

MODIFIED LAGUERRE COLLOCATION BLOCK METHOD FOR SOLVING INITIAL VALUE PROBLEMS OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS.

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Abstract

In this paper, an implicit k-step linear multi-step methods using the Laguerre polynomials as the basis functions is proposed. The given discrete methods were used in block and implemented for solving the initial value problems, being continuous interpolant derived and collocated at grid points. Numerical examples of initial value problems (IVPs) of ordinary differential equations (ODEs) were solved using the derived methods, and it is observed that the results obtained converged faster and the consistency and the zero stability are validated.

Keywords : Block methods; Laguerre polynomials; collocation and convergent.

1 Introduction

The analytical solutions of ordinary differential equations of the form :

$$f(x, y, y', y'', \dots, y^{(n)}) = 0, y(a) = y_0, y'(a) = y_1, \dots, y^{(n-1)}(a) = y_n, \quad (1)$$

have been difficult to obtain, especially when the equation is nonlinear. In

order to obtain an approximate solution to the problem, numerical methods are resorted to. Also, solutions to differential equations which always occur in series form, have been possibly obtained in closed forms as a result of numerical methods. Hence, various numerical techniques such as finite difference methods (FDM), integral transform methods (ITM), linear multi-step methods (LMM) e.t.c., have been developed by many researchers to provide approximate solutions to both ordinary and partial differential equations. For instance, the power series expansion solution in the form of convergent series were considered in El-Ajou et al (2015) to obtain analytical solutions of partial differential equations. Also, in Abu Arqub (2015), iterative methods were developed for solving initial value problems using approximate solutions with series form in terms of parametric form. The methods are always developed based on the nature and the type of the differential equation under consideration.

Ordinary differential equations of higher orders always give rise to discrete variable methods known as the single step and multi-step methods which are otherwise known as linear multi-step methods. The single step methods such as Euler's and Runge-Kutta methods among others, are purposely developed to solve first order (IVP) of ordinary differential equations. Solving these kinds of differential equations by implementing the single-step methods require that the differential is reduced to a system of first order (IVP) of ODEs. However, eminent researchers like Awoyemi (2001,2003), Sarafan (1965,1990), Simos (2002), Abualnaja (2015), Lambert (1973,1976,1991), Kayode (2004) and Fatunla (2004), have attempted to solve the differential equation using (LMM) without reducing to a system of a first order ODEs. Moreover, Onumanyi et al (1999) and Kayode (2005), independently proposed (LMM) for the first order differential equations in the predictor-

corrector mode, using the power series as basis function.

Continuous (LMM) ensures easy approximation of solution at all points of the integration interval (Awoyemi, 1999). so based on the collocation method, Awoyemi (1995) proposed a two-step hybrid method for the solution of a first order (IVP) at chosen grid points which was implemented on the hybrid predictor-corrector mode. Other researchers who also studied the hybrid method include Onumanyi (2001), Yahaya and Badmus (2009). The development of a class of methods called block method, is an outcome of improving the numerical solution of (IVPs) of ordinary differential equations. It was first proposed by Milne (1965). Afterwards, many scholars have been working on the implementation of the block methods for the numerical solution of ordinary differential equations. Among them are Chu and Hamilton (1987), Fatunla (1994), Omar and Suleiman (1999,2003 and 2005), and Ismail et al (2009).

In the recent years, Jator (2007) and Jator and Li (2009) proposed five-step and four-step (LMM) respectively to obtain a (FDM) as a block for the direct solution of first order (IVP). Very recently, a block technique with (LMM), adopting Legendre polynomials as basis function was recently developed by Abualnaja (2015), but the block schemes were not included. Thus, in this work, Laguerre polynomials is considered as basis function to be implemented using a block method.

