

# Solution to the Dirichlet Problem on Irregular Domains using Wavelet-Based Approach

Francis Ohene Boateng<sup>1</sup>, Joseph Ackora-Prah<sup>2</sup> and Anthony. Y. Aidoo<sup>3</sup>

<sup>1</sup>*Department of Mathematics Education, University of Education, Winneba  
Kumasi, Ghana*

*Corresponding author: fanbotgh@yahoo.com*

<sup>2</sup>*Department of Mathematics, Kwame Nkrumah University of Science and Technology  
Kumasi, Ghana  
jaackoraprah.cos@knust.edu.gh*

<sup>3</sup>*Department of Mathematics and Computer Science, Eastern Connecticut State University,  
Willimantic, USA  
AidooA@easternct.edu*

## Abstract

We present a Finite Difference Fictitious Domain Wavelet Method (FDFDWM) with penalty for solving two dimensional (2D) Dirichlet problem for linear elliptic PDE on irregular geometric domains. In this method, the 2-D Dirichlet problem is discretized along one of the spatial variables, reducing it to a 1-D problem. The problem and the boundaries of the irregular domain are approximated using compactly supported wavelets. Results from the numerical analysis indicate that, our method performs better in terms of accuracy and convergence of the approximate solution compared with finite element method.

**Keywords:** Dirichlet problem, penalty, fictitious domain, PDEs, Daubechies wavelet function, irregular domain, finite difference

## 1 Introduction

The linear elliptic partial differential equation (PDE) with Dirichlet boundary condition, commonly referred to as the Dirichlet problem (DP) is an age long problem that has significant application in myriads of fields. For instance, in the area of fluid dynamics it is applied in calculating forces and moments on aircraft, predicting weather patterns, determining the mass flow rate of petroleum through pipeline and modeling fission weapon denotation ([1], [8] and [12]). Other applications are found in electrostatics, Newtonian gravity, hydrodynamics, diffusion, etc.

In recent times, the emphasis on solving the Dirichlet problem using analytical methods is diminishing. This is due to the availability of high speed computers and workstations which

make numerical solutions to this problem more attractive [14]. However, numerical computations is known to generate approximate solutions. The Finite Difference Method(FDM), Spectral Method (SM) and Finite Element Method (FEM) are some of the classical methods used to solve this problem. The SM is used for the discretization of spatial variables in Dirichlet Problem for elliptic PDE in 2D, in the Hilbert space. Notwithstanding, this method has some limitations which makes FDM and FEM preferable in many areas ([3] and [15]). One drawback is that it generates large systems of linear or non-linear equations involving full matrices. In contrast, FDM and FEM, lead to systems involving sparse matrices. Another limitation of spectral methods is its inability to handle irregularly shaped domains. The FDM also has some difficulties in handling problems with complex geometric domains. This is due to the fact that the FDM uses topologically square network of lines to discretize the Dirichlet problem. In the case of FEM, irregular domains can be handled by generating complex grid adapted to the geometry of the domain, however this process is cumbersome and time consuming [18].

Due to the difficulties associated with problems defined on irregular domains, methods that are efficient and offer high accurate and stable solution to the Dirichlet Problem are desired. Over the past two decades, wavelets have been used by a number of researchers including ([4],[6], [11], [13], [16] and [17]) as preferred functions for the approximation of PDEs. In wavelet methods, we are able to obtain information in both time and frequency domains [2]. Notably, the vanishing moment property of wavelet contributes significantly to the rapid convergence of wavelet series solution to a point in the domain as compared to the classical numerical methods mentioned earlier for solving the Dirichlet Problem for linear elliptic PDE in 2D on irregular domains. This paper therefore seeks to develop a wavelet-based numerical method that generates more accurate approximation solution to the DP on irregular domains compared with the aforementioned traditional methods.

## 2 The FDFDWM with Penalty on Irregular Domain

We present in this paper a Finite Difference Fictitious Domain Wavelet Method (FDFDWM) with penalty term to approximate the Dirichlet problem for linear elliptic PDE defined on irregular domains. The penalty term incorporated in this method regularizes the irregular boundary. Boundary measure theory is apply to the boundary integral obtained from the boundaries of the irregular domains. These boundary integral are approximated using wavelet functions.

We consider first, the Dirichlet Problem in an irregular domain which is defined as follows

**Definition 2.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega$ . The Dirichlet problem for linear second order elliptic PDE in two dimensional coordinates is given as*

$$\begin{cases} -\nabla \cdot (a\nabla\phi) + b\nabla\phi + c\phi = f & \text{in } \Omega \\ \phi(x, y) = g & \text{on } \partial\Omega \end{cases} \quad (1)$$

where the coefficients  $a = a(x, y)$ ,  $b = b(x, y)$  and  $c = c(x, y)$  are smooth in  $\partial\bar{\Omega}$  which satisfy

$$a(x, y) \geq a_0 > 0, \quad c(x, y) - \frac{1}{2} \nabla \cdot b(x, y) \geq 0, \quad \text{for all } x, y \in \Omega \quad (2)$$

and where  $f$  is a given function and  $g$  is the boundary data.

A function  $\phi(x, y) \in C^2(\bar{\Omega})$  that satisfies the second order elliptic PDE and the Dirichlet boundary condition in (1) is called the classical solution of the Dirichlet problem. The existence, uniqueness and stability of the solution obtained from the Dirichlet problem heavily depends on its well-posedness. The notion of well-posedness in the sense of Hadamard can be defined as follows.

**Definition 2.2 (Hadamard's well-posedness).** A Dirichlet problem (1) is said to be well-posed if:

1. a solution exists,
2. the solution is unique,
3. the solution depends continuously on the given data,

otherwise it is ill-posed.

If the Dirichlet problem is well-posed, obtaining a numerical approximation of the exact solution is possible as long as the boundary data to the problem are approximated suitably.

