

Mathematical and numerical analysis for Neumann boundary value problem of the poisson equation

Abstract

This paper falls within the framework of mathematical modelling and that of numerical analysis. The analysis to be developed through this paper deals with three Neumann boundary value problems: one pure, one modified and the other with conduction term for the Poisson equation. We introduced Dirichlet and Neumann problems with conduction valuables to prove the continuity in comparison with conduction term of the Neumann problem with conduction term. We demonstrated the existence and uniqueness of the modified Neumann problem. For simplicity and concreteness, it was appropriate to choose the finite element and classical methods to find the numerical and the explicit solutions, respectively so that numerical simulations were implemented in *Scilab*. **Keywords:** Neumann's problem; Conduction term; Continuity; Finite element method; Numerical simulations.

Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, η is the exterior normal to the boundary $\partial\Omega$. We shall consider the following three Neumann boundary value problems: one pure, one modified and the other with conduction term for the Poisson equation to develop our analysis. The Pure Neumann boundary value problem for the Poisson equation [3]

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = g & \text{on } \partial\Omega \end{cases} \quad (0.1)$$

where $\frac{\partial u}{\partial \eta} = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} \eta_i$ and $\eta = (\eta_i)_{1 \leq i \leq n}$, does not admit a unique solution because if u is solution, $u + c$ (c constant) still solution. On the other hand, the Modified Neumann boundary value problem for the Poisson equation [4]

$$\begin{cases} -\Delta u + u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = g & \text{on } \partial\Omega \end{cases} \quad (0.2)$$

admits a unique solution that we will prove by the Lax-Milgram theorem. We shall introduce the homogeneous Dirichlet problem with conduction term and unhomogeneous Neumann problem with conduction term those will be valuable to prove the continuity in comparison with conduction term of the solution of the following Neumann boundary value problem with conduction term for Poisson equation [1]

$$\begin{cases} -\nabla \cdot (\sigma_i \nabla u_i) = f & \text{in } \Omega \\ \frac{\partial u_i}{\partial \eta} = g & \text{on } \partial\Omega \end{cases} \quad (0.3)$$

where $\sigma_i \in \mathbb{R}$, ($i = 1, 2$) represents the term of conduction of the model. The aim of this paper is to prove the existence, uniqueness, continuity and to find the exact and numerical solutions of the above problems. The resolution algorithm and the implementation of numerical simulations depend on the type of the solution and require a search of the exact solution. For simplicity and concreteness, it will be appropriate to use the one-dimensional Pure Neumann boundary value problem for the Poisson equation. Then, the finite element method and the classical method shall be developed to find the exact and numerical solutions so that numerical simulations will be implemented in *Scilab*. The analysis to be presented through the paper makes a strong use of the results and arguments of [2, 3, 4, 5, 6, 8, 10].

The organization of the paper is as follows. The homogeneous Dirichlet problem and the unhomogeneous Neumann problem with conduction term will be presented for developing the continuity analysis of problem (0.3). A variational formulation will be presented to demonstrate the existence and uniqueness of the problem (0.2). The one-dimensional Pure Neumann boundary value problem for the Poisson equation will be solved numerically and analytically by using the finite element method and the classical method, respectively. Finally using those exact and numerical solutions, numerical simulations will be implemented in *Scilab*.

1 Continuity of the solution u of modified Neumann problem with respect to σ

We first prove the continuity of the solution of the following homogeneous Dirichlet problem with conduction term

$$\begin{cases} -\sigma_i \Delta u_i = f & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

A variational formulation of the problem (1.1) is written as

$$\int_{\Omega} \sigma_i \nabla u_i \nabla v d\Omega = \int_{\Omega} f v d\Omega \quad \forall v \in H_0^1(\Omega)$$

For $i = 1$

$$\int_{\Omega} \sigma_1 \nabla u_1 \nabla v d\Omega = \int_{\Omega} f v d\Omega \quad (1.2)$$