2 Derivation of the Method

In this section, discrete methods are derived to solve

$$y'(x) = f(x, y(x)), y(x_0) = y_0, \quad (2)$$

Considering the approximate solution of the perturbed form of (2) as

$$y_k(x) = \sum_{i=0}^k c_i \psi_i'(x), x_n \leq x \leq x_{n+k}, \quad (3)$$

and letting the Laguerre polynomials be defined on $[-1, 1]$, which can be determined with the aid of the recurrence formula

$$Y(x) = \sum_{i=0}^k a_i L_i(x) = y(x), \quad (4)$$

and,

$$\psi_i(x) = x^i, i = 0, 1, \dots, k \quad (5)$$

from equation (5), it can be deduced that $\psi'_0(x) = 0, \psi'_1(x) = 1$. Substituting equation (3) into equation (2) leads to

$$\sum_{i=0}^k c_i \psi'_i(x) = f(x, y) + \lambda L_k(x), \quad (6)$$

such that the approximate solution of the perturbed form of (1) is expressed in the form

$$y_k(x) = \sum_{i=0}^k a_i L_i(x) \quad (7)$$

where,

$$\psi_i(x) = x^i, i = 0, 1, \dots, k \quad (8)$$

substituting (3) into (2), we have

$$\sum_{i=0}^k c_i \psi'_i(x) = f(x, y) + \lambda L_k(x) \quad (9)$$

where, $L_k(x)$ is the Laguerre polynomial of degree k valid in the interval $x_n \leq x \leq x_{n+k}$ and λ is the perturbed parameter. Hence, deducing $\psi_0(x) = 0$, $\psi_1(x) = 1$, then (5) will reduce to the form

$$f(x, y) = c_1 - \lambda L_1(x) \quad (10)$$

collocating (6) at $x_{n+i}, i = 0, 1$ and interpolate (3) at $x = x_n$, to get a system of three equations with $c_i (i = 0, 1)$ and parameter, λ

$$\begin{aligned} y_n &= c_0 + c_1 x_n \\ f_n &= c_1 + 2\lambda \\ f_{n+1} &= c_1 \end{aligned} \tag{11}$$

Solving (11), we obtain

$$\lambda = \frac{1}{2}(f_n - f_{n+1}), c_1 = f_{n+1}, c_0 = y_n - x_n(f_{n+1}).$$

From (3), we have

$$\bar{y}(x) = c_0 + c_1 x \tag{12}$$

Hence, the required numerical scheme of the proposed method is obtained by collocating (12) at $x = x_{n+1}$ and substitute c_0, c_1, λ as follows

$$y_{n+1} = y_n + h f_{n+1} \tag{13}$$

CASE K=2

In this case, we take the polynomial $L_2 = \frac{1}{2}(x^2 - 4x + 2)$ and use the shifted polynomial

$$x = \frac{2\bar{x} - (x_{n+k} + x_n)}{(x_{n+k} - x_n)}, \tag{14}$$

then collocating this equation at x_n, x_{n+1} and x_{n+2} , we get

$$L_2(x_n) = 7, L_2(x_{n+1}) = 2, L_2(x_{n+2}) = -1.$$

In addition, from (4), we can deduce that $\psi'_0(x) = 0, \psi'_1(x) = 1, \psi'_2(x) = 2x$.

Then (5) will reduce to the form

$$f(x, y) = c_1 + 2xc_2 - \lambda L_2(x) \tag{15}$$

Collocate (15) at $x_{n+i}, i = 0, 1, 2$ and interpolate (3) at $x = x_n$ to get a

system of four equations with c_i , ($i = 0, 1, 2$) and parameter, λ

$$\begin{aligned}
y_n &= c_0 + c_1 x_n + c_2 x_n^2 \\
f_n &= c_1 + 2x_n c_2 - 7\lambda \\
f_{n+1} &= c_1 + 2x_{n+1} c_2 - 2\lambda \\
f_{n+2} &= c_1 + 2x_{n+2} c_2 + \lambda
\end{aligned} \tag{16}$$

Solving (16), we get

$$\begin{aligned}
\lambda &= \frac{1}{2}(2f_{n+1} - f_n - f_{n+2}) \\
c_0 &= y_n - \frac{x_n}{2}(14f_{n+1} - 5f_n - 7f_{n+2}) - \frac{x_n^2}{2h}(3f_n - 8f_{n+1} + 5f_{n+2}) \\
c_1 &= \frac{1}{2}(14f_{n+1} - 5f_n - 7f_{n+2}) - \frac{x_n}{2h}(3f_n - 8f_{n+1} + 5f_{n+2}) \\
c_2 &= \frac{1}{4h}(3f_n - 8f_{n+1} + 5f_{n+2})
\end{aligned} \tag{17}$$