We begin the FDFDWM solution process by reducing equation (1) to a system of ordinary differential equations. This is accomplished by discretising along one of the spatial variable (say  $y$ ) having equally spaced sample,  $y^i = i\Delta y$  with  $N_y = 2^m$  subintervals, where  $m$  is the resolution of the wavelet function employed. The discretization of the spatial variable  $y$  is carried out using central difference approximation to obtain

$$-\nabla \cdot (\beta_1 \nabla \phi^i) + \beta_2 \nabla \phi^i + \phi_{\Sigma}^i = \beta_6 f^i \quad \text{for } i = 1, 2, \dots, N_y. \quad (3)$$

where

$$\begin{aligned} \beta_1 &= a\Delta y^2 \\ \beta_2 &= b\Delta y^2 \\ \beta_3 &= b\frac{\Delta y}{2} - a \\ \beta_4 &= 2a + c\Delta y^2 \\ \beta_5 &= -(a + b\frac{\Delta y}{2}) \\ \beta_6 &= \Delta y^2 \end{aligned}$$

and

$$\phi_{\Sigma}^i(x) = \beta_3 \phi^{i+1} + \beta_4 \phi^i + \beta_5 \phi^{i-1}. \quad (4)$$

## 2.1 Fictitious Domain and Penalty Formulation of the DP

We proceed to the next stage of the FDFWDM by writing equation (3) in a weak form and seek a solution in a larger class  $H^1$ . This is achieved by multiplying (3) by a test function,  $v \in H_0^1$  and integrating over the domain  $\Omega$  to give

$$\int_{\Omega} (-\nabla \cdot (\beta_1 \nabla \phi^i) + \beta_2 \nabla \phi^i + \phi_{\Sigma}^i) v dx = \int_{\Omega} \beta_6 f^i v dx \quad (5)$$

We apply the divergence theorem and setting  $v = 0$  on the  $\partial\Omega$  to obtain the following weak form,

$$\begin{cases} \text{find } \phi^i \in H^1(\Omega_F) \text{ such that} \\ \int_{\Omega} (\beta_1 \nabla \phi^i \nabla v + \beta_2 \nabla \phi^i v + \phi_{\Sigma}^i v) dx = \int_{\Omega} \beta_6 f^i v dx \quad \forall v \in H_0^1. \end{cases} \quad (6)$$

We introduce a bilinear form  $\alpha : V \times V \rightarrow \mathbb{R}$  and a linear functional  $L : V \rightarrow \mathbb{R}$ , such that  $\alpha(\cdot, \cdot)$  is continuous over  $V$ . Then we express equation (6) in a bilinear form as

$$\alpha(v, \eta) = \int_{\Omega} (\beta_1 \nabla v \nabla \eta + \beta_2 \nabla v \eta + v \eta) dx \quad \forall v, \eta \in V \quad (7)$$

and

$$L(v) = \beta_6 \int_{\Omega} f v dx \quad (8)$$

We note from (1) that when  $b = 0$ , the bilinear becomes symmetrical. That is  $\alpha(v, \eta) = \alpha(\eta, v)$ .

We apply the fictitious domain approach to the DP formulated in a weak form by embedding the original domain,  $\Omega$  of the DP (1) in a slightly larger but simple (rectangular) domain,  $\Omega_F$ . The fictitious domain  $\Omega_F$  is defined as a subspace of the Hilbert space. That is, we let  $V$  be a closed subspace  $H_p^1(\Omega_F)$  such that

$$\{v : v = \tilde{v}|_{\Omega}, \tilde{v} \in V\} = H_p^1(\Omega_F)$$

The choice for  $V$ , considering the Dirichlet boundary condition is  $H_0^1(\Omega)$  and the space  $V_p(\Omega_F)$  defined by

$$V_p(\Omega_F) = \{v \in H_0^1(\Omega_F) : v \text{ is periodic on } \partial\Omega\} \quad (9)$$

Given some  $s > 0$ , suppose that  $\Omega_F = (0, s)^2$  then the periodicity property in (9) implies that  $v(0, y) = v(s, y)$  and  $v(x, 0) = v(x, s)$ . Combining equations (7) and (8), the problem (6) can be formulated as a variational problem. That is,

$$\begin{cases} \text{find } \phi^i \in H^1(\Omega_F), \forall v \in H_0^1 \text{ so that} \\ \alpha(\phi^i, v) = L(v). \end{cases} \quad (10)$$

Now, we consider the already embedded reduced dimension DP (10) with an irregular domain. We reformulate the problem by applying a penalty term,  $\epsilon$  to regularize the irregular domain [9]. For  $\epsilon > 0$ ,

$$\begin{cases} \text{find } \phi_{\epsilon}^i \in V \text{ so that} \\ \epsilon \alpha(\phi_{\epsilon}^i, v) + \rho(\phi_{\epsilon}^i, v) = \epsilon L(v) + l(v) \quad \forall v \in V. \end{cases} \quad (11)$$

Then we show by the following theorem that  $\phi_\epsilon^i$  converges to a function  $\phi$  whose restriction to  $\Omega$  is the solution we seek.

**Theorem 2.1.** *Suppose that  $\Omega$  is a  $C^{0,1}$  domain, suppose also that the hypothesis on  $\alpha, \rho, L, l$  in (11) hold. Then we have*

$$\lim_{\epsilon \rightarrow 0} \|\phi_\epsilon^i - \phi^i\|_{H^1(\Omega_F)} = 0, \quad (12)$$

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-1/2} \|\phi_\epsilon^i - \phi\|_{H^1(\Omega)} = 0, \quad (13)$$

where  $\phi, \phi^i$  and  $\phi_\epsilon^i$  are the solution to DP (1), (10) and (11) respectively.

*Proof.* We apply Lax-Milgram theorem for the proof of this theorem [18].

