Then $\hat{G} \ni \lim_{n\to\infty} g^{r_n}$ where $\{r_n\}_{n=1}^{\infty} \subset \mathbb{Q}\{\varpi\}$ converges in the completion* of $\mathbb{Q}\{\varpi\}$ *with respect to the absolute value $|\cdot|$ associated to $\|\cdot\|$.*

Proof: First,let $\{r_n\}_{n=1}^{\infty} \subset \mathbb{Q}\{\varpi\}$, then for any $g \in G$ we have $\{g^{r_n}\}_{n=1}^{\infty} \subset G$. Thus

$$\|g^{r_n} \cdot (g^{r_m})^{-1}\| = \|g^{r_n - r_m}\| = \|g\||r_n - r_m|$$

Consequently, the sequence $\{g^{r_n}\}_{n=1}^{\infty}$ converges in$\hat{G}$ with respect to (the natural metric induced by) the norm$\|\cdot\|$ ifthe sequence $\{r_n\}_{n=1}^{\infty}$ converges in the completion of $\mathbb{Q}\{\varpi\}$ with respect to (the natural metric induced by) the absolute value $|\cdot|$. **QED**

COROLLARY 3.4: *Let $(\varrho; C, \mu_0)$ be a close proximity function on $(G,\cdot,\|\cdot\|)$. Then there exists an absolute constant $L_\varrho > 0$ such that* $\lim\ \inf_{n\to 0} \varrho(g_n)^{\mu_0}\|g_n\| \geq L_\varrho$ *for every null sequence $\{g_n\}_{n=1}^{\infty} \subset G\backslash\{e\}$.*

Proof: Suppose to the contrary that there exists no such absolute constant L_ϱ. Indeed, then for every integer $m \geq 1$, there is a null sequence $\{g_n(m)\}_{n=1}^{\infty} \subset G\backslash\{e\}$ such that $\lim\ \inf_{n\to\infty} \varrho\big(g_n(m)\big)^{\mu_0}\|g_n(m)\| < 1/m$. It follows that for every *m*there are infinitely many$h_m \in \{g_n(m)\}_{n=1}^{\infty}$sothat $\varrho(h_m)^{\mu_0}\|h_m\| < 1/m$.But since $\{g_n(m)\}_{n=1}^{\infty}$ and $\{g_n(m+1)\}_{n=1}^{\infty}$ are null sequences,then we can choose h_{m+1} suchthat $\|h_{m+1}\| < \|h_m\|$. It then implies that$\{h_m\}_{m=1}^{\infty}$ is a null sequence withinf$\{\varrho^{\mu_0}(h_m)\|h_m\|\} = 0$, whichis acontradiction to the fact thatϱ is a close proximity function. **QED**

COROLLARY 3.5:*A close proximity function on a ϖ-divisible group induces close proximity functionson $\mathbb{Q}\{\varpi\}$.*

*Proof:*Indeed fix a non-identity element*g* belonging to the ϖ-divisible group*G*. Now given any null sequence $\{r_n\}_{n=1}^{\infty} \subset \mathbb{Q}\{\varpi\}\backslash\{0\}$and a close proximity function ϱ on *G*, then $\{g^{r_n}\}_n^{\infty}$ is a null sequence in *G* and thusinf$\{\varrho(g^{r_n})\|g^{r_n}\|\} > 0$. But then $\inf\{\varrho(g^{r_n})\|g^{r_n}\|\} = \|g\| \inf\{\varrho(g^{r_n})|r_n|\}$. Hence if $\varrho_g(r_n) \coloneqq \varrho(g^{r_n})$ then we have $\inf\{\varrho_g(r_n)|r_n|\} > 0$, implying that$\varrho_g$ is a close proximity function on $\mathbb{Q}\{\varpi\}$. Since we can do same for every non-identity element *g* in *G*, the conclusion follows. **QED**