For $i = 2$

$$\int_{\Omega} \sigma_2 \nabla u_2 \nabla v d\Omega = \int_{\Omega} f v d\Omega \quad (1.3)$$

By subtracting (1.3) from (1.2) we get

$$\int_{\Omega} (\sigma_1 \nabla u_1 - \sigma_2 \nabla u_2) \nabla v d\Omega = 0$$

which can also be written

$$\int_{\Omega} \sigma_1 \nabla(u_1 - u_2) \nabla v d\Omega + (\sigma_1 - \sigma_2) \int_{\Omega} \nabla u_2 \nabla v d\Omega = 0 \quad (1.4)$$

Let $v = u_1 - u_2 \in H_0^1(\Omega)$, the equality (1.4) becomes

$$\int_{\Omega} \sigma_1 \nabla(u_1 - u_2)^2 d\Omega + (\sigma_1 - \sigma_2) \int_{\Omega} \nabla u_2 \nabla(u_1 - u_2) d\Omega = 0$$

$$\int_{\Omega} \sigma_1 |\nabla(u_1 - u_2)|^2 d\Omega \leq |\sigma_1 - \sigma_2| \int_{\Omega} |\nabla u_2| |\nabla(u_1 - u_2)| d\Omega$$

$$\sigma_1 \|u_1 - u_2\|_{H_0^1(\Omega)}^2 \leq |\sigma_1 - \sigma_2| \|u_2\|_{H_0^1(\Omega)} \|u_1 - u_2\|_{H_0^1(\Omega)}$$

$$\|u_1 - u_2\|_{H_0^1(\Omega)} \leq \frac{|\sigma_1 - \sigma_2|}{\sigma_1} \|u_2\|_{H_0^1(\Omega)} \quad (1.5)$$

Otherwise

$$\begin{aligned}\int_{\Omega} \sigma_2 \nabla u_2 \nabla v d\Omega &= \int_{\Omega} f v d\Omega \\ \left| \int_{\Omega} \sigma_2 \nabla u_2 \nabla v d\Omega \right| &= \left| \int_{\Omega} f v d\Omega \right| \\ \int_{\Omega} |\sigma_2 \nabla u_2 \nabla v| d\Omega &\leq \int_{\Omega} |f v| d\Omega\end{aligned}$$

Applying Cauchy-Schwartz inequality [5, 10], we have

$$\sigma_2 \|\nabla u_2\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

$$\sigma_2 \|u_2\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)} \leq C \|v\|_{L^2(\Omega)}$$

By applying the inclusion of standards in $H^1(\Omega)$ and $L^2(\Omega)$ [5, 10] we have

$$\|u_2\|_{H_0^1(\Omega)} \leq \frac{MC}{\sigma_2}$$

So (1.5) becomes

$$\|u_1 - u_2\|_{H_0^1(\Omega)} \leq \frac{|\sigma_1 - \sigma_2|}{\sigma_1 \sigma_2} MC$$

$$\|u_1 - u_2\|_{H_0^1(\Omega)} \leq K \frac{|\sigma_1 - \sigma_2|}{\sigma_1 \sigma_2} \quad (\text{with } K = MC) \quad (1.6)$$

So if $\sigma_1, \sigma_2 \geq \mu > 0$ with μ fixed, that is to say

$$\sigma_1 \sigma_2 \geq \mu^2 \Rightarrow \frac{1}{\sigma_1 \sigma_2} \leq \frac{1}{\mu^2};$$

The inequality (1.6) becomes

$$\|u_1 - u_2\|_{H_0^1(\Omega)} \leq \frac{K}{\mu^2} |\sigma_1 - \sigma_2|$$

Which proves the continuity of u

$$\begin{aligned}[\mu, +\infty[&\longrightarrow H_0^1(\Omega) \\ \sigma &\longmapsto u_\sigma\end{aligned}$$

In the following we use the above result to prove the continuity of the solution of the problem (0.3)