1. *Boundedness of  $\{\phi_\epsilon^i\}_{\epsilon>0}$ :*

Let  $\phi^i$  be the solution of the problem (10). Setting  $v = \phi_\epsilon^i - \phi^i$  in (11), we obtain

$$\begin{aligned} \epsilon \alpha(\phi_\epsilon^i - \phi^i, \phi_\epsilon^i - \phi^i) + \rho(\phi_\epsilon^i - \phi^i, \phi_\epsilon^i - \phi^i) &= \epsilon [L(\phi_\epsilon^i - \phi^i) - \alpha(\phi^i, \phi_\epsilon^i - \phi^i)] \\ &+ l(\phi_\epsilon^i - \phi^i) - \rho(\phi^i, \phi_\epsilon^i - \phi^i). \end{aligned} \quad (14)$$

Since  $\phi^i$  in the domain  $\Omega$  give the solution  $\phi$ , it follows from (6), (7) and (9) that

$$\rho(\phi^i, v) = l(v), \quad v \forall v \in V. \quad (15)$$

Replacing  $v$  by  $\phi_\epsilon^i - \phi^i$  in (9) and combining with (14) we have

$$\alpha(\phi_\epsilon^i - \phi^i, \phi_\epsilon^i - \phi^i) + \frac{1}{\epsilon} \rho(\phi_\epsilon^i - \phi^i, \phi_\epsilon^i - \phi^i) = L(\phi_\epsilon^i - \phi^i) - \alpha(\phi^i, \phi_\epsilon^i - \phi^i). \quad (16)$$

Considering the ellipticity and continuity properties of  $\alpha, \rho, L$  and the positivity of  $\epsilon$ , it follows from (16) that

$$\gamma \|\phi_\epsilon^i - \phi^i\|_V^2 \leq (\|L\| + \|\alpha\| \|\phi^i\|_V) \|\phi_\epsilon^i - \phi^i\|_V, \quad \forall \epsilon > 0, \quad (17)$$

where in (17),  $\gamma > 0$  and where  $\|L\|$  and  $\alpha$  are defined by

$$\|L\| = \sup_v \frac{|L(v)|}{\|v\|_V}, \quad \forall v \in V \setminus \{0\},$$

and

$$\|\rho\| = \sup_{v, \eta} \frac{|\alpha(v, \eta)|}{\|v\|_V \|\eta\|_V}, \quad \forall v \in V \setminus \{0\}, \quad \forall \eta \in V \setminus \{0\},$$

The inequality (17) implies

$$\|\phi_\epsilon^i - \phi^i\|_V \leq C, \quad \forall \epsilon > 0,$$

hence

$$\|\phi_\epsilon^i\|_{H^1(\Omega)} \leq C, \quad \forall \epsilon > 0, \quad (18)$$

2. *Weak convergence of  $\{\phi_\epsilon^i\}_{\epsilon>0}$ :*

It follows from (18), and from the closedness of  $V$  in  $H(\Omega_F)$  that there exist  $\phi^* \in V$  and a subsequence, also represented by  $\{\phi_\epsilon^i\}_{\epsilon>0}$ , such that

$$\lim_{\epsilon \rightarrow 0} \phi_\epsilon^i = \phi^* \text{ weakly in } H^1(\Omega_F). \quad (19)$$

Combining (19) and (11) we obtain at the limit in (11) that

$$\rho(\phi^*, v) = l(v), \quad \forall v \in V,$$

That is

$$\phi^* \in H \quad (20)$$

Substituting  $v = \phi_\epsilon^i - \phi^i$  in (11) and considering the ellipticity of  $\rho(\cdot, \cdot)$ , we obtain

$$\begin{aligned} \alpha(\phi_\epsilon^i, v) &= L(v - \phi_\epsilon^i) + \alpha(\phi_\epsilon^i, \phi_\epsilon^i) + \frac{1}{\epsilon} \rho(\phi_\epsilon^i, \phi_\epsilon^i) \\ &+ \frac{1}{\epsilon} [l(v - \phi_\epsilon^i) - \rho(v, v - \phi_\epsilon^i)] \geq L(v - \phi_\epsilon^i) + \alpha(\phi_\epsilon^i, \phi_\epsilon^i) \\ &+ \frac{1}{\epsilon} [l(v - \phi_\epsilon^i) - \rho(v, v - \phi_\epsilon^i)], \quad \forall v \in V. \end{aligned} \quad (21)$$

Suppose now that  $v \in H$ , then we have  $v|_\Omega = \phi$  which implies that

$$\rho(v, v - \phi_\epsilon^i) = l(v - \phi_\epsilon^i), \quad \forall v \in H,$$

which combined with (21) implies in turn that

$$\alpha(\phi_\epsilon^i, v) \geq L(v - \phi_\epsilon^i) + \alpha(\phi_\epsilon^i, \phi_\epsilon^i) \quad \forall v \in H. \quad (22)$$

Since  $\alpha(\cdot, \cdot)$  is positive over  $V \times V$  we have, from (19),

$$\liminf_{\epsilon \rightarrow 0} \alpha(\phi_\epsilon^i, \phi_\epsilon^i) \geq \alpha(\phi^*, \phi^*),$$

which combined with (22) implies

$$\alpha(\phi^*, v) \geq L(v - \phi^*) + \alpha(\phi^*, \phi^*) \quad \forall v \in H.$$

Hence, considering (20)  $\phi^*$  is a solution of (10). Since (10) has a unique solution, we have  $\phi^* = \phi^i$ , which implies that the sequence  $\{\phi_\epsilon^i\}_{\epsilon>0}$  converges to  $\phi^i$ .

3. *Strong convergence of  $\{\phi_\epsilon^i\}_{\epsilon>0}$ :*

Following from inequality (16); we have established that  $\phi_\epsilon^i$  converges weakly to  $\phi^i$ . Combining this result with (16), and taking into account the ellipticity properties of  $\alpha(\cdot, \cdot)$  and  $\rho(\cdot, \cdot)$ , we finally obtain the convergence properties (12) and (13).

