

Algorithm for Internal Rate of Return

ABSTRACT

Internal Rate of Return (IRR) is defined as the discount rate that would make Net Present Value (NPV) zero. Many favor IRR over other methods of ranking projects because it considers the time value of money, cash generated over the life of a project, and because IRR may be directly compared to external investments.

The drawback to IRR is that it requires solution of non-linear equations beyond the ability of most finance and accounting practitioners. Textbooks recommend trial and error to find a project's IRR. In the alternative, they recommend use of financial calculators. The problem with financial calculators or preprogramed spreadsheet functions is that they are not transparent. Without transparency, it is impossible to confirm that IRR problems have been properly framed, that data has been input correctly, and that calculations are valid.

This paper provides a transparent, testable algorithm for computing IRR that can be implemented with spreadsheet software.

Keywords: Internal Rate of Return, Capital Budgeting, Return on Capital

1. INTRODUCTION

Allocation of limited capital is an ongoing issue for many businesses. Techniques for ranking projects include (i) payback, (ii) discounted payback, (iii) net present value, and (iv) internal rate of return.

Payback is the number of years it takes to recover invested capital. It does not consider the time value of money or cash inflows after payback is reached. Discounted payback is the number of years it takes to recover invested capital if cash inflows are discounted to their present value. It does not consider cash flows after payback is reached.

Net present value is the present value of cash inflows less the present value of cash outflows. It is favored over payback and discounted payback by many analysts because it considers cash flows generated over a project's life, not just cash flows needed to recover invested capital. Cash flows are discounted at a company's cost of capital. Net Present Value (NPV) represents the new wealth created by a project. However, ranking projects by NPV does not always provide the best allocation of capital.

Suppose project A had a net present value of \$100,000 and project B had a net present value of \$200,000 which should a company fund? Given net present value alone, B would be selected. Now suppose project A required \$500,000 in capital and project B required \$5,000,000 in capital. Now which one should a company select? Given the additional information, it is clear that project A provides a greater return on invested capital.

Internal rate of return is defined as the discount rate that will make the present value of cash inflows and outflows equal. This is another way of saying that IRR is the discount rate that will drive NPV to zero. IRR is favored over net present value because it considers the present value of returns in relationship to invested capital and it is directly comparable to external investments. The drawback to IRR is that it requires solution of non-linear equations beyond the skill of most accounting and finance practitioners.

Many articles and textbooks recommend finding IRR through trial and error [1][2][3][4]. Trial and error is daunting for complex projects. It may require writing custom software and it is intellectually unsatisfying.

Some articles have suggested Newton's Method [5] which is fine if you took calculus and remember it. Other methods have been proposed for computing IRR including Hillier's method and a Taylor series expansion [6]. These methods require familiarity with advanced mathematics.

The alternative to such complex math is use of preprogramed spreadsheet functions like Excel's IRR or XIRR or financial calculators. The problem with financial calculators is lack of transparency. What exactly is the calculator doing? How is it doing it? How can we confirm there are no input errors? Spreadsheets raise the same questions. It seems careless to make multimillion dollar decisions based on results that cannot be validated.

This paper develops a transparent algorithm for computing the internal rate of return that does not resort to complex mathematics, financial calculators or preprogrammed spreadsheet functions. Spreadsheets can be used to facilitate computations, but each step in such computations will be transparent and verifiable.

2. Theory

Consider the generalized equation for Net Present Value (NPV) shown in equation (1). The left side of the equation is the NPV, the right side of the equation is the sum of the present value of the cash inflows less the sum of the present value of the cash outflows.

$$NPV = \sum PVIF(k, i) \times CFI - \sum PVIF(k, i) \times li \quad (1)$$

In this equation, k is a company's cost of capital. $PVIF(k, i)$ is the Present Value Interest Factor function at discount rate k , for i periods. CFI is the cash inflow for year i . li is the cash invested in year i . In the simplest case, there is only one cash outflow, in year 0. However, many projects require environmental clean-up when a project is terminated. Failure to consider terminal cash outflows will overstate a project's net present value.

