

Compounding Commuting Matrices

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Abstract

We formulate a method of constructing families of commuting matrices of increasing order by sequential compounding. Formulas are derived for sequentially constructing their Jordan forms and singular-value decompositions. Examples are given to illustrate our methods including construction of commuting Latin squares.

Introduction. Although matrices (as arrays) existed much earlier, matrix algebra originated with Arthur Cayley [1] in 1858 and was developed further by many others. As noted by Cayley, in general two matrices $\mathbf{A}$ and $\mathbf{B}$ do not commute, i.e. $\mathbf{AB} \neq \mathbf{BA}$. However, certain special families of matrices do commute, e.g. circulant matrices.

Other families of commuting matrices are constructed in the present paper together with their Jordan form and singular-value decomposition (SVD). Our constructions employ the method of sequential compounding which has a long history as discussed by Rogers, et.al. [6] and used by them in constructions of magic squares and Latin squares. **A great variety of commuting matrices can be formed by sequential compounding as our examples illustrate. The Jordan form and SVD of these commuting matrices are easy to construct, again by compounding. Our compounded commuting matrices should be regarded as complementary to circulant matrices for which formulas are available for their Jordan form. However, there are no known formulas for the SVD of circulant matrices and they do not compound to new circulant matrices as an example shows. Thus, our compounded matrices have certain advantages over circulant matrices.**

The Kronecker product, denoted by $\otimes$, and its associated identities (see Meyer [4]) are employed in our construction. All matrices in the present paper are square.

Definition. The matrix $\mathbf{A}$ is said to be **commutal** if its elements a_{ij} can be replaced by new variables a'_{ij} to form a matrix $\mathbf{A}'$ of the same structural form as $\mathbf{A}$ such that $\mathbf{A}'$ commutes with $\mathbf{A}$. For example, circulant matrices are commutal, as this order-2 one illustrates:

$$\mathbf{C}_2 = \begin{bmatrix} c_0 & c_1 \\ c_1 & c_0 \end{bmatrix}, \quad \mathbf{C}'_2 = \begin{bmatrix} c'_0 & c'_1 \\ c'_1 & c'_0 \end{bmatrix}, \quad \mathbf{C}_2 \mathbf{C}'_2 = \mathbf{C}'_2 \mathbf{C}_2. \quad (1)$$

Note that $\mathbf{C}_2$ is a Latin square (c_0 and c_1 appear in all its rows and columns).

Compounding. We compound members of two families of commutal matrices to form a third family of commutal matrices according to the following theorem:

Theorem 1 *Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be a commutal matrix and let $\mathbf{B} \in \mathbb{R}^{n \times n}$ be a commutal matrix. Let*

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B}, \quad \mathbf{C}' = \mathbf{A}' \otimes \mathbf{B}'. \quad (2)$$

Then $\mathbf{C} \in \mathbb{R}^{mn \times mn}$ is a commutal matrix.

Proof. From (2) we have

$$\begin{aligned} \mathbf{C}\mathbf{C}' &= (\mathbf{A} \otimes \mathbf{B}) (\mathbf{A}' \otimes \mathbf{B}') = (\mathbf{A}\mathbf{A}') \otimes (\mathbf{B}\mathbf{B}') \\ &= (\mathbf{A}'\mathbf{A}) \otimes (\mathbf{B}'\mathbf{B}) = (\mathbf{A}' \otimes \mathbf{B}') (\mathbf{A} \otimes \mathbf{B}) = \mathbf{C}'\mathbf{C}, \end{aligned} \quad (3)$$

whence $\mathbf{C}\mathbf{C}' = \mathbf{C}'\mathbf{C}$, i.e. $\mathbf{C}$ is commutal. ■

For example, compounding of two order-2 circulant matrices (1) gives the following commutal order-4 matrix:

$$\mathbf{C}_4 = \begin{bmatrix} a_0 & a_1 \\ a_1 & a_0 \end{bmatrix} \otimes \begin{bmatrix} b_0 & b_1 \\ b_1 & b_0 \end{bmatrix} = \begin{bmatrix} a_0b_0 & a_0b_1 & a_1b_0 & a_1b_1 \\ a_0b_1 & a_0b_0 & a_1b_1 & a_1b_0 \\ a_1b_0 & a_1b_1 & a_0b_0 & a_0b_1 \\ a_1b_1 & a_1b_0 & a_0b_1 & a_0b_0 \end{bmatrix} \quad (4)$$

the elements of which can be relabeled as

$$\mathbf{C}_4 = \begin{bmatrix} c_0 & c_1 & c_2 & c_4 \\ c_1 & c_0 & c_4 & c_2 \\ c_2 & c_4 & c_0 & c_1 \\ c_4 & c_2 & c_1 & c_0 \end{bmatrix}. \quad (5)$$