From (3), we have

$$\bar{y} = c_0 + c_1 x + c_2 x^2 \tag{18}$$

Hence, the required numerical scheme of the proposed method is obtained by collocating (18) at $x = x_{n+1}$ and substitute c_0, c_1, c_2, λ as follows

$$y_{n+1} = y_n + \frac{h}{4}(-7f_n + 20f_{n+1} - 9f_{n+2}) \tag{19}$$

CASE K=3

In this case, we take the polynomial $L_3 = x^3 - 9x^2 + 18x - 6$ and in conjunction with (10). Then collocating at x_n, x_{n+1}, x_{n+2} and x_{n+3} , we obtain

$$L_3(x_n) = -34, L_3(x_{n+1}) = -\frac{352}{27}, L_3(x_{n+2}) = -\frac{26}{27}, L_3(x_{n+3}) = 4.$$

In addition, from (4), it can be deduced that $\psi'_0(x) = 0, \psi'_1(x) = 1, \psi'_2(x) = 2x, \psi'_3(x) = 3x^2$.

Then (5) will reduce to the form

$$f(x, y) = c_1 + 2xc_2 + 3x^2c_3 - \lambda L_3(x) \quad (20)$$

We now collocate (25) at $x_{n+i}, i = 0, 1, 2, 3$ and interpolate (3) at $x = x_n$ to get a system of equations with $c_i, (i = 0, 1, 2, 3)$ and parameter, λ

$$\begin{aligned} y_n &= c_0 + c_1x_n + c_2x_n^2 + c_3x_n^3 \\ f_n &= c_1 + 2c_2x_n + c_3x_n^2 + 34\lambda \\ f_{n+1} &= c_1 + 2c_2x_{n+1} + 3c_3x_{n+1}^2 + \frac{352}{27}\lambda \\ f_{n+2} &= c_1 + 2c_2x_{n+2} + 3c_3x_{n+2}^2 + \frac{26}{27}\lambda \\ f_{n+3} &= c_1 + 2c_2x_{n+3} + 3c_3x_{n+3}^2 - 4\lambda \end{aligned} \quad (21)$$

Solving equation (25) above, we get

$$\begin{aligned} \lambda &= \frac{9}{16}(f_n - 3f_{n+1} - 3f_{n+2} - f_{n+3}) \\ c_0 &= \frac{-1}{48h^2}(-2754t_nh^2f_{n+2} + 918t_nf_{n+3}h^2 - 307ht_n^2f_n + 81t_nf_{n+1}h^2 - 870t_nf_nh^2 + \\ &\quad 981ht_n^2f_{n+1} + 343ht_n^2f_{n+3} - 1017ht_n^2f_{n+2} - 48y_nh^2 - 112t_n^3f_{n+2} + 32t_n^3f_n + 40t_n^3f_{n+3} + \\ &\quad 2754t_nh^2f_{n+1} + 104t_n^3f_{n+1}) \\ c_1 &= \frac{1}{24h^2}(-435f_nh^2 + 156t_n^2f_{n+1} - 48t_n^2f_n + 60t_n^2f_{n+3} - 168t_n^2f_{n+2} + 981ht_nhf_{n+1} - \\ &\quad 307ht_nf_{n+1} + 343ht_nf_{n+3} - 1017t_nhf_{n+2} + 1377h^2f_{n+1} + 459h^2f_{n+3} - 1377f_{n+2}h^2 \\ c_2 &= \frac{-1}{48}(312t_nf_{n+1} + 96t_nf_n + 120t_nf_{n+3} - 336t_nf_{n+2} + 981hf_{n+1} - 307hf_n + \\ &\quad 343hf_{n+3} - 1017hf_{n+2} \\ c_3 &= \frac{1}{6}\left(\frac{-4f_n + 13f_{n+1} - 14f_{n+2} + 5f_{n+3}}{h^2}\right) \end{aligned}$$