The function $PVIF(k, i)$ expands into equation (2) where k is the period discount rate and i is the number of periods.

$$PVIF(k, i) = 1/(1+k)^i \quad (2)$$

Equation (1) may be rewritten as the equation for Internal Rate of Return (IRR) by replacing k with IRR and replacing NPV with zero as shown in equation (3).

$$0 = \sum PVIF(IRR, i) \times CFI - \sum PVIF(IRR, j) \times lj \quad (3)$$

Let us make some reasonable assumptions about the nature of a typical project. One assumption is that most of the investment will be made at the beginning of the project as is common practice. The next assumption is that the undiscounted cash inflows are greater than the undiscounted cash outflows. If they are not, NPV cannot be greater than zero and the project would be dismissed before an IRR calculation is needed.

Consider a thought experiment in which the zero in equation (3) is replaced by $y1$ and IRR is replaced by IRR1 giving equation (4).

$$y1 = \sum PVIF(IRR1, i) \times CFI - \sum PVIF(IRR1, i) \times li \quad (4)$$

Suppose an IRR1 were selected that was higher than any reasonable internal rate of return, say 200%. Future cash inflows and outflows would be discounted to practically nothing whereas the initial cash outflow at the beginning of the project would not be discounted at all. That initial cash outflow would dominate other terms and drive $y1$ negative.

Now suppose IRR1 were gradually reduced. Eventually, the present value of cash inflows and outflows would be equal each other and $y1$ would be zero. At that point, we will have discovered the

internal rate of return, IRR. If IRR1 continued to be reduced, it would eventually reach the cost of capital. At the cost of capital, y1 would be the NPV.

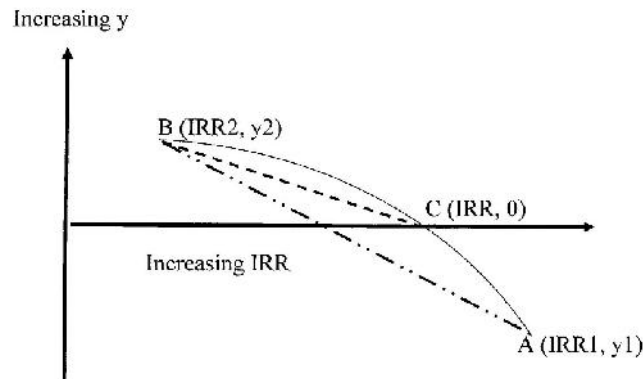
A computer program could be written to decrement the initial IRR1 to the cost of capital to discover IRR. However, there is an algorithm to solve for IRR without the need to write a such a program. No sophisticated mathematics is needed to use this algorithm.

Suppose we replace IRR1 in equation (3) with IRR2 and replace y1 with y2. Further, let us stipulate that IRR2 is the cost of capital. For a project to be considered, its NPV must be positive. Therefore, there must be a positive value for y2 as shown in equation (5).

$$y2 = \sum PVIF (IRR2, i) \times CFi - \sum PVIF (IRR2, i) \times li \quad (5)$$

We may now conceive of a graph with three points on it as shown in Figure 1. The coordinates of A are (IRR1, y1), the coordinates of B are (IRR2, y2) and the coordinates of C are (IRR, 0). Point C represents the solution to the internal rate of return equation because y = 0. This is the same as saying NPV = 0.