It can be verified directly that $\mathbf{C}_4$ is commutal. However, $\mathbf{C}_4$ is not a circulant matrix, i.e. two circulant matrices do not compound to a circulant matrix. It is noteworthy that $\mathbf{C}_4$ is a symmetric Latin square. Also, $\mathbf{C}_4$ is a representation of the Klein four-group (Viergruppe) given by Felix Klein [2] in 1884.

Jordan Form. The Jordan form of a matrix is of wide interest as discussed in many textbooks, e.g. Meyer [4]. It can be formed from the eigenvalues and eigenvectors of the matrix (when they exist). Here, the families of commutal matrices with members $\mathbf{A}_i \in \mathbb{R}^{m \times m}$ and $\mathbf{B}_i \in \mathbb{R}^{n \times n}$ are restricted to be diagonalizable with Jordan forms

$$\mathbf{A}_i = \mathbf{S}_A \mathbf{D}_{Ai} \mathbf{S}_A^{-1}, \quad \mathbf{B}_i = \mathbf{S}_B \mathbf{D}_{Bi} \mathbf{S}_B^{-1}, \quad (6)$$

where $\mathbf{D}_{Ai} \in \mathbb{C}^{m \times m}$ and $\mathbf{D}_{Bi} \in \mathbb{C}^{n \times n}$ are diagonal matrices and the columns of the matrices $\mathbf{S}_A \in \mathbb{C}^{m \times m}$ and $\mathbf{S}_B \in \mathbb{C}^{n \times n}$ are their corresponding eigenvectors. Since $\mathbf{A}_i$ and $\mathbf{B}_i$ are diagonalizable and members of commuting families, it is known [4] that $\mathbf{S}_A$ and $\mathbf{S}_B$ are the same for all members of each family but $\mathbf{D}_{Ai}$ and $\mathbf{D}_{Bi}$ are different for each member of the family. The Jordan form of the compounded matrix $\mathbf{C}$ of (2) is given by the following theorem:

Theorem 2 *The compounded commutal matrix $\mathbf{C}_i = \mathbf{A}_i \otimes \mathbf{B}_i$ has the Jordan form*

$$\mathbf{C}_i = \mathbf{S}_C \mathbf{D}_{Ci} \mathbf{S}_C^{-1}, \quad (7)$$

where

$$\mathbf{S}_C = \mathbf{S}_A \otimes \mathbf{S}_B, \quad \mathbf{D}_{Ci} = \mathbf{D}_{Ai} \otimes \mathbf{D}_{Bi} = \mathbf{S}_C^{-1} \mathbf{C}_i \mathbf{S}_C. \quad (8)$$

Proof. Using (6), we have

$$\begin{aligned} \mathbf{C}_i &= \mathbf{A}_i \otimes \mathbf{B}_i = (\mathbf{S}_A \mathbf{D}_{Ai} \mathbf{S}_A^{-1}) \otimes (\mathbf{S}_B \mathbf{D}_{Bi} \mathbf{S}_B^{-1}) \\ &= (\mathbf{S}_A \otimes \mathbf{S}_B) (\mathbf{D}_{Ai} \otimes \mathbf{D}_{Bi}) (\mathbf{S}_A^{-1} \otimes \mathbf{S}_B^{-1}), \\ \mathbf{S}_C^{-1} &= (\mathbf{S}_A \otimes \mathbf{S}_B)^{-1} = \mathbf{S}_A^{-1} \otimes \mathbf{S}_B^{-1}. \end{aligned} \quad (9)$$

$\mathbf{D}_{Ci}$ is diagonal since $\mathbf{D}_{Ai}$ and $\mathbf{D}_{Bi}$ are. The relation $\mathbf{D}_{Ci} = \mathbf{S}_C^{-1} \mathbf{C}_i \mathbf{S}_C$ follows from (7) ■