□

Having shown that the solution of the FDFDWM with penalty formulation given by (11) exist, we proceed to write the problem in an expanded form as follows: For  $\epsilon > 0$ , find  $\phi_\epsilon^i \in V$  such that

$$\int_{\Omega} (\beta_1 \nabla \phi_\epsilon^i \nabla v + \beta_2 \nabla \phi_\epsilon^i v + \phi_{\Sigma, \epsilon}^i v) dx + \frac{1}{\epsilon} \int_{\partial\Omega} \phi_\epsilon^i v ds = \beta_3 \int_{\Omega_F} \tilde{f} v dx + \frac{1}{\epsilon} \int_{\partial\Omega} g v ds \quad (23)$$

for every  $v \in H_p^1(\Omega)$ , where  $\tilde{f}$  is an arbitrary  $L^2$ -extension of  $f$  in  $\Omega_F$ . Substituting equation (4) into (23) we obtain

$$\begin{aligned} \int_{\Omega} (\beta_1 \nabla \phi_\epsilon^i \nabla v + \beta_2 \nabla \phi_\epsilon^i v) dx &+ \int_{\Omega_F} (\beta_3 \phi_\epsilon^{i+1} + \beta_4 \phi_\epsilon^i + \beta_5 \phi_\epsilon^{i-1}) v_\epsilon dx \\ &+ \frac{1}{\epsilon} \int_{\partial\Omega} \phi_\epsilon^i v ds = \beta_3 \int_{\Omega_F} \tilde{f} v dx + \frac{1}{\epsilon} \int_{\partial\Omega} g v ds \end{aligned} \quad (24)$$

## 2.2 Wavelet Approximation of the DP

Now that we are done with the penalty formulation of the DP embedded in a fictitious domain, we proceed to approximate the weak formulation using compactly supported wavelet introduced by Daubechies [5]. We assume  $\Omega_F = (0, s)^2$ , where  $s$  is a positive integer. Let

$$V_p^i = \{v \in L^2(\Omega) : v(x) = \sum_k c_k \varphi_{k,j}(x), x \in (0, s), \text{ with } c_k = c_{k+2^i s}\} \quad (25)$$

We write the approximate solution to the penalty problem at a fixed value of  $y^i$  as

$$\tilde{\phi}_\epsilon(x, y^i) = 2^{\frac{m}{2}} \sum_k \xi_{k,m}^{\epsilon, i} \varphi(2^m x - k) \in V_p^i \quad (26)$$

where  $m$  and  $k$  represent resolution and scaling parameter respectively. Similarly, the boundary measure  $\mu^i$ ,

$$\mu(x, y^i) = 2^{\frac{m}{2}} \sum_k \mu_{k,m}^i \varphi(2^m x - j) \in V_p^i \quad (27)$$

where the coefficient  $\mu_{k,m}^i$  can be computed from the approximation of the boundary measure  $\|\partial\Omega\|$  at a fixed value of  $y^i$  and level  $m$ , given by

$$\|\partial\Omega\|_m^i = \sum_k \mu_{k,m}^i (2^m - k) \quad (28)$$

We recall from [18] that

$$\|\partial\Omega\|_m^i = -\nabla \chi_{\Omega, m}^i \cdot n_m^i \quad (29)$$

We can write

$$\begin{aligned} \nabla \chi_{\Omega, m}^i &= \frac{\partial \chi_{\Omega, m}^i}{\partial x}(x) \\ &= \sum_k \left( \frac{\partial \chi}{\partial x} \right)_{k,m}^i \varphi(2^m x - k) \end{aligned} \quad (30)$$

The coefficient of (30) can be written in terms of the connection coefficients as

$$\left(\frac{\partial \chi}{\partial x}\right)_{k,m}^i = \sum_k \Omega_{k-m}^d \chi_{k,m}^i \quad (31)$$

We approximate the normal vector  $\vec{n}$  at level  $m$  and fixed value of  $y^i$  by

$$n_m^i(x) = \sum_k \frac{\left(\frac{\partial \Upsilon}{\partial x}\right)_{k,m}^i}{|\nabla \Upsilon|_{k,m}^i} \varphi(2^m - k) \quad (32)$$

where

$$|\nabla \Upsilon|_{k,m}^i = \sqrt{\left(\left(\frac{\partial \Upsilon}{\partial x}\right)_{k,m}^i\right)^2} \quad (33)$$

Therefore from (29), (30) and (32), we have

$$-\nabla \chi_{\Omega,m}^i \cdot n_m^i(x) = -\sum_k \left[ \left(\left(\frac{\partial \chi}{\partial x}\right)_{k,m}^i \left(\frac{\partial \Upsilon}{\partial x}\right)_{k,m}^i\right) / |\nabla \Upsilon|_{k,m}^i \right] \varphi(2^m - k) \quad (34)$$

Hence

$$\mu_{k,m}^i = - \left[ \left(\left(\frac{\partial \chi}{\partial x}\right)_{k,m}^i \left(\frac{\partial \Upsilon}{\partial x}\right)_{k,m}^i\right) / |\nabla \Upsilon|_{k,m}^i \right] \quad (35)$$

Now substituting equations (26) and (27) into (24) we obtain,

$$\begin{aligned} & \beta_1 \sum_k \xi_{k,m}^{\epsilon,i} \int \varphi'(2^m x - k) \varphi'(2^m x - j) dx \\ & + \beta_2 \sum_k \xi_{k,m}^{\epsilon,i} \int \varphi'(2^m x - k) \varphi(2^m x - j) dx \\ & + \beta_3 \sum_k \xi_{k,m}^{\epsilon,i+1} \int \varphi(2^m x - k) \varphi(2^m x - j) dx \\ & + \beta_4 \sum_k \xi_{k,m}^{\epsilon,i+1} \int \varphi(2^m x - k) \varphi(2^m x - j) dx \\ & + \beta_5 \sum_k \xi_{k,m}^{\epsilon,i-1} \int \varphi(2^m x - k) \varphi(2^m x - j) dx \\ & + \frac{1}{\epsilon} \sum_k \mu_{k,m}^i \xi_{k,m}^{\epsilon,i} \int \varphi(2^m x - k) \varphi(2^m x - j) dx \\ & = \beta_6 \sum_k \tilde{f}^i \int \varphi(2^m x - k) \varphi(2^m x - j) dx \\ & + \frac{1}{\epsilon} \sum_k \mu_{k,m}^i g^i \int \varphi(2^m x - k) \varphi(2^m x - j) dx \end{aligned} \quad (36)$$