Figure 1. Internal Rate of Return



Looking at Figure 1, we see the line segment AB has almost the same slope as line segment CB. As A approaches C, the slopes of the lines will converge and IRR1 will, in the limit, become IRR. The general formula for the slope of a line is shown in equation (6).

$$\text{Slope} = (y2 - y1) / (x2 - x1) \quad (6)$$

Applying that to the data in the Figure 1 we find:

$$\text{Slope AB} = (y2 - y1) / (IRR2 - IRR1)$$

$$\text{Slope CB} = (y2 - 0) / (IRR2 - IRR)$$

Since the slopes of AB and CB are similar, we can set the slope equations equal to each other as shown in equation (7).

$$(y2 - y1) / (IRR2 - IRR1) = (y2 - 0) / (IRR2 - IRR) \quad (7)$$

There is only one unknown in this equation, the internal rate of return, IRR. We selected IRR1 at the beginning of the thought experiment and computed the resulting y1. IRR2 is a company's cost of capital and we computed y2.

Cross multiplying both sides of equation (7) by $(IRR2 - IRR)$ and by $(IRR2 - IRR1)$ yields equation (8).

$$(IRR2 - IRR) \times (y2 - y1) = (y2 - 0) \times (IRR2 - IRR1) \quad (8)$$

Dividing both sides of (8) by $(y2 - y1)$ gives equation (9).

$$IRR2 - IRR = ((y2 - 0) / (y2 - y1)) \times (IRR2 - IRR1) \quad (9)$$

Subtracting $IRR2$ from both sides of the equation and multiplying both sides of the equation by -1 yields equation (10).

$$IRR = -((y2 - 0) / (y2 - y1)) \times (IRR2 - IRR1) + IRR2 \quad (10)$$

Equation (10) provides a first approximation of the Internal Rate of Return. The estimate is approximate because the slope of AB in Figure 1 is not exactly the same as the slope of CB. The error in the estimate of IRR may be computed by using this first estimate of IRR in equation (4). If the resulting $y1$ is non-zero, the solution is not perfect and more iterations of the algorithm may be needed. Each iteration of the algorithm will drive point A closer to point C and reduce the difference in the slopes of lines AB and CB.

Initially, we selected an $IRR1$ that is higher than any reasonable IRR. After making this first estimate, we have a more reasonable IRR to use in computing a new $y1$. But, in computing the error from the first estimate, we have already computed a new $y1$. We can then use the initial estimate of IRR and the new $y1$ to compute a second estimate of IRR using equation (10). Successive approximations can drive the error to as small a value as needed. Simulations suggest that after the fourth estimate, the change in IRR from one iteration to the next is on the order of a few tenths of a percent. Beyond that, additional iterations of this algorithm produce an error, a non-zero $y1$, that is so small it is likely to be swamped by errors in a project's estimated cash flows.

3. Application

Consider a project with an initial cash outflow of \$1,000,000 and a \$250,000 cash outflow in year 6 for environmental clean-up. In this example let cash inflows from the project be \$250,000 in year 1, and \$500,000 in years 2 through 5.

For purposes of this exercise, let us select an $IRR1$ higher than any reasonable IRR for projects of this nature. Suppose an $IRR1$ of 40% is selected. Applying these facts to equation (3) gives equation (11).

$$\begin{aligned} y1 &= [\$250,000 \times 1 / (1+40\%)^1 + \$500,000 \times 1 / (1+40\%)^2 + \$500,000 \times 1 / (1+40\%)^3 \\ &\quad + \$500,000 \times 1 / (1+40\%)^4 + \$500,000 \times 1 / (1+40\%)^5] \\ &\quad - [\$1,000,000 + \$250,000 \times 1 / (1+40\%)^6] \\ &= [\$178,571 + \$255,102 + \$182,216 + \$130,154 + \$92,967] \\ &\quad - [\$1,000,000.00 + \$33,203] \\ &= -\$194,192 \end{aligned} \quad (11)$$

This makes the initial coordinates of point A in Figure 1 (40%, -\$194,192).