Furthermore, if $\mathbf{A}_i$ and $\mathbf{B}_i$ are normal matrices ($\mathbf{A}_i \mathbf{A}_i^T = \mathbf{A}_i^T \mathbf{A}_i$, $\mathbf{B}_i \mathbf{B}_i^T = \mathbf{B}_i^T \mathbf{B}_i$), it can be shown that $\mathbf{C}_i$ is normal and [4]

$$\mathbf{S}_A^{-1} = \mathbf{S}_A^*, \quad \mathbf{S}_B^{-1} = \mathbf{S}_B^*, \quad \mathbf{S}_C^{-1} = \mathbf{S}_C^*, \quad (10)$$

where $\mathbf{S}^*$ denotes the conjugate transpose of $\mathbf{S}$. In addition, if $\mathbf{A}_i$ and $\mathbf{B}_i$ are symmetric matrices (always normal), then [4]

$$\mathbf{S}_A^{-1} = \mathbf{S}_A^T, \quad \mathbf{S}_B^{-1} = \mathbf{S}_B^T, \quad \mathbf{S}_C^{-1} = \mathbf{S}_C^T, \quad (11)$$

i.e. they are orthogonal and $\mathbf{C}_i$ is symmetric too. When $\mathbf{A}_i$ and $\mathbf{B}_i$ are real and symmetric, all $\mathbf{S}$'s and $\mathbf{D}$'s are real too. When $\mathbf{A}_i$ and $\mathbf{B}_i$ also are positive definite or positive semi-definite, so is $\mathbf{C}_i$.

For example, in the Jordan form for $\mathbf{A}_2$ of (1),

$$\mathbf{S}_A = \mathbf{S}_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{D}_A = \text{diag}(a_0 \pm a_1), \quad \mathbf{D}_B = \text{diag}(b_0 \pm b_1), \quad (12)$$

and in the Jordan form for $\mathbf{C}_4$ of (5), by (8),

$$\begin{aligned} \mathbf{S}_C &= \mathbf{S}_A \otimes \mathbf{S}_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \\ \mathbf{D}_C &= \mathbf{D}_A \otimes \mathbf{D}_B = \mathbf{S}_C^{-1} \mathbf{C}_4 \mathbf{S}_C = \text{diag}[\lambda_1, \lambda_2, \lambda_3, \lambda_4], \end{aligned} \quad (13)$$

where

$$\begin{aligned} \lambda_1 &= c_0 + c_1 + c_2 + c_4, & \lambda_2 &= c_0 - c_1 + c_2 - c_4, \\ \lambda_3 &= c_0 + c_1 - c_2 - c_4, & \lambda_4 &= c_0 - c_1 - c_2 + c_4. \end{aligned}$$

The Jordan form of $\mathbf{C}_4$ can be verified directly from (13). Other examples of the Jordan form construction are given in what follows.

SVD. The singular-value decomposition (SVD) of a matrix also is of general interest. It has many applications as discussed by Martin and Porter [3] and Meyer [4], where additional references are given. We first review the known facts concerning the SVD and then we consider the SVD for compounded commutal matrices. The SVD of any matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ reads

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^T, \quad (14)$$

where $\mathbf{U} \in \mathbb{R}^{m \times m}$ and $\mathbf{V} \in \mathbb{R}^{m \times m}$, are orthogonal matrices ($\mathbf{U} \mathbf{U}^T = \mathbf{V} \mathbf{V}^T = \mathbf{I}$) and $\mathbf{\Sigma} \in \mathbb{R}^{m \times m}$ is a diagonal matrix with $r = \text{rank}(\mathbf{A})$ positive diagonal elements σ_k (called the singular values) and $m - r$ zero elements.¹ From (14) we have

$$\mathbf{A} \mathbf{A}^T = \mathbf{U} \mathbf{\Sigma}^2 \mathbf{U}^T, \quad \mathbf{A}^T \mathbf{A} = \mathbf{V} \mathbf{\Sigma}^2 \mathbf{V}^T \quad (15)$$

from which $\mathbf{U}$ and $\mathbf{V}$ can be determined subject to (14) being satisfied. For example, if $\mathbf{U}$ and $\mathbf{\Sigma}^2$ are obtained from (15) and $\mathbf{A}$ is nonsingular ($r = m$), then $\mathbf{V}$ is given by

$$\mathbf{V} = \mathbf{A}^T \mathbf{U} \mathbf{\Sigma}^{-1} \quad (16)$$

which can be shown to be orthogonal. If $\mathbf{A}$ is singular, then other considerations are necessary, see Meyer [4]. The construction of $\mathbf{V}$ will not be mentioned in the examples that follow.