From (3), we have

$$\bar{y} = c_0 + c_1x + c_2x^2 + c_3x^3 \quad (23)$$

Now the required numerical scheme of the proposed method will be obtained if we collocate (3) at $x = x_{n+1}$ and substitute $c_0, c_1, c_2, c_3, \lambda$ as follows
Thus,

$$y_{n+1} = y_n + \frac{h}{48}(-595f_n + 1877f_{n+1} - 1849f_{n+2} + 615f_{n+3}) \quad (24)$$

CASE K=4

In this case, we take the polynomial $L_4 = \frac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24)$ and use (10). Then collocating at $x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$, we obtain
 $L_4(x_n) = \frac{209}{24}, L_4(x_{n+1}) = -\frac{1473}{384}, L_4(x_{n+2}) = 1, L_4(x_{n+3}) = -\frac{127}{384}, L_4(x_{n+4}) = -\frac{5}{8}$.

In addition, from (4), we can deduce that $\psi'_0(x) = 0, \psi'_1(x) = 1, \psi'_2(x) = 2x, \psi'_3(x) = 3x^2, \psi'_4(x) = 4x^3$.

Then (5) will reduce to the following form

$$f(x, y) = c_1 + 2xc_2 + 3x^2c_3 + 4x^3c_4 - \lambda L_4(x) \quad (25)$$

We now collocate (25) at $x_{n+i}, i = 0, 1, 2, 3, 4$ and interpolate (3) at $x = x_n$ to get a system of equations with $c_i, (i = 0, 1, 2, 3, 4)$ and parameter, λ

$$\begin{aligned} y_n &= c_0 + c_1x_n + c_2x_n^2 + c_3x_n^3 + c_4x_n^4 \\ f_n &= c_1 + 2c_2x_n + 3c_3x_n^2 + 4c_4x_n^3 - \frac{209}{24}\lambda \\ f_{n+1} &= c_1 + 2c_2x_{n+1} + 3c_3x_{n+1}^2 + 4c_4x_{n+1}^3 - \frac{1473}{384}\lambda \\ f_{n+2} &= c_1 + 2c_2x_{n+2} + 3c_3x_{n+2}^2 + 4c_4x_{n+2}^3 - \lambda \\ f_{n+3} &= c_1 + 2c_2x_{n+3} + 3c_3x_{n+3}^2 + 4c_4x_{n+3}^3 + \frac{127}{384}\lambda \\ f_{n+4} &= c_1 + 2c_2x_{n+4} + 3c_3x_{n+4}^2 + 4c_4x_{n+4}^3 + \frac{5}{8}\lambda \end{aligned} \quad (26)$$