Select an $IRR2$ equal to the cost of capital. Any IRR less than the cost of capital would be rejected because it would generate a negative NPV. Suppose the cost of capital is 8%. Applying the cash inflows and outflows to equation (5) along with the cost of capital gives equation (12).

$$y2 = [\$250,000 \times 1 / (1+8\%)^1 + \$500,000 \times 1 / (1+8\%)^2 + \$500,000 \times 1 / (1+8\%)^3] \quad (12)$$

$$\begin{aligned}
 & + \$500,000 \times 1 / (1+8\%)^4 + \$500,000 \times 1 / (1+8\%)^5] \\
 & - [\$1,000,000 + \$250,000 \times 1 / (1+8\%)^6] \\
 & = [\$231,481 + \$428,669 + \$396,916 + \$367,515 + \$340,292] \\
 & - [\$1,000,000.00 + \$157,542] \\
 & = \$607,311
 \end{aligned}$$

This makes the coordinates of point B in Figure 1 (8%, \$607,311).

Applying these data to equation (10) gives equation (13).

$$\begin{aligned}
 \text{IRR} &= - ((\$607,311 - 0) / (\$607,311 - \$194,192)) \times (8\% - 40\%) + 8\% \\
 &= - (\$607,311 / \$801,503) \times (-32\%) + 8\% \\
 &= .7577 \times 32\% + 8\% \\
 &= 24.2464\% + 8\% \\
 &= 32.2464\%
 \end{aligned} \tag{13}$$

The quality of this solution can be tested by using IRR as IRR1 in equation (4) along with the cash flows in this example. Using this first estimate of IRR, y1 is -\$68,554. This is far from the hoped-for answer of zero. However, it is a significant step closer to a useful solution.

The act of testing the quality of the first estimate of IRR has generated new coordinates for point A that are (32.2464%, -\$68,554). These new data can be used in equation (10) to find a second estimate of IRR as shown in equation (14).

$$\begin{aligned}
 \text{IRR} &= - ((\$607,311 - 0) / (\$607,311 - (-\$68,554))) \times (8\% - 32.2464\%) + 8\% \\
 &= (\$607,311 / \$675,865) \times 24.2464\% + 8\% \\
 &= .8986 \times 24.2464\% + 8\% \\
 &= 21.7878\% + 8\% \\
 &= 29.7878\%
 \end{aligned} \tag{14}$$

Using equation (4) we find the error in the second iteration of y1 is -\$22,173 as shown in Table I, row y, Column 5. This is still less than perfect, but it is better than the prior iteration. The new coordinates for point A that are (29.7878%, -\$22,173). Each succeeding iteration moves point A closer to point C. Table 1 summarizes the computations for the first few iterations of this algorithm.

Table 1. IRR Computations

Column (1) are undiscounted cash inflows and outflows. Column (2) are cash inflows and outflows discounted at the cost of capital. Column (3) are cash flows discounted at the highest likely IRR. Columns (4) through (8) are cash flows discounted at the rate indicated. Each successive estimate of IRR yields a new y1 that can be used to compute a new IRR.

Col.	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Year	Cash Inflows	IRR2 8.000%	IRR1 40.000%	IRR1 32.247%	IRR1 29.788%	IRR1 29.020%	IRR1 28.782%	IRR1 28.708%
1	\$250,000	\$231,481	\$178,571	\$189,040	\$192,622	\$193,768	\$194,126	\$194,237
2	\$500,000	\$428,669	\$255,102	\$285,889	\$296,827	\$300,368	\$301,480	\$301,825
3	\$500,000	\$396,916	\$182,216	\$216,178	\$228,702	\$232,807	\$234,101	\$234,503
4	\$500,000	\$367,515	\$130,154	\$163,465	\$176,212	\$180,442	\$181,780	\$182,197
5	\$500,000	\$340,292	\$92,967	\$123,606	\$135,769	\$139,856	\$141,153	\$141,558
Totals	\$2,250,000	\$1,764,874	\$839,011	\$978,179	\$1,030,132	\$1,047,240	\$1,052,640	\$1,054,322
Year	Outflows							
0	\$1,000,000	\$1,000,000	\$1,000,000	\$1,000,000	\$1,000,000	\$1,000,000	\$1,000,000	\$1,000,000
6	\$250,000	\$157,542	\$33,203	\$46,733	\$52,304	\$54,199	\$54,803	\$54,992
Totals	\$1,250,000	\$1,157,542	\$1,033,203	\$1,046,733	\$1,052,304	\$1,054,199	\$1,054,803	\$1,054,992
y		\$607,331	-\$194,192	-\$68,554	-\$22,173	-\$6,959	-\$2,163	-\$670
IRR			32.247%	29.788%	29.020%	28.782%	28.708%	28.686%