If $\mathbf{A}$ is normal, then by (6) and (10) the Jordan form of $\mathbf{A}$ reads

$$\mathbf{A} = \mathbf{S} \mathbf{D} \mathbf{S}^* \quad \text{and} \quad \mathbf{A} \mathbf{A}^T = \mathbf{A} \mathbf{A}^* = \mathbf{S} |\mathbf{D}|^2 \mathbf{S}^*, \quad (17)$$

¹The SVD also applies to rectangular and complex matrices, see [3, 4].

where the (possibly complex) eigenvalues of $\mathbf{D}$ are denoted by λ_k . From (15) and (17) we see that the unique eigenvalues of $\mathbf{A}\mathbf{A}^T$ are contained in both Σ^2 and $|\mathbf{D}|^2$, thus

$$\sigma_k = |\lambda_k|. \quad (18)$$

However, when $\mathbf{S}$ is complex it cannot be used for $\mathbf{U}$ (nor $\mathbf{S}^*$ for $\mathbf{V}^T$). For symmetric (normal) matrices, $\mathbf{S}$ and $\mathbf{D}$ are real and $\mathbf{S}$ is orthogonal, hence $\mathbf{U} = \mathbf{S}$ is allowed. If $\mathbf{A}$ is symmetric and positive semi-definite, then $\lambda_k \geq 0$ and the Jacobi form and SVD of $\mathbf{A}$ are identical.

Now, let $\mathbf{A}_i \in \mathbb{R}^{m \times m}$ be a member of a family of normal commutal matrices ($\mathbf{A}_i \mathbf{A}_i^T = \mathbf{A}_i^T \mathbf{A}_i$). It follows that $\mathbf{A}_i \mathbf{A}_j^T = \mathbf{A}_j^T \mathbf{A}_i$ for all i, j and it can be shown that $\mathbf{A}_i \mathbf{A}_i^T$ and $\mathbf{A}_j \mathbf{A}_j^T$ commute. Therefore, $\mathbf{U}_{A_i}$ in the SVD of $\mathbf{A}_i$ is the same for all $\mathbf{A}_i$ ($\mathbf{U}_{A_i} = \mathbf{U}_A$), hence

$$\mathbf{A}_i \mathbf{A}_i^T = \mathbf{U}_A \Sigma_{A_i}^2 \mathbf{U}_A^T \quad (19)$$

and we have the following theorem:

Theorem 3 For $\mathbf{A}_i$ and $\mathbf{B}_i$ members of families of normal commutal matrices, the compounded commutal matrix $\mathbf{C}_i = \mathbf{A}_i \otimes \mathbf{B}_i$ has the SVD

$$\mathbf{C}_i = \mathbf{U}_C \Sigma_{C_i} \mathbf{V}_{C_i}^T, \quad (20)$$

where

$$\mathbf{U}_C = \mathbf{U}_A \otimes \mathbf{U}_B, \quad \Sigma_{C_i} = \Sigma_{A_i} \otimes \Sigma_{B_i} = (\mathbf{U}_A^T \mathbf{A}_i \mathbf{A}_i^T \mathbf{U}_A)^{\frac{1}{2}}, \quad \mathbf{V}_{C_i}^T = \mathbf{V}_{A_i}^T \otimes \mathbf{V}_{B_i}^T. \quad (21)$$

Proof.

$$\begin{aligned} \mathbf{C}_i &= \mathbf{A}_i \otimes \mathbf{B}_i = (\mathbf{U}_A \Sigma_{A_i} \mathbf{V}_{A_i}^T) \otimes (\mathbf{U}_B \Sigma_{B_i} \mathbf{V}_{B_i}^T) \\ &= (\mathbf{U}_A \otimes \mathbf{U}_B) (\Sigma_{A_i} \otimes \Sigma_{B_i}) (\mathbf{V}_{A_i}^T \otimes \mathbf{V}_{B_i}^T) = \mathbf{U}_C \Sigma_{C_i} \mathbf{V}_{C_i}^T. \end{aligned} \quad (22)$$