Weak form of the problem (0.3) is written as

$$\int_{\Omega} \nabla u_i \nabla v d\Omega = \frac{1}{\sigma_i} \int_{\Omega} f v d\Omega + \int_{\partial\Omega} g v d\sigma \quad \forall v \in H^1(\Omega)$$

For $i = 1$

$$\int_{\Omega} \nabla u_1 \nabla v d\Omega = \frac{1}{\sigma_1} \int_{\Omega} f v d\Omega + \int_{\partial\Omega} g v d\sigma \quad (1.7)$$

For $i = 2$

$$\int_{\Omega} \nabla u_2 \nabla v d\Omega = \frac{1}{\sigma_2} \int_{\Omega} f v d\Omega + \int_{\partial\Omega} g v d\sigma \quad (1.8)$$

By making the difference of (1.7) and (1.8) we get

$$\begin{aligned} \int_{\Omega} (\nabla u_1 - \nabla u_2) \nabla v d\Omega &= \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) \int_{\Omega} f v d\Omega \\ \int_{\Omega} \nabla(u_1 - u_2) \nabla v d\Omega &= \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) \int_{\Omega} f v d\Omega \end{aligned} \quad (1.9)$$

Let $v = u_1 - u_2 \in H^1(\Omega)$, the problem (1.9) becomes

$$\int_{\Omega} \nabla(u_1 - u_2)^2 d\Omega = \frac{\sigma_2 - \sigma_1}{\sigma_1 \sigma_2} \int_{\Omega} f v d\Omega$$

Using the semi-norm

$$\begin{aligned} \left| \int_{\Omega} \nabla(u_1 - u_2)^2 d\Omega \right| &= \left| \frac{\sigma_2 - \sigma_1}{\sigma_1 \sigma_2} \int_{\Omega} f v d\Omega \right| \\ \int_{\Omega} |\nabla(u_1 - u_2)^2| d\Omega &\leq \left| \frac{\sigma_2 - \sigma_1}{\sigma_1 \sigma_2} \right| \int_{\Omega} |f v| d\Omega \end{aligned}$$

And applying the Cauchy-Schwartz inequality

$$\|u_1 - u_2\|_{H^1(\Omega)}^2 \leq \left| \frac{\sigma_2 - \sigma_1}{\sigma_1 \sigma_2} \right| \|f\|_{L^2(\Omega)} \|u_1 - u_2\|_{L^2(\Omega)}$$

$$\|u_1 - u_2\|_{L^2(\Omega)} \leq B \|u_1 - u_2\|_{H^1(\Omega)}$$

We then obtain

$$\begin{aligned} \|u_1 - u_2\|_{H^1(\Omega)} &\leq NB \frac{|\sigma_2 - \sigma_1|}{\sigma_1 \sigma_2} \\ \|u_1 - u_2\|_{H^1(\Omega)} &\leq L \frac{|\sigma_1 - \sigma_2|}{\sigma_1 \sigma_2} \quad (L = NB) \end{aligned} \quad (1.10)$$

So if $\sigma_1, \sigma_2 \geq \mu > 0$ with μ fixed, that is to say

$$\sigma_1 \sigma_2 \geq \mu^2 \Rightarrow \frac{1}{\sigma_1 \sigma_2} \leq \frac{1}{\mu^2};$$

The inequality (1.10) becomes

$$\|u_1 - u_2\|_{H^1(\Omega)} \leq \frac{L}{\mu^2} |\sigma_1 - \sigma_2|$$

Which proves the continuity of the solution u of Neumann's problem with conduction term with respect to σ that is to say

$$\begin{aligned} [\mu, +\infty[&\longrightarrow H^1(\Omega) \\ \sigma &\longmapsto u_\sigma \end{aligned}$$

1.1 Existence and uniqueness of the solution of the Modified Neumann problem

We will use the Modified problem (0.2) to prove the existence and uniqueness of the solution u using the Lax-Milgram theorem [5, 10].