We introduce a shift to handle the exterior nodes  $\xi_{k,m}^{i+1}$  and  $\xi_{k,m}^{i-1}$ , which are given by

$$\xi_{k,m}^{i+1}\varphi(2^m x - k)\varphi(2^m x - j) = \xi_{k,m}^i\varphi(2^m x - k + 1)\varphi(2^m x - j) \quad (37)$$

and

$$\xi_{k,m}^{i-1}\varphi(2^m x - k)\varphi(2^m x - j) = \xi_{k,m}^i\varphi(2^m x - k - 1)\varphi(2^m x - j) \quad (38)$$

In addition, we introduce the following connection coefficients into equation (36) ,

$$\Omega_{k-j}^2 = \int_{-\infty}^{\infty} \varphi'(X - k)\varphi'(X - j)dx, \quad \text{for derivative } d = 2,$$

$$\Omega_{k-j} = \int_{-\infty}^{\infty} \varphi'(X - k)\varphi(X - j)dx, \quad \text{for derivative } d = 1$$

and the Kronecker-delta function,

$$\delta_{k,j} = \int_{-\infty}^{\infty} \varphi(X - k)\varphi(X - j)dx$$

where we set  $X = 2^m x$

Equation (36) becomes,

$$\begin{aligned} & \beta_1 \sum_k \xi_{k,m}^{\epsilon,i} \Omega_{k-j}^2 + \beta_2 \sum_k \xi_{k,m}^{\epsilon,i} \Omega_{k-j} + \beta_3 \sum_k \xi_{k,m}^{\epsilon,i} \delta_{k+1,j} \\ & + \beta_4 \sum_k \xi_{k,m}^{\epsilon,i} \delta_{k,j} + \beta_5 \sum_k \xi_{k,m}^{\epsilon,i} \delta_{k-1,j} + \frac{1}{\epsilon} \sum_k \xi_{k,m}^{\epsilon,i} \mu_{k,m}^i \delta_{k,j} \\ & = \beta_6 \sum_k \tilde{f}^i \delta_{k,j} + 2^{\frac{m}{2}} \frac{1}{\epsilon} \sum_k \mu_{k,m}^i g^i \delta_{k,j} \end{aligned} \quad (39)$$

We express equation (39) as a linear system in a vector form, given by

$$A_1 \vec{\xi} + A_2 \vec{\xi} + A_3 \vec{\xi} + A_4 \vec{\xi} + A_5 \vec{\xi} + \frac{1}{\epsilon} M \vec{\xi} = F + \frac{1}{\epsilon} M G \quad (40)$$

where  $A_1, A_2, A_3, A_4$  and  $A_5$  are sparse coefficient matrices. The matrix  $M$  corresponds to the boundary integral. We can write equation (40) compactly as

$$A \vec{\xi} + \frac{1}{\epsilon} M \vec{\xi} = F + \frac{1}{\epsilon} M G \quad (41)$$

where

$$A = \sum_i A_i, \quad i = 1, \dots, 5$$

### 3 Numerical Results

In this section, we present the results obtained from numerical experiments performed using the FDFDWM with penalty on two dimensional Dirichlet boundary value problems defined on irregular domains. In these experiments we consider two irregular domains; the star shaped and the diamond shaped domains which in each of the test cases are embedded in a slightly larger rectangular domain often described as the fictitious domain. The experiments are done using Daubechies wavelet of order  $D6$ , with varying levels of resolution (i.e:  $m = 1, m = 2$  and  $m = 3$ ) at each penalty parameter value,  $\epsilon = 10^0, 10^{-1}, \dots, 10^{-10}$ . The results from the FDFDWM with penalty are compared with the results from the classical FEM, to determine the level of accuracy and the rate of convergence of the approximate solutions. The FDFDWM algorithm is implemented by pre-computing the connection coefficients, resulting in the generation of the sparse coefficient matrix  $A$  in the linear system (41). In addition the boundary integrals are computed from the differentials of the characteristic function of the domain. Consequently, we present the results of the experiments in a form of two dimensional graphs and tables. The MATLAB software was used to carry out all the numerical experiments in the paper.

#### Test Case 1

In this test, we consider a Dirichlet problem defined on a star shaped domain which is centered at the origin and described by the equation,  $2e^{-a^2} \leq e^{-b^2x^2} + e^{-c^2y^2}$ , where the constants,  $a, b$  and  $c$  are positive real numbers and only the constant  $a$  is restricted by the following condition;  $|a| < \sqrt{\log_e 2} = 0.8326$ . This domain is embedded in a slightly larger rectangular domain  $\Omega_F = [-8, 8] \times [-8, 8]$ . We write the problem as

$$\begin{cases} -\Delta\phi + \phi = x^2 + y^2 - 4 & \text{in } \Omega \\ \phi(x, y) = g & \text{on } \partial\Omega \end{cases} \quad (42)$$

The analytical exact solution of (42) is given by

$$\phi(x, y) = \begin{cases} x^2 + y^2 & (x, y) \in \Omega \\ 0 & (x, y) \notin \Omega \end{cases} \quad (43)$$

where the domain is described by

$$\Omega = \{(x, y) : 2e^{-a^2} \leq e^{-b^2x^2} + e^{-c^2y^2}, \quad a = 0.82, b = 0.39 \text{ and } c = 0.39\}.$$

Using the FDWDWM with penalty parameter,  $\epsilon = 1$ , Daubechies wavelet coefficient of order  $D6$  and varying levels of resolution (i.e:  $m = 1, m = 2$  and  $m = 3$ ) the approximate solutions are displayed in figure 1.