Σ_{C_i} is diagonal with nonzero elements since Σ_{A_i} and Σ_{B_i} are. It can be shown that $\mathbf{U}_C$ and $\mathbf{V}_{C_i}$ are orthogonal. The relation $\Sigma_{C_i} = (\mathbf{U}_A^T \mathbf{A}_i \mathbf{A}_i^T \mathbf{U}_A)^{\frac{1}{2}}$ follows from (19). ■

As already noted, $\mathbf{C}_i$ is normal when $\mathbf{A}_i$ and $\mathbf{B}_i$ are normal and thus its singular values are related to its eigenvalues by a relation of the form (18). From (21) and (8) we have

$$\mathbf{U}_C = (\mathbf{U}_A \mathbf{S}_A^* \mathbf{S}_A) \otimes (\mathbf{U}_B \mathbf{S}_B^* \mathbf{S}_B) = (\mathbf{U}_A \mathbf{S}_A^*) \otimes (\mathbf{U}_B \mathbf{S}_B^*) (\mathbf{S}_A \otimes \mathbf{S}_B), \quad (23)$$

i.e. the SVD matrix $\mathbf{U}_C$ is related to the eigenvector matrix $\mathbf{S}_C$ by

$$\mathbf{U}_C = \mathbf{L} \mathbf{S}_C, \quad \mathbf{L} = (\mathbf{U}_A \mathbf{S}_A^*) \otimes (\mathbf{U}_B \mathbf{S}_B^*) \quad \mathbf{L} \mathbf{L}^* = \mathbf{I}. \quad (24)$$

Examples of the SVD construction are given in what follows.

Example 1. Here is an order-8 normal commutal matrix constructed from compounding $\mathbf{C}_4$ of (5) with $\mathbf{C}_2$ of (1) and relabeling the elements as before:

$$\mathbf{C}_8 = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ c_1 & c_0 & c_3 & c_2 & c_5 & c_4 & c_7 & c_6 \\ c_2 & c_3 & c_0 & c_1 & c_6 & c_7 & c_4 & c_5 \\ c_3 & c_2 & c_1 & c_0 & c_7 & c_6 & c_5 & c_4 \\ c_4 & c_5 & c_6 & c_7 & c_0 & c_1 & c_2 & c_3 \\ c_5 & c_4 & c_7 & c_6 & c_1 & c_0 & c_3 & c_2 \\ c_6 & c_7 & c_4 & c_5 & c_2 & c_3 & c_0 & c_1 \\ c_7 & c_6 & c_5 & c_4 & c_3 & c_2 & c_1 & c_0 \end{bmatrix}. \quad (25)$$

Again, it is noteworthy that $\mathbf{C}_8$ is a symmetric Latin square. In view of its symmetry, $\mathbf{U}_{C8} = \mathbf{S}_{C8}$ and $\sigma_i = |\lambda_i|$. Furthermore, repeated application of our compounding construction using (1) produces symmetric Latin squares of order 2^k . Also, commutal normal Latin squares of any nonprime order can be produced by compounding two circulant matrices which are commutal and normal. The Jordan form and SVD of such matrices can be constructed sequentially from our formulas.

Example 2. We construct an order-6 commutal matrix by compounding an order-3 commutal matrix with an order-2 commutal matrix. We start with the symmetric order-3 commutal matrix

$$\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_0 + a_2 & a_1 \\ a_2 & a_1 & a_0 \end{bmatrix}. \quad (26)$$

The Jordan form of $\mathbf{A}$ has

$$\mathbf{S}_A = \frac{1}{2} \begin{bmatrix} -\sqrt{2} & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \end{bmatrix}, \quad \mathbf{D}_A = \text{diag} \left(a_0 - a_2, a_0 + a_2 \pm \sqrt{2}a_1 \right). \quad (27)$$

From $\mathbf{D}_A$ we see that $\mathbf{A}$ with integer elements is nonsingular ($\lambda_i \neq 0$) unless $a_0 = a_2$ or $a_0 = -a_2$, $a_1 = 0$. Furthermore, $\mathbf{A}$ is positive semi-definite if $a_0 \geq a_2$ and $a_0 + a_2 \geq \pm\sqrt{2}a_1$. The SVD $\mathbf{A}$ has

$$\mathbf{U}_A = \mathbf{S}_A, \quad \mathbf{\Sigma}_A = \text{diag}(\sigma_1, \sigma_2, \sigma_3), \quad \sigma_i = |\lambda_i|. \quad (28)$$