Initially, an IRR1 is selected that is higher than any reasonable IRR for a given project, in this example 40% (Column 3). IRR2 is the cost of capital, in this example 8% (Column 2). The present value of cash flows at 8% is \$607,331 (row y, Column 2). The present value of cash flows at 40% is – \$194,192 (row y, Column 3). The values (40%, -194,192) and (8%, \$607,331) become the coordinates of points A and B respectively in Figure 1. These coordinates are used in equation (10), to make the first estimate of IRR which is 32.247% (row IRR, Column 3).

Is this an accurate estimate of IRR? The proof is to discount cash flows at the estimated IRR using equation (4) and see how close the result is to zero. In this case the error (row y, Column 4) is – \$68,554. The first estimate of IRR, 32.247%, and y1, -\$68,554 become the new coordinates of point A of (32.247%, -\$68,554). These data are used in equation (10) to compute a new estimate of IRR of 29.788% (row IRR, Column 4).

This IRR is tested by using equation (4) that provides a new y1 of -\$22,173 (row y, Column 5). These data generate new coordinates for A of (29.788%, -\$22,173). Equation (10) is then used to further refine the estimate of IRR.

In the example worked above, IRR changed by 0.074% (28.782% - 28.708%) between the fourth and fifth iteration (row IRR, Columns 7 and 8). The value y1 in the fifth iteration is -\$670. For a million-dollar project, any error, defined as a non-zero y1, is likely to be swamped by errors in the estimates of a project's underlying cash flow.

With the use of this algorithm, a generalized computer spreadsheet considering twenty or more years of cash inflows and outflows can be easily developed. Once a generalized worksheet is constructed, future IRR calculation can be reduced to inputting annual cash flows and the cost of capital. Any level of precision can be achieved by adding iterations to the algorithm. However, the most important feature of this algorithm is transparency. Each step can be examined for accuracy to assure there are no input or calculation errors.

4. Conclusion

The internal rate of return (IRR) is defined as the return that will drive NPV to zero. IRR considers the time value of money and cash flows over the life of a project. It can be used to rank projects in

situations where net present value (NPV) provides uncertain answers. Internal rate of return can also be directly compared to external investments. A major drawback to IRR is that it requires complex mathematics to solve non-linear equations, or the development of computer programs to find IRR through trial and error or reliance on spreadsheet functions or financial calculators that lack transparency.

The algorithm underlying this paper is based on a thought experiment involving three data points. One of the data points, A, is an IRR that over discounts cash flows, one data point, B, under discounts cash flows and one datapoint, C, discounts cash flows just enough to make NPV zero. We can select the over and under discount rates so we know the coordinates of A and B. The discount rate that drives NPV to zero, C, is unknown. The slope of line AB is roughly equal to the slope of the line CB. Setting the equations for these slopes equal to each other we find there is only one unknown, the IRR that drives NPV to zero. Using algebra, we can find a first estimate of IRR. That estimate can be tested by using it to discount cash flows. The first estimate of IRR and the resulting discounted cash flows can then be used to find better estimates of IRR.

This algorithm can be implemented on spreadsheet software without resort to macros or other specialized programming. The algorithm is transparent and each step can be verified. It can also calculate IRR to any level of precision. Simulations suggest that after a few iterations the difference between IRR estimated using this algorithm and a perfect answer is immaterial. The mathematical technique discussed the theory section might also be useful for solving other classes problems.

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