It can be seen from figure 1 that, the FDFDWM with penalty offers satisfactory approximation to the exact solution,  $\phi = x^2 + y^2$  which has been restricted to a star shaped geometric domain. Varying the resolution from  $m = 1$  to  $m = 3$  at  $\epsilon = 1$  as displayed in figure 1, we observe that the accuracy of the approximation is being enhanced.

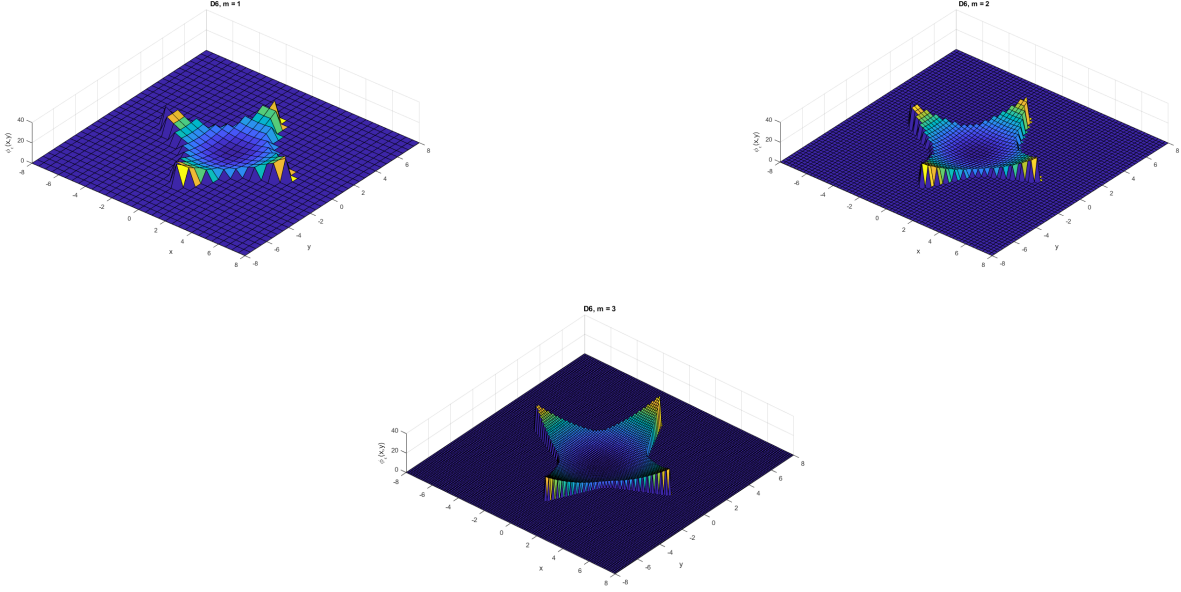


Figure 1: The FWDFDM with penalty approximation for the exact solution  $\phi = x^2 + y^2$  on a star domain for  $\epsilon = 1$ ,  $D6$  and  $m = 1, 2$  and  $3$

### Test Case 2

A diamond shaped domain centered at the origin is considered as the defined domain for solving the Helmholtz equation (42). The diamond domain,  $\Omega = \{(x, y) : x + y < 5\}$  is embedded in a fictitious domain  $\Omega_F = [-8, 8] \times [-8, 8]$ . The analytical exact solution of (42) is given by

$$\phi(x, y) = \begin{cases} x^2 + y^2 & (x, y) \in \Omega \\ 0 & (x, y) \notin \Omega \end{cases} \quad (44)$$

The approximate solution generated from FDWDWM with penalty, using  $\epsilon = 1$ , Daubechies wavelet coefficient of order  $D6$  and resolution at levels  $m = 1, m = 2$  and  $m = 3$  are shown in figure 2.

The FDFDWM with penalty approximation for test case 2 restricted to a diamond shaped geometric domain with radius 5 has also proven to generate reasonable results. It is apparent from the graphs that FDFDWM with penalty provides better approximation to the PDEs under consideration as the resolution of the scaling function gets bigger. Moreover, we need to know the effect of the penalty parameter,  $\epsilon$  on the accuracy of our method. Analysis of  $\epsilon$  and the error of approximation are discussed in the succeeding section.

## 3.1 Error Analysis for FDFDWM with Penalty

In the previous section our concern was on whether the FDFDWM with penalty could give any meaningful results. From figures 1 and 2, it is quite clear that the FDFDWM with penalty also does well with the approximation of solutions of linear elliptic PDEs. Notwithstanding, the level of accuracy needs to be dealt with thoroughly. In this section,