Next, we introduce the following order-2 commutal normal matrix:

$$\mathbf{B} = \begin{bmatrix} b_0 & b_1 \\ -b_1 & b_0 \end{bmatrix} \quad (29)$$

whose Jordan form has

$$\mathbf{S}_B = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \quad [\mathbf{S}_B]^* = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}, \quad \mathbf{D}_B = \text{diag}(b_0 \mp ib_1). \quad (30)$$

From (19), the SVD of $\mathbf{B}$ has

$$\begin{aligned} \mathbf{U}_B &= \mathbf{V}_B = \mathbf{I}_2, \quad \mathbf{\Sigma}_B = \text{diag}(\sigma_1, \sigma_2), \\ \sigma_1 &= \sigma_2 = (b_0^2 + b_1^2)^{\frac{1}{2}} = |\lambda_1| = |\lambda_2|. \end{aligned} \quad (31)$$

On compounding $\mathbf{A} \otimes \mathbf{B}$ we obtain the order-6 commutal matrix $\mathbf{C}$ which after relabeling its elements reads

$$\mathbf{C} = \begin{bmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ -c_1 & c_0 & -c_3 & c_2 & -c_5 & c_4 \\ c_2 & c_3 & c_0 + c_4 & c_1 + c_5 & c_2 & c_3 \\ -c_3 & c_2 & -c_1 - c_5 & c_0 + c_4 & -c_3 & c_2 \\ c_4 & c_5 & c_2 & c_3 & c_0 & c_1 \\ -c_5 & c_4 & -c_3 & c_2 & -c_1 & c_0 \end{bmatrix}. \quad (32)$$

By (8), the Jordan form of $\mathbf{C}$ has

$$\mathbf{S}_C = \frac{1}{4} \begin{bmatrix} -2 & -2 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 2i & -2i & -i\sqrt{2} & i\sqrt{2} & -i\sqrt{2} & i\sqrt{2} \\ 0 & 0 & 2 & 2 & -2 & -2 \\ 0 & 0 & -2i & 2i & 2i & -2i \\ 2 & 2 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ -2i & 2i & -i\sqrt{2} & i\sqrt{2} & -i\sqrt{2} & i\sqrt{2} \end{bmatrix}, \quad \mathbf{S}_C^{-1} = \mathbf{S}_C^*,$$

$$\mathbf{D}_C = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_6), \quad (33)$$

where

$$\begin{aligned} \lambda_1 &= c_0 - c_4 - i(c_1 - c_5), \quad \lambda_2 = \bar{\lambda}_1, \\ \lambda_3 &= c_0 + c_4 + \sqrt{2}c_2 - i(c_1 + c_5 + \sqrt{2}c_3), \quad \lambda_4 = \bar{\lambda}_3, \\ \lambda_5 &= c_0 + c_4 - \sqrt{2}c_2 - i(c_1 + c_5 - \sqrt{2}c_3), \quad \lambda_6 = \bar{\lambda}_5, \end{aligned}$$

as can be verified directly. From $\mathbf{D}_C$, we see that $\mathbf{C}$ with integer elements is nonsingular unless $c_0 = c_4$, $c_1 = c_5$ or $c_0 = -c_4$, $c_1 = -c_5$, $c_2 = c_3 = 0$. By (21) with (28) and (20), the SVD of $\mathbf{C}$ has

$$\mathbf{U}_C = \mathbf{S} \otimes \mathbf{I}_2 = \frac{1}{2} \begin{bmatrix} -\sqrt{2} & 0 & 1 & 0 & 1 & 0 \\ 0 & -\sqrt{2} & 0 & 1 & 0 & 1 \\ 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} \\ \sqrt{2} & 0 & 1 & 0 & 1 & 0 \\ 0 & \sqrt{2} & 0 & 1 & 0 & 1 \end{bmatrix}. \quad (34)$$