As per examples we state, without verification,three closeproximity functions, which we puttogether in the following lemma. We shall verify these, alongside other close proximity functions, in a sequel to this paper

LEMMA 3.6: *The following are close proximity functions on the respective groups defined:*

(i) *Suppose the absolute value function associated to the normed ϖ-divisible group $(G,\cdot,\|\cdot\|)$ is the usual one on the real numbers. Assume S is a normal subgroup of G such that the quotient group G/S is Abelian and torsion, and that the norm$\|\cdot\|$ is a discrete norm on S—i.e., there is anabsolute constant l suchthat $\|g \in S\backslash\{e\}\| \geq l$. Then the function $\varrho_{G/S}(g) = ord(g \cdot S) \coloneqq min\{n \in \mathbb{Z}_{>0}: g^n \in S\}$ is a close order function on G with $\mu_0 = 1$, $C = 1$; moreover, if ϖ is a singleton set then ϱ is ultra-metric. (We refer to this as a ϖ-ary order function on G).*

(ii) *Given a prime p and the group $\mathbb{Q}\{p\}$, then the function $\varrho_p(q \neq 0) = \lceil p^{\lfloor log(|q|_\infty)/\, log\, p \rfloor} \rceil$ (where $\lfloor\cdot\rfloor$ (resp. $\lceil\cdot\rceil$) denotes the floor (resp. ceiling) function and where $|\cdot|_\infty$ is the usual absolute value on the real numbers) is a close ultra-metric proximity function on $\mathbb{Q}\{p\}$ with $\mu_0 = 1$ and $C = p$ given the usual p-adic norm on $\mathbb{Q}$. (We refer to this proximity function as the p-adic proximity function on $\mathbb{Q}\{p\}$).*

(iii) *For an algebraic number field $\mathbb{K}$ with the usual normalised absolute values $|\cdot|_v$ over all places v such that $\prod_v |\alpha|_v = 1$ for every $\alpha \in \mathbb{K}\backslash\{0\}$, the function $\varrho_{\mathbb{K}}(\alpha) \coloneqq \prod_v max\{1, |\alpha|_v\}$—i.e., the Weil height—is a close proximity function on $\mathbb{K}^+$ with $\mu_0 = 1$ and $C = 2$ given the normdefined by the usual absolute value on the complex numbers. (We shall refer to this as the $\mathbb{K}$-proximity function).*

EXAMPLE 3.7: A particular example of case (i) above is given by $G = \mathbb{Q}\{\varpi\}$and $S = \mathbb{Z}$, where the function $\varrho_{G/S}$ is a close order function on $\mathbb{Q}\{\varpi\}$ given the usual norm on the real numbers. Indeed $|n \in \mathbb{Z}| \geq 1$ and so $|\cdot|$ is discrete on $\mathbb{Z}$. On the other hand, a non-example is given by$G = \mathbb{Q}_\varpi^\times$, the multiplicative group of (the positive real values of the) $\mathbb{Q}\{\varpi\}$-powers of the positive rational numbers $\mathbb{Q}_{>0} \coloneqq S$ with norm $\|\cdot\| \coloneqq |\log(\cdot)|$—that is, $\mathbb{Q}_\varpi^\times \coloneqq \{q^r \in \mathbb{R}_{>0}: q \in \mathbb{Q}_{>0}, r \in \mathbb{Q}\{\varpi\}\}$.Here the so-defined ϖ-ary order function $\varrho_{G/S}$ is an open order function on $\mathbb{Q}_\varpi^\times$. This is so, obviously, as the norm is not a discrete normon $\mathbb{Q}_{>0}$;indeed, for instance, $\left\{1 + \frac{1}{n}\right\}_{n=1}^{\infty} \subset \mathbb{Q}_{>0}$ and yet $\left|\log\left(1 + \frac{1}{n}\right)\right| \to 0$ as $n \to \infty$.

4.0 PROOF OF MAIN RESULTS

We now establish the main resultsof this paper, culminating in the proof of the main theorem stated in the introduction. We start with the following lemma.