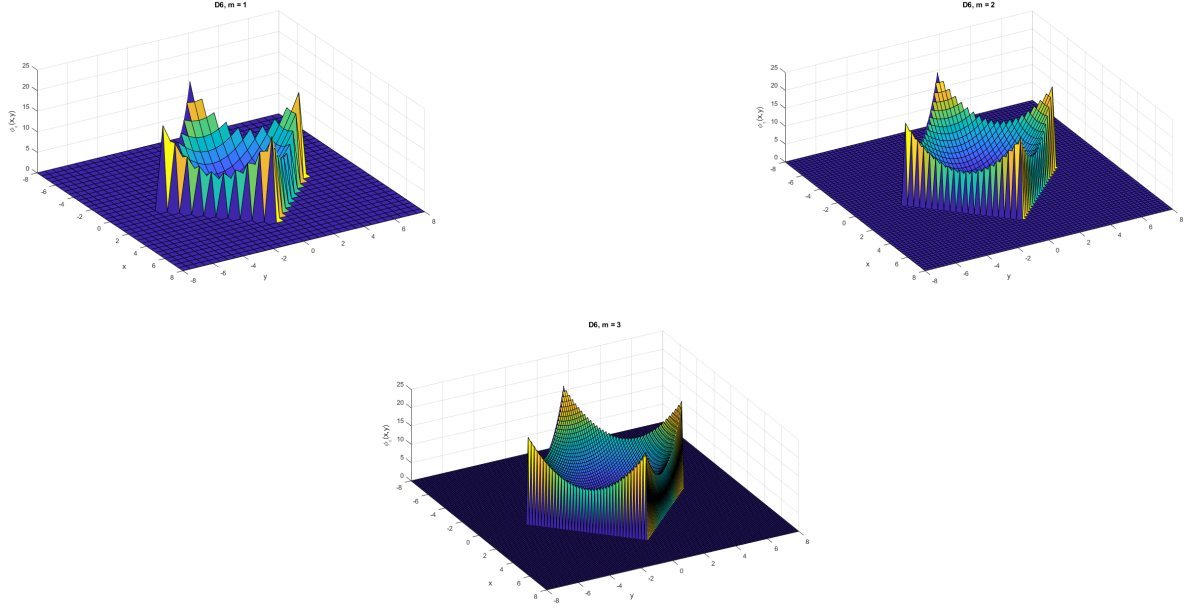


Figure 2: The FWDFDM with penalty approximation for the exact solution  $\phi = x^2 + y^2$  on a star domain for  $\epsilon = 1$ ,  $D6$  and  $m = 1, 2$  and  $3$

we look at the impact of the variations in the value of the penalty parameter,  $\epsilon$  on the error of approximation. We use the relative  $L^2$  norm error defined as

$$E = \frac{\|\phi - \phi_\epsilon^i\|_{L^2(\Omega)}}{\|\phi\|_{L^2(\Omega)}},$$

to estimate the error incurred as a result of the FDFDWM with penalty approximations. In table 1 and 2, varying value of  $\epsilon = 1, 10^{-1}, \dots, 10^{-10}$  used at resolution  $m = 1, 2$  and  $3$ , with  $D6$  for test case 1 and 2 respectively, are presented.

Table 1: Relative  $L^2$  error for  $\phi = x^2 + y^2$  on a star domain for FDFDWM with penalty

| $\epsilon$ | $m = 1$           | $m = 2$           | $m = 3$           |
|------------|-------------------|-------------------|-------------------|
| 1          | 7.58149693473e-01 | 5.74277100332e-01 | 2.67247288780e-01 |
| $10^{-1}$  | 2.81518911453e-01 | 1.57494742054e-01 | 9.11708734045e-02 |
| $10^{-2}$  | 9.02774513126e-02 | 7.91268150045e-02 | 3.12707768725e-02 |
| $10^{-3}$  | 5.22544449106e-02 | 1.25749497856e-02 | 9.29208462855e-03 |
| $10^{-4}$  | 1.62747016103e-02 | 6.57439692367e-03 | 1.38665331242e-03 |
| $10^{-5}$  | 8.30166592833e-03 | 1.49495664433e-03 | 6.40764505323e-04 |
| $10^{-6}$  | 6.14114138804e-03 | 8.27628725474e-04 | 2.29516508279e-04 |
| $10^{-7}$  | 3.79895398076e-03 | 3.40098639815e-04 | 8.11607886747e-05 |
| $10^{-8}$  | 1.93646526326e-03 | 1.70588726909e-04 | 5.78102675002e-05 |
| $10^{-9}$  | 9.70349204229e-04 | 3.76045664285e-05 | 9.30121680285e-06 |
| $10^{-10}$ | 8.94335308894e-04 | 3.00351056878e-05 | 8.68786307716e-06 |

Table 2: Relative  $L^2$  error for  $\phi = x^2 + y^2$  on a star domain for FDFDWM with penalty

| $\epsilon$ | $m = 1$           | $m = 2$           | $m = 3$           |
|------------|-------------------|-------------------|-------------------|
| $10^0$     | 3.60393185120e-01 | 1.57526431008e-01 | 2.80294932785e-02 |
| $10^{-1}$  | 9.14834373283e-02 | 2.79415498199e-02 | 9.78191153461e-03 |
| $10^{-3}$  | 8.34997200329e-03 | 5.08279077499e-03 | 1.02863049976e-03 |
| $10^{-4}$  | 6.12478672023e-03 | 1.16822499992e-03 | 8.85375864579e-04 |
| $10^{-5}$  | 3.49964476727e-03 | 9.05540325570e-04 | 2.67528659132e-04 |
| $10^{-6}$  | 1.06020473732e-03 | 6.78838868034e-04 | 7.53398665800e-05 |
| $10^{-7}$  | 8.34793559933e-04 | 5.89344934266e-04 | 1.78517329863e-05 |
| $10^{-8}$  | 4.61203933311e-04 | 1.60769210262e-04 | 9.84145837122e-06 |
| $10^{-9}$  | 1.18175358943e-04 | 7.58924274663e-05 | 3.81498060525e-06 |
| $10^{-10}$ | 9.43750721999e-05 | 4.86807935714e-05 | 3.78343920292e-07 |

From tables 1 and 2, we can deduce that the variations in the value of  $\epsilon$  from  $10^1$  to  $10^{-10}$  has positive impact on the accuracy of the FDFDWM with penalty approximation. The error decays along the variations of  $\epsilon$  and across the resolution of the scaling function (i.e  $m = 1, 2$  and  $3$ ). This is in accordance with theorem 2.1, which shows that as  $\epsilon \rightarrow 0$  the error diminishes and the approximate solution approaches the exact solution. These results agree with the findings of a number of studies including, ([7], [9], [10] and [18]).

We now look at the performance of the FDFDWM with penalty as against FEM. Here we are not considering the FDM because it is not a suitable method for solving PDEs defined on complex domains as discussed in the introductory chapter. The relative  $L^2$  norm error has been estimated for both the penalized FDFDWM and FEM at varying penalty parameter,  $\epsilon = 1, 10^{-1}, \dots, 10^{-10}$  using resolution  $m = 1$  and scaling function order,  $D6$  for test case 1 and 2 respectively as shown in figure 3.

