Original Research Article

APROXIMATIONS IN DIVISIBLE GROUPS: PART II

ABSTRACT

We verify some assertions in the prequel to this paper, in which certain functions which are referred to as proximity functions were introduced in order to study Dirichlet-type approximations in normed divisible groups and similar groups that enjoy a form of divisibility, for instance *p*-divisible groups.

Keywords: Divisible Groups, Cauchy Sequences, Group Norms, Proximity Functions

1.0 INTRODUCTION

A *divisible* group $(G,.)$ is defined as a group such that for every $g \in \{G\}$ and natural number *n* there is an $h \in \{G\}$ suchthat $g = h^n := h.h^{n-1}$; informally, we say that G has *n*-th roots for all *n*. A foremost example is the group of rational numbers $\mathbb{Q}$under addition. Similarly, *p-divisible* group is a group with *p*-th roots. Now let ϖ denote a subset of the prime numbers $\{2,3,5,7,\ldots\}$. In the prequel [1] to this paper, we studied the ϖ-*divisible* groups, which are groups with *p*-th roots for all $p \in \varpi$. Archetypal examples are the additive subgroups of $\mathbb{Q}$ given by $\mathbb{Q}\{\varpi\} = \{q \in \mathbb{Q}: p|\mathrm{D}(q) \Rightarrow p \in \varpi\}$where $\mathrm{D}(q)$ is the denominator of *q*. We say a group is uniquely ϖ-divisible if it is a ϖ-divisible group with unique roots. We recall the following definitions given in [1]:

DEFINITION 1.1 (Norm on ϖ-Divisible Groups): For a set of primes ϖ, let $(G,\cdot)$ be a ϖ-divisible group with identity element *e* and let $|\cdot|: \mathbb{Q}\{\varpi\} \to \mathbb{R}$ be an absolute value function. Then a function $\|\cdot\|: G \to \mathbb{R}$ is a *norm* on *G* if it satisfies:

i. $\|g\| = 0$ only if $g = e$
ii. $\|gh\| \leq \|g\| + \|h\|$
iii. $\|g^r\| = |r|\|g\|, r \in \mathbb{Q}\{\varpi\}$

The absolute value$|\cdot|: \mathbb{Q}\{\varpi\} \to \mathbb{R}$, essentially via Ostrowski's Theorem [2], is the usual one on the real numbers or on the *p*-adic numbers.We denote by $(G,\cdot,\|\cdot\|)$ a ϖ-divisible group with a norm $\|\cdot\|$.

DEFINITION1.2(Proximity Function on Groups): Let *G* be a group with identity *e*. Then a function $\varrho: G\backslash\{e\} \to \mathbb{R}$ is a *proximity function* on *G* if for all $g \neq h$:

i. $\varrho(g \neq e) = \varrho(g^{-1}) > 0$
ii. $\varrho(gh^{-1}) \leq C\varrho(g)\varrho(h)$

iii. $\varrho(gh^{-1}) \leq C\varrho(g)$ if $\varrho(g) = \varrho(h)$

where $C > 0$ is an absolute constant. If in (ii) we have the stronger bound $\varrho(gh^{-1}) \leq C\max\{\varrho(g), \varrho(h)\}$, then we say ϱ is an *ultra-metric proximity function*. Furthermore, if ϱ is integer-valued with $C = 1$ and that (ii) and (iii) read $\varrho(gh^{-1}) | \operatorname{lcm}(\varrho(g), \varrho(h))$ and $\varrho(gh^{-1}) | \varrho(g)$ if $\varrho(g) = \varrho(h)$ respectively, then we say ϱ is an *order function*.

For Abelian torsion groups G, the function $\varrho(.) = \operatorname{ord}(.)$ is an orderfunction (seeExample 1.4 in [1] for more examples).

DEFINITION1.3(Proximity Function on Normed ϖ-Divisible Groups): Let $(G,\cdot,\|\cdot\|)$ be a normed ϖ-divisible group with identity *e* and let ϱ be a proximity function on *G*. Then ϱ is said to be a *close proximity* function on *G* if there exists a $\mu_0 > 0$ such that $\inf\{\varrho(g_n)^{\mu}\|g_n\|\} = 0$ for some null sequence $\{g_n\}_{n=1}^{\infty} \subset G\backslash\{e\}$ if and only if $\mu < \mu_0$; otherwise, then ϱ is an *openproximity* function on *G*.We shall say that the elements in *G* are *in close proximity* (and*in close order*) to each other; else, where necessary, we shall say the elements are *in open proximity* (resp. *in open order*) to each other.

We typify a close proximity function on *G*by $(\varrho; C, \mu_0)$. The main result proved in [1] is the following theorem.