Lemma 4.1: *Let $(\varrho; C, \mu_0)$ be aclose proximity function on $(G, \cdot, \|\cdot\|)$. Then for every distinguished Cauchy sequence $\{g_n\}_{n=1}^{\infty} \subset G\backslash\{e\}$ (i.e., $g_n \neq \lim_{n\to\infty} g_n$ for all n) we have* $\lim_{n\to\infty} \varrho(g_n) = \infty$.

Proof: Given that$\{g_n\}_{n=1}^{\infty}$ is distinguished and Cauchy, then it contains an infinite subsequence of distinct elements; thus for every $\varepsilon > 0$, there exists N such that for all $m, n \geq N$where $g_m \neq g_n$we have $0 < \|g_m g_n^{-1}\| < \varepsilon$;in that casesince$L \coloneqq \inf\{\varrho(g_m g_n^{-1})^{\mu}\|g_m g_n^{-1}\|\} > 0$for every $\mu \geq \mu_0$, then it follows that

$$\left(C\varrho(g_m)\varrho(g_n)\right)^{\mu} \geq \varrho(g_m g_n^{-1})^{\mu} \geq \frac{\inf \varrho(g_m g_n^{-1})^{\mu}\|g_m g_n^{-1}\|}{\|g_m g_n^{-1}\|} = \frac{L}{\|g_m g_n^{-1}\|} > \frac{L}{\varepsilon}$$

But$\lim_{\varepsilon\to 0}\frac{L}{\varepsilon} = L\lim_{\varepsilon\to 0}\frac{1}{\varepsilon} = \infty$, thus$\lim_{\substack{m,n\geq N\to\infty \\ g_m\neq g_n}}\left(C\varrho(g_m)\varrho(g_n)\right)^{\mu} = \infty$. Now suppose to the contrary that $\lim\,\inf_{n\to\infty}\varrho(g_n) < \infty$.It follows that there exists an infinite subsequence of $\{g_n\}_{n=1}^{\infty}$, say $\{g_n^*\}_{n=1}^{\infty}$, such that $\varrho(g_n^*) \leq U$ for some upper bound U.But since $\{g_n\}_{n=1}^{\infty}$ is a distinguished Cauchy sequence, then $\{g_n^*\}_{n=1}^{\infty}$ is also a distinguished Cauchy sequence converging to the same limit,thus (by the same argument as above)we have $\lim_{\substack{m,n\geq N\to\infty \\ g_m^*\neq g_n^*}}\left(C\varrho(g_m^*)\varrho(g_n^*)\right)^{\mu} = \infty$.But then given any disjoint partitions A and B of $\{g_n^*\}_{n=1}^{\infty}$—i.e. $A \cup B = \{g_n^*\}_{n=1}^{\infty}butA \cap B = \emptyset$—then we arrive at

$$\lim_{\substack{m,n\geq N\to\infty \\ g_m^*\neq g_n^*}}\left(C\varrho(g_m^*)\varrho(g_n^*)\right)^{\mu} = C^{\mu}\left(\lim_{\substack{m\geq N\to\infty \\ g_m^*\in A}}\varrho(g_m^*)^{\mu}\right)\left(\lim_{\substack{n\geq N\to\infty \\ g_n^*\in B}}\varrho(g_n^*)^{\mu}\right) \leq (CU^2)^{\mu}$$

which is a contradiction to the fact that left-hand side is unbounded. Consequently, $\lim\,\inf_{n\to\infty}\varrho(g_n) = \infty$ and so $\lim_{n\to\infty}\varrho(g_n) = \infty$. **QED**