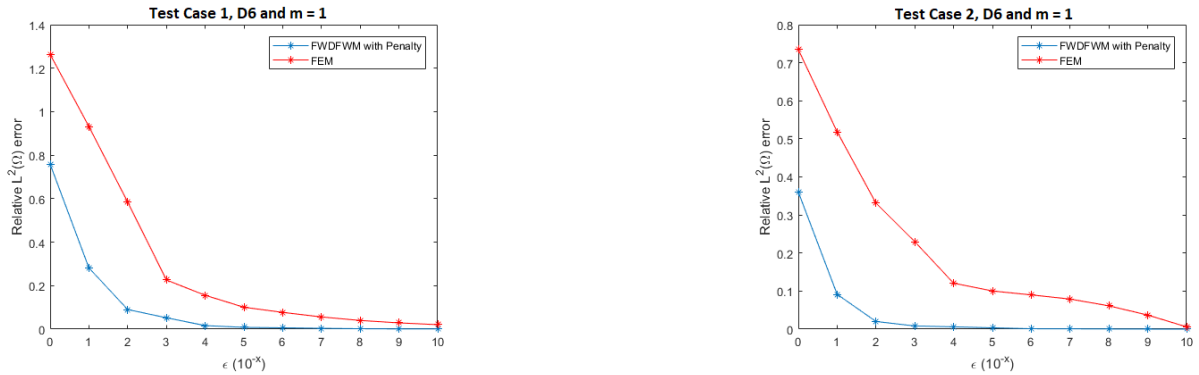


Figure 3: Relative  $L^2$  error of FDFDWM with penalty and FEM for Test Cases 1 and 2 with varying penalty parameter,  $\epsilon = 1$  to  $\epsilon = 10^{-10}$  at resolution  $m = 1$

Considering figure 3, both penalized FDFDWM and FEM decay in  $L^2$  norm error with varying  $\epsilon = 10^{-0}$  to  $\epsilon = 10^{-10}$ . However, FDFDWM with penalty converges rapidly to

the exact solution than that of the FEM. This to buttress the trend that emblematic with FDFDWM with penalty approximations.

## 4 Conclusion

In this paper, we have shown that the FDFDWM with penalty generates reasonable approximation to the Dirichlet problem defined on irregular domains. The FDFDWM with penalty has proven to converge rapidly and generates more accurate approximate solutions compare with the classical FEM. Handling the boundary integrals resulting from the boundaries of the irregular domains using derivatives of the characteristic function and wavelet approximation has been implemented successfully.

## References

- [1] Anderson, J.D. (2007) Fundamentals of Aerodynamics, 4th Ed., McGrawHill.
- [2] Barnes, B., Osei-Frimpong, E., Boateng, F. O. and Ackora-Prah, J. (2016). *A Two-Dimensional Chebyshev Wavelet Method for Solving Partial Dierential Equations*, Journal of Mathematical Theory and Modeling. 6(8), 124 - 138.
- [3] Bueno-Orovio, A., Perez-Garcia, V. M. and Fenton F. H. (2006). Spectral methods for partial differential equations in irregular domains: The spectral smoothed boundary method. SIAM Journal on Scientic Computing, 28(3), 886900.
- [4] Burrus, C. S., Gophinath, R. A. and Guo, H., (2015). *Wavelets and Wavelet Transforms*, Creative Commons Attribution License, OpenStax-CNX.
- [5] Daubechies, I. (1992), Ten Lectures on Wavelets, NSF-CBMS Regional Conference Series in Applied Math. 61, SIAM Publications, Philadelphia.
- [6] Debnath, L. and Shah, F. A. (2015). *Wavelet Transforms and Their Application*, 2nd Edition, Springer Science+Business Media New York, pp 151-152
- [7] Deka, R. K. and Choudhury, A. H. (2009). Solution of Two Point Boundary Value Problems Using Wavelet Integrals. *Int. J. Contemp. Math. Sciences*, Vol. 4, No. 15, 695 - 717
- [8] Eckert, M. (2006). The Dawn of Fluid Dynamics: A Discipline Between Science and Technology. Wiley. p. ix.
- [9] Glowinski R., Pan T.W., Wells R.O. and Zhou X(1996). *Wavelet and finite element solutions for the Neumann problem using fictitious domains*, J. Comput. Phys. , 126, 40-51.
- [10] Katunin, A. (2012). Solution of Plane Dirichlet Problem using Compactly Supported 2D Wavelet Scaling Functions. *Scientific Research of the Institute of Mathematics and Computer Science*, 1(11), 31 - 40.

- [11] Mishra, V. and Sabina (2011). Wavelet-Galerkin Finite Difference Solutions of ODEs. *AMO-Advanced Modeling and Optimization*, Vol. 13, No. 3, 539 - 549.
- [12] Nazarenko, S. (2014). Fluid Dynamics via Examples and Solutions, CRC Press (Taylor and Francis group)
- [13] Sabina and Mishra, V. (2011). Wavelets and the Evaluation of Filter Coefficients. *International Journal of Mathematical Archieve*. Vol 2(10), pp 1776 - 1793.
- [14] Strauss, W. A. (2008). Partial Differential Equations an Introduction, 2nd ed., Wiley, New Jesse.
- [15] Süli, E. (2012). Finite Element Methods for Partial Differential Equations. Lecture Notes. Mathematical Institute University of Oxford.
- [16] Wale, S. S. and Asutkar, G. V. (2014). Evaluation of Wavelet Connection Coefficients by Wavelet Galerkin Approximation. Annual IEEE India Conference (INDICON).
- [17] Walnut, D. F. (2002). *An Introduction to Wavelet Analysis*, Birkhäuser Boston
- [18] Wells Jr., R.O. and Zhou, X. (1995). Wavelet solutions for the Dirichlet problem, *Numer. Math.* 70, 379396.