THEOREM 1.4: *Let* $(\varrho; C, \mu_0)$ *be a close proximity function on* $(G,\cdot,\|\cdot\|)$ *and let* $g \in G$. *Then for every* $\mu > \mu_0$ *and Cauchy sequence* $\{g_n\}_{n=1}^{\infty} \subset G\backslash\{g, e\}$*converging to g, there exists N such that* $\|gg_n^{-1}\| = O(\varrho(g_n)^{-\mu})$ *if and only if* $n \leq N$, *where the implied constant is independent of n or g; moreover, this is also true for* $\mu = \mu_0$ *if* ϱ*is ultra-metric and the implied constant is less than* $\frac{1}{C^{\mu_0}} \inf_{g \neq g_n}\{\varrho(gg_n^{-1})^{\mu_0}\|gg_n^{-1}\|\}$.

Theorem 1.4 implies that there can be only finitely many elements ofG in close proximity to any element in G with respect to the given estimates; or equivalently, Cauchy sequences in G do not converge insideG with respect to the given estimates. A converse to this theorem,would give a Dirichlet-type approximation for (incomplete) ϖ-divisiblegroups. In the present paper, we verify some assertions on examples ofproximity functions given in [1].

2.0 PRELIMINARIES

We require the following definitions and results. A norm $\|.\|$ on an arbitrary group G with identity *e* is said to be *discrete* if

(1) $\|.\|: G \to \mathbb{R}_{\geq 0}$
(2) $\|ab\| \leq \|a\| + \|b\|, \forall a, b \in G$

(3) $\|a^n\| = |n|\|a\|, a \in G, n \in \mathbb{Z}$

(4) $\inf_{a \in G\{e\}} \|a\| > 0$

Let $\mathbb{K}$ be an algebraic number field and let $\overline{\mathbb{Q}}$ be the field of algebraicnumbers. Theabsolute Weil height $h: \mathbb{K} \to \mathbb{R}_{\geq 0}$is given by

$$h(\cdot) \coloneqq \prod_{v} \max\{1, |\cdot|_v\}$$

where v runs through all places of $\mathbb{K}$ and $|\cdot|_v$is a normalised absolutevalue, hence$\prod_v |\alpha|_v = 1$. We know (see [3]) that $h(\alpha\beta) \leq 2h(\alpha)h(\beta)$and also$h(\alpha^{-1}) = h(\alpha)if\alpha \neq 0$.

The p-adic norm $|\cdot|_p$of a rational number $q = \frac{a}{b}$, where a, b areintegers with $b \neq 0$ is given by

$$|q|_p = p^{-\left(v_p(a) - v_p(b)\right)}$$

where$p^{v_p(a)}$is the greatest power dividing a and similarly $p^{v_p(b)}$is the greatest power dividing b.

3.0 MAIN RESULT

We now establish the main resultof this paper, which was stated without proof in [1].

LEMMA 3.1: *The following are close proximity functions on the respective groups defined:*

(i) *Suppose the absolute value function associated to the normed ϖ-divisible group $(G, \cdot, \|\cdot\|)$ is the usual one on the real numbers. Assume S is a normal subgroup of G such that the quotient group G/S is Abelian and torsion, and that the norm$\|\cdot\|$ is a discrete norm on S—i.e., there is an absolute constant l suchthat $\|g \in S \backslash \{e\}\| \geq l$. Then the function $\varrho_{G/S}(g) = ord(g \cdot S) \coloneqq min\{n \in \mathbb{Z}_{>0} : g^n \in S\}$ is a close order function on G with $\mu_0 = 1$, $C = 1$; moreover, if ϖ is a singleton set then ϱ is ultra-metric. (We refer to this as a ϖ-ary order function on G).*

(ii) *Given a prime p and the group $\mathbb{Q}\{p\}$, then the function $\varrho_p(q \neq 0) = \left\lceil p^{\lfloor log(|q|_\infty)/\, log\, p \rfloor} \right\rceil$ (where $\lfloor\cdot\rfloor$ (resp. $\lceil\cdot\rceil$) denotes the floor (resp. ceiling) function and where $|\cdot|_\infty$ is the usual absolute value on the real numbers) is a close ultra-metric proximity function on $\mathbb{Q}\{p\}$ with $\mu_0 = 1$ and $C = p$ given the usual p-adic norm on $\mathbb{Q}$. (We refer to this proximity function as the p-adic proximity function on $\mathbb{Q}\{p\}$).*

(iii) *For an algebraic number field $\mathbb{K}$ with the usual normalised absolute values $|\cdot|_v$ over all places v such that $\prod_v |\alpha|_v = 1$ for every $\alpha \in \mathbb{K}\backslash\{0\}$, the function $\varrho_{\mathbb{K}}(\alpha) \coloneqq \prod_v max\{1, |\alpha|_v\}$—i.e., the Weil height—is a close*

proximity function on $\mathbb{K}^+$ with $\mu_0 = 1$ and $C = 2$ given the norm defined by the usual absolute value on the complex numbers. (We shall refer to this as the $\mathbb{K}$-proximity function).