Solving equation (26) above, we get

$$\begin{aligned}
\lambda &= (64f_{n+1} - 16f_n - 96f_{n+2} + 64f_{n+3} + 64f_{n+4}) \\
c_0 &= \frac{1}{144h^3}(-80256x_nh^3f_{n+3} + 20064x_nf_{n+4}h^3 + 120384h^3x_nf_{n+2} - 27744x_n^2f_{n+1}h^2 - \\
&\quad 27936x_n^2f_{n+3}h^2 + 6990h^2x_n^2f_{n+4} + 6858h^2x_n^2f_n + 41832h^2x_n^2f_{n+2} - 3824x_n^3hf_{n+1} - \\
&\quad 3920x_n^3hf_{n+3} + 938x_n^3hf_n + 986x_n^3hf_{n+4} + 5820x_n^3hf_{n+2} + 19920x_nh^3f_n - 80256x_nh^3f_{n+1} + \\
&\quad 144y_n - 198x_n^4f_{n+1} + 45x_n^4f_n + 51x_n^4f_{n+4} + 288x_n^4f_{n+2}) \\
c_1 &= \frac{-1}{24h^3}(-9248x_nh^2f_{n+1}h^2 - 9312x_nh^2f_{n+3} + 2330x_nh^2f_{n+4} + 2286x_nh^2f_n + 13944x_nh^2f_{n+2} - \\
&\quad 1912hx_n^2f_{n+1} - 1960hx_n^2f_{n+3} + 469hx_n^2f_n + 493x_n^2hf_{n+4} + 2910hx_n^2f_{n+2} - 13376h^3f_{n+1} + \\
&\quad 3344f_{n+4}h^3 + 3320h^3f_n + 20064h^3f_{n+2} - 13376h^3f_{n+3} - 132x_n^3f_{n+3} - 124x_n^3f_{n+1} + 30f_nx_n^3 + \\
&\quad 34f_{n+4}x_n^3 + 192f_{n+2}x_n^3) \\
c_2 &= \frac{1}{24}(2910x_nf_{n+2}h + 493x_nf_{n+4}h + 469x_nf_nh - 1960x_nh f_{n+3} - 1912x_nf_{n+1}h + 1672f_{n+2}h^2 \\
&\quad + 1143f_nh^2 + 1165f_{n+4}h^2 - 198x_n^2f_{n+3} - 186x_n^2f_{n+1} + 45x_n^2f_n + 51x_n^2f_{n+4} + 288x_n^2f_{n+2} \\
&\quad - 4656h^2f_{n+3} - 4624f_{n+1}h^2) \\
c_3 &= \frac{-1}{72h^3}(-396x_nf_{n+3} - 372x_nf_{n+1} + 90x_nf_n + 102x_nf_{n+4} + 576x_nf_{n+2} - 1960hf_{n+3} - \\
&\quad 1912f_{n+1}h + 469f_nh + 493f_{n+4}h + 2910f_{n+2}h) \\
c_4 &= \frac{1}{48h^3}(15f_n - 62f_{n+1} + 96f_{n+2} - 66f_{n+3} + 17f_{n+4})
\end{aligned}$$

From (3), we have

$$\bar{y} = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 \quad (28)$$

.

Now the required numerical scheme of the proposed method will be obtained if we collocate (28) at $x = x_{n+1}$ and substitute $c_0, c_1, c_2, c_3, c_4, \lambda$ as follows

Thus,

$$y_{n+1} = y_n + \frac{h}{144}(-13955f_n + 56150f_{n+1} - 84084f_{n+2} + 56042f_{n+3} - 14009f_{n+4}) \quad (29)$$

Formulating the Block Scheme of Cases K=2,3 and 4

Case K=2

We collocate equation (14) at $x = x_{n+1}, x_{n+2}, x_{n+3}$ to give us

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{4}(-7f_n + 20f_{n+1} - 9f_{n+2}) \\ y_{n+2} &= y_n + \frac{h}{4}(-2f_n + 6f_{n+1} - 2f_{n+2}) \\ y_{n+3} &= y_n + \frac{h}{4}(12f_{n+1} - 3f_n + 3f_{n+2}) \end{aligned} \quad (30)$$

Case K=3

We collocate equation (18) at $x = x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}$ to give us

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{48}(-595f_n + 1877f_{n+1} - 1849f_{n+2} + 615f_{n+3}) \\ y_{n+2} &= y_n + \frac{h}{3}(-48f_n + 151f_{n+1} - 146f_{n+2} + 49f_{n+3}) \\ y_{n+3} &= y_n + \frac{h}{16}(-237f_n + 747f_{n+1} - 711f_{n+2} + 249f_{n+3}) \\ y_{n+4} &= y_n + \frac{h}{6}(-77f_n + 247f_{n+1} - 239f_{n+2} + 93f_{n+3}) \end{aligned} \quad (31)$$

Case K=4

We collocate equation (22) at $x = x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ to give us

$$\begin{aligned} y_{n+1} &= y_n + \frac{h}{144}(-13955f_n + 56150f_{n+1} - 84084f_{n+2} + 56042f_{n+3} - 14009f_{n+4}) \\ y_{n+2} &= y_n + \frac{h}{18}(-2399f_n + 9644f_{n+1} - 14424f_{n+2} + 9620f_{n+3} - 2405f_{n+4}) \\ y_{n+3} &= y_n + \frac{h}{16}(-2191f_n + 8806f_{n+1} - 13164f_{n+2} + 8794f_{n+3} - 2197f_{n+4}) \\ y_{n+4} &= y_n + \frac{h}{9}(-1154f_n + 4640f_{n+1} - 6936f_{n+2} + 4640f_{n+3} - 1154f_{n+4}) \\ y_{n+5} &= y_n + \frac{h}{144}(-17275f_n + 69430f_{n+1} - 13620f_{n+2} + 69130f_{n+3} - 16945f_{n+4}) \end{aligned} \quad (32)$$