Theorem4.2: *Let $(\varrho; C, \mu_0)$ be a closeproximity function on $(G, \cdot, \|\cdot\|)$ with$\hat{G}$as itscompletion. Let$\{g_n\}_{n=1}^{\infty} \subset G\backslash\{e\}$be a Cauchy sequence converging to$\hat{g} \in \hat{G}$so that*$0 < \|\hat{g}g_n^{-1}\| = O(\varrho(g_n)^{-\mu})$*for all n, where $\mu > \mu_0$. Thenfor all sufficiently large m and n,$\varrho(g_m) = \varrho(g_n)$if and only if$g_m = g_n$; moreover, thisis also true for $\mu = \mu_0$ ifthe implied constant is less than* $\frac{1}{2C^{\mu_0}}\inf_{g_m\neq g_n}\{\varrho(g_m g_n^{-1})^{\mu_0}\|g_m g_n^{-1}\|\}$.

*Proof:*Let M be the implied constant in the estimate $O(\varrho(g_n)^{-\mu})$. Now from the sub-additivity of $\|\cdot\|$, we have $\|g_m g_n^{-1}\| \leq \|g_m\hat{g}^{-1}\| + \|\hat{g}g_n^{-1}\| = \|\hat{g}g_m^{-1}\| + \|\hat{g}g_n^{-1}\| \leq M\varrho(g_m)^{-\mu} + M\varrho(g_n)^{-\mu}$. Let us assume that$\varrho(g_m) = \varrho(g_n)$ but that $g_m \neq g_n$.Thus$\|g_m g_n^{-1}\| \leq 2M\varrho(g_n)^{-\mu}$or equivalently $\varrho(g_n)^{\mu-\mu_0}\varrho(g_n)^{\mu_0}\|g_m g_n^{-1}\| \leq 2M$ andsince$\varrho(g_m g_n^{-1}) \leq C\varrho(g_n)$then$\varrho(g_n)^{\mu-\mu_0}(\varrho(g_m g_n^{-1})^{\mu_0}\|g_m g_n^{-1}\|) \leq 2C^{\mu_0}M$.Finally, via the bound $\varrho(g_m g_n^{-1})^{\mu_0}\|g_m g_n^{-1}\| \geq \inf_{g_m\neq g_n}\{\varrho(g_m g_n^{-1})^{\mu_0}\|g_m g_n^{-1}\|\} \coloneqq L$, then we arrive at$\varrho(g_n)^{\mu-\mu_0} \leq \frac{1}{L}2C^{\mu_0}M$ and as such$\varrho(g_n)$ is bounded above by$\left(\frac{1}{L}2C^{\mu_0}M\right)^{1/(\mu-\mu_0)}if\mu>\mu_0$or that$M\geq\frac{L}{2C^{\mu_0}}$when$\mu=\mu_0$. Henceif $\mu>\mu_0$, then$g_m=g_n$if$\varrho(g_m)=\varrho(g_n)>\left(\frac{1}{L}2C^{\mu_0}M\right)^{1/(\mu-\mu_0)}$, which latter condition holds for all sufficiently large *n*due toLemma4.1; similarlyif$\mu=\mu_0$ and $M<L/2C^{\mu_0}$,then necessarily$g_m=g_n$if$\varrho(g_m)=\varrho(g_n)$. Hence the proof.**QED**

THEOREM 4.3: *Let* $(\varrho; C, \mu_0)$ *be a close proximity function on* $(G,\cdot,\|\cdot\|)$ *and let* $g\in G$. *Then for every* $\mu>\mu_0$ *and Cauchy sequence* $\{g_n\}_{n=1}^{\infty}\subset G\backslash\{g,e\}$*converging to g, there exists N such that* $\|gg_n^{-1}\|=O(\varrho(g_n)^{-\mu})$ *if and only if* $n\leq N$, *where the implied constant is independent of n or g; moreover, this is also true for* $\mu=\mu_0$ *if* ϱ*is ultra-metric and the implied constant is less than* $\frac{1}{C^{\mu_0}}\inf_{g\neq g_n}\{\varrho(gg_n^{-1})^{\mu_0}\|gg_n^{-1}\|\}$.