By (19) with (34), the singular values of $\hat{A}_6$ are

$$\boldsymbol{\Sigma}_C = [\mathbf{U}_C^T \mathbf{A}_C \mathbf{A}_C^T \mathbf{U}_C]^{\frac{1}{2}} = \text{diag}(\sigma_1, \sigma_1, \sigma_3, \sigma_3, \sigma_5, \sigma_5), \quad \sigma_k = |\lambda_k|, \quad k = 1, 3, 5, \quad (35)$$

as can be verified directly. The relation between σ_k and λ_k follows from (18) since $\mathbf{C}$ is a normal matrix (because $\mathbf{A}$ and $\mathbf{B}$ are normal). However, $\mathbf{S}_C$ is complex and therefore it cannot be used as $\mathbf{U}_C$ instead of (34). The relation (24) between $\mathbf{S}_C$ and $\mathbf{U}_C$ has

$$\mathbf{L}_C = (\mathbf{U}_A \mathbf{S}_A^*) \otimes (\mathbf{U}_B \mathbf{S}_B^*) = \mathbf{I}_3 \otimes \mathbf{S}_B^* = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & i & 0 & 0 & 0 & 0 \\ 1 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 & 1 & -i \end{bmatrix}. \quad (36)$$

There are many other possibilities for constructing an order-6 commutal matrix using various combinations of (26), (1), (5), and an order-3 circulant matrix. One could start with an order-2 matrix and use two order-3 commutal matrices in the construction. There are endless possibilities for constructing higher-order commutal matrices sequentially by repeated compounding as the reader should enjoy doing.

Example 3. For comparison purposes we record the Jordan form of the order-4 circulant matrix

$$\hat{\mathbf{C}}_4 = \begin{bmatrix} c_0 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_3 \\ c_3 & c_2 & c_1 & c_0 \end{bmatrix}, \quad (37)$$

From known formulas [4], we find that

$$\hat{\mathbf{C}}_4 = \hat{\mathbf{S}}_4 \hat{\mathbf{D}}_4 \hat{\mathbf{S}}_4^*, \quad (38)$$

where

$$\begin{aligned} \hat{\mathbf{S}}_4 &= \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}, \quad \hat{\mathbf{D}}_4 = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \\ \lambda_1 &= c_0 + c_1 + c_2 + c_3, \quad \lambda_2 = c_0 - c_2 - i(c_1 - c_3), \\ \lambda_3 &= c_0 - c_1 + c_2 - c_3, \quad \lambda_4 = c_0 - c_2 + i(c_1 - c_3). \end{aligned} \quad (39)$$

In addition, the SVD of $\hat{\mathbf{C}}_4$ is found to be

$$\hat{\mathbf{C}}_4 = \hat{\mathbf{U}}_4 \hat{\mathbf{\Sigma}}_4 \hat{\mathbf{V}}_4^T, \quad (40)$$

where, on noting that $|\lambda_2| = |\lambda_4|$ and recombining their associated eigenvectors in $\hat{\mathbf{S}}_4$, we find that

$$\begin{aligned} \hat{\mathbf{U}}_4 &= \frac{1}{2} \begin{bmatrix} 1 & \sqrt{2} & 1 & 0 \\ 1 & 0 & -1 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 & 0 \\ 1 & 0 & -1 & \sqrt{2} \end{bmatrix}, \quad \hat{\mathbf{U}}_4 \hat{\mathbf{U}}_4^T = \mathbf{I}, \\ \hat{\mathbf{\Sigma}}_4 &= \text{diag}(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \quad \sigma_k = |\lambda_k|. \end{aligned} \quad (41)$$

However, formulas for the SVD of a circulant matrix of arbitrary order are lacking. Perhaps further research will discover them.

References

- [1] A. Cayley A memoir on the theory of matrices, Phil. Trans. 148 (1858) 17-37 (<https://archive.org/stream/philtrans05474612/05474612#page/n0/mode/2up>).
- [2] F. Klein, *Lectures on the Icosahedron and the Solution of Equations of the Fifth Degree* (1884), Translated by G. G. Morrice, 2nd and rev. ed., Kegan Paul, Trench, Trübner & Co. Ltd., London (1913).
- [3] C. Martin, M. Porter, The extraordinary SVD, MAA Monthly 119 (2012) 838-851.
- [4] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, Society for Industrial and Applied Mathematics, Philadelphia (2000).
- [5] C. A. Pickover, *The Zen of Magic Squares, Circles, and Stars*, second printing and first paperback printing, Princeton University Press, Princeton, NJ, 2003 (original printing and e-book: 2002).
- [6] A. Rogers, I. Cameron, P. Loly, Compounding doubly affine matrices, <https://arxiv.org/abs/1711.11084>, 2017.