3 Basic Properties of the Method

3.1 Order, Error Constant and Consistency of the Methods

The schemes developed belong to the class of Linear Multi-step Method (LMM) of the form

$$\sum_{j=0}^k \alpha_j(x) y(x_{n+j}) = h \sum_{j=0}^k \alpha_j(x) f(x_{n+j}) \quad (33)$$

Equation (33) is a method associated with a linear difference operator

$$L[y(x); h] = \sum_{j=0}^k (\alpha_j y(x + jh) - h \alpha_j y'(x + jh)) \quad (34)$$

where $y(x)$ is continuously differentiable on the interval $[a, b]$, and the Taylor series expansion about the point x is expressed as

$$L[y(x); h] = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \cdots + c_q h^q y^{(q)}(x) \quad (35)$$

In line with [12], schemes (29, 24, 19) are said to be of order P if $C_0 = C_1 = C_2 = \dots = C_p = 0$ and the error constant is $C_{p+1} \neq 0$. Hence, we establish that (29), (24), and (19) are of the following orders respectively

when $k = 2$, $P = 3$ and $C_{p+1} = 0.041667$

when $k = 3$, $P = 3$ and $C_{p+1} = 0.016$

when $k = 4$, $P = 1$ and $C_{p+1} = 0.041667$

3.2 Stability Analysis

The scheme can be expressed as:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + h \begin{bmatrix} 5 & -\frac{9}{4} & 0 \\ 6 & -2 & 0 \\ 3 & \frac{3}{4} & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\frac{7}{4} \\ 0 & 0 & -2 \\ 0 & 0 & -\frac{3}{4} \end{bmatrix}$$

$$\begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

where,

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B^{(0)} = \begin{bmatrix} 5 & -\frac{9}{4} & 0 \\ 6 & -2 & 0 \\ 3 & \frac{3}{4} & 0 \end{bmatrix}$$

and

$$B^{(1)} = \begin{bmatrix} 0 & 0 & -\frac{7}{4} \\ 0 & 0 & -2 \\ 0 & 0 & -\frac{3}{4} \end{bmatrix}$$

The first characteristics polynomial of the scheme is

$$\rho(\lambda) = \det[\lambda A^0 - A^1]$$

$$\rho(\lambda) = \det \left[\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right] = \det \begin{bmatrix} \lambda & 0 & -1 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda-1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda-1 \end{vmatrix} = 0$$

$$\lambda^2(\lambda - 1) = 0$$

$$\lambda_1 = \lambda_2 \text{ or } \lambda_3 = 1$$

3.3 Zero-stability for case K=3

A block method is said to be stable as $h \rightarrow 0$ if the roots of the first characteristics polynomial defined by

$$\rho\lambda = \det[\lambda A^0 - A^1]$$

satisfies $|r_s| = 1$

The scheme can be expressed as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + h \begin{bmatrix} \frac{1877}{48} & -\frac{1849}{48} & \frac{615}{48} & 0 \\ \frac{151}{3} & -\frac{146}{3} & \frac{49}{3} & 0 \\ \frac{747}{16} & -\frac{711}{16} & \frac{249}{16} & 0 \\ \frac{247}{6} & -\frac{239}{6} & \frac{93}{6} & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -\frac{595}{48} \\ 0 & 0 & 0 & -16 \\ 0 & 0 & 0 & -\frac{237}{16} \\ 0 & 0 & 0 & -\frac{77}{6} \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

where,

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B^{(0)} = \begin{bmatrix} \frac{1877}{48} & -\frac{1849}{48} & \frac{615}{48} & 0 \\ \frac{151}{3} & -\frac{146}{3} & \frac{49}{3} & 0 \\ \frac{747}{16} & -\frac{711}{16} & \frac{249}{16} & 0 \\ \frac{247}{6} & -\frac{239}{6} & \frac{93}{6} & 0 \end{bmatrix}$$

and

$$B^{(1)} = \begin{bmatrix} 0 & 0 & 0 & -\frac{595}{48} \\ 0 & 0 & 0 & -16 \\ 0 & 0 & 0 & -\frac{237}{16} \\ 0 & 0 & 0 & -\frac{77}{6} \end{bmatrix}$$