*Proof:*Given $\|gg_n^{-1}\|\leq M\varrho(g_n)^{-\mu}$ for some absolute constant$M$, then multiplying through by $\left(\varrho(g)\varrho(g_n)\right)^{\mu_0}$ gives us$\varrho(g_n)^{\mu-\mu_0}\left(\varrho(g)\varrho(g_n)\right)^{\mu_0}\|gg_n^{-1}\|\leq M\varrho(g)^{\mu_0}$. But$\varrho(gg_n^{-1})\leq C\varrho(g)\varrho(g_n)$, hence$\varrho(g_n)^{\mu-\mu_0}\varrho(gg_n^{-1})^{\mu_0}\|gg_n^{-1}\|\leq C^{\mu_0}M\varrho(g)^{\mu_0}$.Since$g\notin\{g_n\}_{n=1}^{\infty}$, then for some infimum L we have $L\leq\varrho(gg_n^{-1})^{\mu_0}\|gg_n^{-1}\|$;thus$\varrho(g_n)^{\mu-\mu_0}\leq C^{\mu_0}M\varrho(g)^{\mu_0}/L$and as suchfor $\mu>\mu_0$ it follows that $\varrho(g_n)$ is bounded above by $(C^{\mu_0}M\varrho(g)^{\mu_0}/L)^{\mu_0/(\mu-\mu_0)}$.Hence Lemma 4.1 tells us that there is no distinguished Cauchy sequence $\{g_n\}_{n=1}^{\infty}$ converging to g and satisfying the estimate in the lemma,so we can choose $N\coloneqq\max\left\{n:\varrho(g_n)\leq(C^{\mu_0}M\varrho(g)^{\mu_0}/L)^{\mu_0/(\mu-\mu_0)}\right\}$. Nowlet$\mu=\mu_0$ withϱbeingultra-metric andsuppose$\varrho(g_n)>\varrho(g)$suchthat $\|gg_n^{-1}\|\leq M\varrho(g_n)^{-\mu_0}$. Here,note that$\varrho(gg_n^{-1})\leq C\max\{\varrho(g_n),\varrho(g)\}=C\varrho(g_n)$ and consequently we have$L\leq\varrho(gg_n^{-1})^{\mu_0}\|gg_n^{-1}\|\leq C^{\mu_0}\varrho(g_n)^{\mu_0}\|gg_n^{-1}\|\leq C^{\mu_0}M$, implying that$M\geq L/C^{\mu_0}$; hence if we require that $M<L/C^{\mu_0}$, then necessarily we must have the bound $\varrho(g_n)\leq\varrho(g)$. It thus follows from Lemma 4.1 that there is no distinguished Cauchy sequence $\{g_n\}_{n=1}^{\infty}$ converging to g and satisfying the estimate in the Lemma. In this case we can choose $N\coloneqq\max\{n:\varrho(g_n)\leq\varrho(g)\}$.**QED**

5.0 CONCLUSION

In conclusion, we note that if a close proximity function exhibits the extra property of being uniform—that is, if there exists some absolute constant $L_\varrho>0$ such that $\varrho(g_n)^{\mu_0}\|g_n\|\geq L_\varrho$ for every null sequence $\{g_n\}_{n=1}^{\infty}\subset G\backslash\{e\}$—then the latter parts of Theorems 1 and 2 would have$\frac{1}{2C^{\mu_0}}L_\varrho$ and $\frac{1}{C^{\mu_0}}L_\varrho$ respectively instead of the terms $\frac{1}{2C^{\mu_0}}\inf_{g\neq g_n}\{\varrho(gg_n^{-1})^{\mu_0}\|gg_n^{-1}\|\}$ and $\frac{1}{C^{\mu_0}}\inf_{g\neq g_n}\{\varrho(gg_n^{-1})^{\mu_0}\|gg_n^{-1}\|\}$. In this way, the implied constants in the theorems above would be independent of *n* or *G* when $\mu=\mu_0$. We make use of this uniformity in the sequel to this paper.

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