Proof. For (i), it is easy to see that since $\varrho_{G/S}(g) = ord(g \cdot S) \coloneqq min\{n \in \mathbb{Z}_{>0}: g^n \in S\}$, that is since $\varrho_{G/S}$ denotes the order of agroup, then straightforwardly, it suffices for the definition of a proximity(indeed, an order function). To see that it is a close order function, welet $\{g_n\}_{n\geq 1} \subset G\backslash\{e\}$be any null sequence; then we observe that for$\mu \geq \mu_0 = 1$, we have

$$\inf\{\varrho(g_n)^{\mu}\|g_n\|\} \geq \inf\|g_n\| > 0$$

which is so since $\varrho(g_n) \geq 1$.

For (ii), we observe that for $q \neq r$and $q, r \neq 0$, we have

$$\varrho_p(q) = \left\lceil p^{\lfloor log(|q|_\infty)/\, log\, p \rfloor} \right\rceil = \left\lceil p^{\lfloor log(|-q|_\infty)/\, log\, p \rfloor} \right\rceil = \varrho_p(-q)$$

and

$$\begin{aligned} \varrho_p(q-r) &= \left\lceil p^{\lfloor log(|q-r|_\infty)/\, log\, p \rfloor} \right\rceil \\ &\leq \left\lceil p^{\lfloor log(|q|_\infty) + log(|r|_\infty)/\, log\, p \rfloor} \right\rceil \\ &\leq \left\lceil p^{1+\lfloor log(|q|_\infty) + log(|r|_\infty)/\, log\, p \rfloor} \right\rceil \\ &\leq p \left\lceil p^{\lfloor log(|q|_\infty)/\, log\, p \rfloor} \right\rceil \left\lceil p^{\lfloor log(|r|_\infty)/\, log\, p \rfloor} \right\rceil \\ &= p\varrho_p(q)\varrho_p(r) \end{aligned}$$

If $\varrho_p(q) = \varrho_p(r)$, we easily see that $\varrho_p(q-r) \leq p\varrho_p(q)$. Finally, if$\{q_n\}_{n\geq 1} \subset \mathbb{Q}\{p\}$ is a non-zero null sequence, the we see that for all$\mu \geq \mu_0$= 1 and with the p-adic norm $|.|_p$, we have

$$\inf\left\{\varrho_p(q_n)^{\mu}|q_n|_p\right\} \geq 1$$

which is so since by definition we have the inequality $\varrho_p(q) \geq |q|_p^{-1}$.

For (iii), we know that

$$\varrho_{\mathbb{K}}(\alpha) = \varrho_{\mathbb{K}}(\alpha^{-1})$$

and that

$$\varrho_{\mathbb{K}}(\alpha\beta^{-1}) \leq 2\varrho_{\mathbb{K}}(\alpha)\varrho_{\mathbb{K}}(\beta^{-1}) = 2\varrho_{\mathbb{K}}(\alpha)\varrho_{\mathbb{K}}(\beta)$$

It is easy to see that $\varrho_{\mathbb{K}}(\alpha\beta^{-1}) \leq 2\varrho_{\mathbb{K}}(\alpha)$when $\varrho_{\mathbb{K}}(\alpha) = \varrho_{\mathbb{K}}(\beta)$. Finally,if $\{\alpha_n\}_{n\geq 1} \subset \mathbb{K}$ is a non-zero null sequence, then for all $\mu \geq \mu_0$= 1and norm $|.|$, we have

$$\inf\{\varrho_{\mathbb{K}}(\alpha_n)^{\mu}|\alpha_n|\} \geq 1$$

which is so since normalisation of absolute values implies that

$$|\alpha_n|\varrho_{\mathbb{K}}(\alpha_n) \prod_{\substack{v \\ |\alpha_n|_v < 1}} |\alpha_n|_v = 1$$

which completes the proof.

References

1. Ezearn, J. and Obeng-Denteh, W., (2015): *Approximations in Divisible Groups:Part I*, Physical Sciences Journal International, **6** (2), 112-118.

2. Ostrowski, A., (1916):*ÜbereinigeLösungen der Funktionalgleichung$\varphi(x)\cdot\varphi(y)=\varphi(xy)$*. Acta Mathematica (2nd ed.) **41** (1):271–284.

3. Waldschmidt, M., (2000): “Diophantine approximation on linear algebraicgroups”, Grundlehren 326, Springer.