The first characteristics polynomial of the scheme is

$$\rho(\lambda) = \det[\lambda A^0 - A^1]$$

$$\rho(\lambda) = \det \left[\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] = \det \begin{bmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda-1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda & 0 & 0 & -1 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda-1 \end{vmatrix} = 0$$

$$\lambda^3(\lambda - 1) = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 0 \text{ or } \lambda_4 = 1$$

3.4 Zero-stability for case K=4

A block method is said to be stable as $h \rightarrow 0$ if the roots of the first characteristics polynomial defined by

$$\rho\lambda = \det[\lambda A^0 - A^1]$$

satisfies $|r_s| = 1$

The scheme can be expressed as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + h \begin{bmatrix} \frac{4162}{5040} & -\frac{1308}{5040} & \frac{382}{5040} & -\frac{43}{5040} \\ \frac{9644}{18} & -\frac{14424}{18} & \frac{9620}{18} & -\frac{2405}{18} \\ \frac{8806}{16} & -\frac{13164}{16} & \frac{8794}{16} & -\frac{2197}{16} \\ \frac{4640}{9} & -\frac{6936}{9} & \frac{4640}{9} & -\frac{1154}{9} \\ \frac{69430}{144} & -\frac{103620}{144} & \frac{69130}{144} & -\frac{16945}{144} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1847}{5040} \\ 0 & 0 & 0 & 0 & -\frac{2399}{18} \\ 0 & 0 & 0 & 0 & -\frac{2191}{16} \\ 0 & 0 & 0 & 0 & -\frac{1154}{9} \\ 0 & 0 & 0 & 0 & -\frac{17275}{144} \end{bmatrix} \begin{bmatrix} f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

where,

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, B^{(0)} = \begin{bmatrix} \frac{4162}{5040} & -\frac{1308}{5040} & \frac{382}{5040} & -\frac{43}{5040} & 0 \\ \frac{9644}{18} & -\frac{14424}{18} & -\frac{9620}{18} & -\frac{2405}{18} & 0 \\ \frac{8806}{16} & -\frac{13164}{16} & \frac{8794}{16} & -\frac{2197}{16} & 0 \\ \frac{4640}{9} & -\frac{6936}{9} & \frac{4640}{9} & -\frac{1154}{9} & 0 \\ \frac{69430}{144} & -\frac{103620}{144} & \frac{69130}{144} & \frac{16945}{144} & 0 \end{bmatrix}$$

and,

$$B^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1847}{5040} \\ 0 & 0 & 0 & 0 & -\frac{2399}{18} \\ 0 & 0 & 0 & 0 & -\frac{2191}{16} \\ 0 & 0 & 0 & 0 & -\frac{1154}{9} \\ 0 & 0 & 0 & 0 & -\frac{17275}{144} \end{bmatrix}$$

The first characteristics polynomial of the scheme is

$$\rho(\lambda) = \det[\lambda A^0 - A^1]$$

$$\rho(\lambda) = \det \left[\begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right] = \det \begin{bmatrix} \lambda & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \lambda - 1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda & 0 & 0 & 0 & -1 \\ 0 & \lambda & 0 & 0 & -1 \\ 0 & 0 & \lambda & 0 & -1 \\ 0 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & \lambda - 1 \end{vmatrix} = 0$$

$$\lambda^4(\lambda - 1) = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0 \text{ or } \lambda_5 = 1$$

4 Numerical Experiments

In order to confirm the accuracy and efficiency of the scheme, we consider the following initial value problems:

Problem 1:

$$y' = -y(x), h = 0.1, y(0) = 1 \quad (36)$$

Exact solution: $y(x) = e^{-x}$. [see K.M.Abualnaja, 2015]

Problem 2:

$$y'(x) = -x(1 - y), h = 0.1, y(0) = 0 \quad (37)$$

Exact solution: $y(x) = 1 - e^{-\frac{x^2}{2}}$. [see K.M.Abualnaja, 2015]

Table 1: Results and errors of problem 1 for K=3

N	x	Numerical Result	Exact	Error Results
0	0.000000	1.000000	1.000000	0.000000
1	0.100000	0.905953	0.904837	1.116×10^{-3}
2	0.200000	0.820856	0.818730	2.126×10^{-3}
3	0.300000	0.743857	0.740818	3.039×10^{-3}
4	0.400000	0.674185	0.670320	3.865×10^{-3}
5	0.500000	0.611143	0.606531	4.612×10^{-3}
6	0.600000	0.554100	0.548811	5.289×10^{-3}
7	0.700000	0.502486	0.496585	5.901×10^{-3}
8	0.800000	0.455784	0.449328	6.456×10^{-3}
9	0.900000	0.413527	0.406569	6.958×10^{-3}
10	1.000000	0.375290	0.367879	7.411×10^{-3}

Table 2: Results and errors of problem 1 for K=4

N	x	Numerical Result	Exact	Error Results
0	0.000000	1.000000	1.000000	0.000000
1	0.100000	0.908872	0.904837	4.035×10^{-3}
2	0.200000	0.826416	0.818730	7.686×10^{-3}
3	0.300000	0.751807	0.740818	1.099×10^{-2}
4	0.400000	0.684298	0.670320	1.398×10^{-2}
5	0.500000	0.623213	0.606531	1.668×10^{-2}
6	0.600000	0.567941	0.548811	1.913×10^{-2}
7	0.700000	0.517929	0.496585	2.134×10^{-2}
8	0.800000	0.472676	0.449328	2.335×10^{-2}
9	0.900000	0.431729	0.406569	2.516×10^{-2}
10	1.000000	0.394679	0.367879	2.680×10^{-2}

Table 3: Results and errors of problem 2 for K=3

N	x	Numerical Result	Exact	Error Results
0	0.000000	0.000000	0.000000	0.000000
1	0.100000	0.005892	0.004988	9.040×10^{-4}
2	0.200000	0.020465	0.019801	6.664×10^{-4}
3	0.300000	0.037475	0.044003	6.528×10^{-3}
4	0.400000	0.078865	0.076884	1.981×10^{-3}
5	0.500000	0.120033	0.117503	2.530×10^{-3}
6	0.600000	0.159353	0.164729	5.376×10^{-3}
7	0.700000	0.221055	0.217295	3.760×10^{-3}
8	0.800000	0.279245	0.273851	5.394×10^{-3}
9	0.900000	0.328348	0.333023	4.675×10^{-3}
10	1.000000	0.398478	0.393469	5.009×10^{-3}

Table 4: Results and errors of problem 2 for K=4

N	x	Numerical Result	Exact	Error Results
0	0.000000	0.000000	0.000000	0.000000
1	0.100000	0.000792	0.004988	4.196×10^{-3}
2	0.200000	0.014504	0.019801	5.297×10^{-3}
3	0.300000	0.039213	0.044003	4.790×10^{-3}
4	0.400000	0.070658	0.076884	6.226×10^{-3}
5	0.500000	0.109369	0.117503	8.134×10^{-3}
6	0.600000	0.156529	0.164729	8.200×10^{-3}
7	0.700000	0.210379	0.217295	6.916×10^{-3}
8	0.800000	0.267841	0.273851	6.010×10^{-3}
9	0.900000	0.327112	0.333023	5.911×10^{-3}
10	1.000000	0.387823	0.393469	5.646×10^{-3}

5 Conclusion

In this research work, a class of implicit block collocation methods for the direct solution of initial value problems of general first order ordinary differential equations was developed using Laguerre collocation approach. The collocation technique yielded a consistent and zero stable implicit block multi-step methods with continuous coefficients; and the method is implemented without the need for developing correctors.

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