$$f \in L^2(\Omega) \Rightarrow -\Delta u + u \in L^2(\Omega)$$

$$\Delta u \in L^2(\Omega)$$

$$u \in H^1(\Omega)$$

Let

$$V = H^1(\Omega) \quad \text{which is a Hilbert space.}$$

We now find the variational formulation of the problem (0.2)

Let

$$v \in H^1(\Omega)$$

Let us multiply the first equality of (0.2) by v and integrate over Ω

$$\int_{\Omega} -\Delta u \cdot v d\Omega + \int_{\Omega} u \cdot v d\Omega = \int_{\Omega} f \cdot v d\Omega$$

Using Green's formula we have

$$\int_{\Omega} \nabla u \cdot \nabla v d\Omega - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} \cdot v d\sigma + \int_{\Omega} u \cdot v d\Omega = \int_{\Omega} f \cdot v d\Omega$$

$$\int_{\Omega} \nabla u \cdot \nabla v d\Omega - \int_{\partial\Omega} g \cdot v d\sigma + \int_{\Omega} u \cdot v d\Omega = \int_{\Omega} f \cdot v d\Omega$$

$$\int_{\Omega} \nabla u \cdot \nabla v d\Omega + \int_{\Omega} u \cdot v d\Omega = \int_{\partial\Omega} g \cdot v d\sigma + \int_{\Omega} f \cdot v d\Omega$$

By setting

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v d\Omega + \int_{\Omega} u \cdot v d\Omega$$

and

$$L(v) = \int_{\partial\Omega} g \cdot v d\sigma + \int_{\Omega} f \cdot v d\Omega$$

We then obtain

$$a(u, v) = L(v)$$

Let us check the continuity of a since its bilinearity is trivial

$$|a(u, v)| = \left| \int_{\Omega} \nabla u \cdot \nabla v d\Omega + \int_{\Omega} u \cdot v d\Omega \right|$$

$$|a(u, v)| \leq \int_{\Omega} |\nabla u \cdot \nabla v| d\Omega + \int_{\Omega} |u \cdot v| d\Omega$$

Using the Cauchy-Schwartz inequality

$$|a(u, v)| \leq \left(\int_{\Omega} |\nabla u|^2 d\Omega \right)^{\frac{1}{2}} \cdot \left(\int_{\Omega} |\nabla v|^2 d\Omega \right)^{\frac{1}{2}} + \left(\int_{\Omega} |u|^2 d\Omega \right)^{\frac{1}{2}} \cdot \left(\int_{\Omega} |v|^2 d\Omega \right)^{\frac{1}{2}}$$

$$|a(u, v)| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

So that

$$|a(u, v)| \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

And applying the Poincaré inequality [5, 8]

$$\|u\|_{L^2(\Omega)} \leq c_1 \|u\|_{H^1(\Omega)}$$

$$\|v\|_{L^2(\Omega)} \leq c_2 \|v\|_{H^1(\Omega)}$$

Then

$$|a(u, v)| \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + c_1 c_2 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

$$|a(u, v)| \leq (1 + c_1 c_2) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

$$|a(u, v)| \leq c \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

with

$$c = 1 + c_1 c_2$$

Hence a is continuous.

Let us check the continuity of L since its linearity is trivial

$$\begin{aligned} |L(v)| &= \left| \int_{\partial\Omega} g.v d\sigma + \int_{\Omega} f.v d\Omega \right| \\ |L(v)| &\leq \int_{\partial\Omega} |g.v| d\sigma + \int_{\Omega} |f.v| d\Omega \end{aligned}$$

Using the Cauchy-Schwartz inequality we have

$$\begin{aligned} |L(v)| &\leq \left(\int_{\partial\Omega} |g|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |v|^2 d\sigma \right)^{\frac{1}{2}} + \left(\int_{\Omega} |f|^2 d\Omega \right)^{\frac{1}{2}} \left(\int_{\Omega} |v|^2 d\Omega \right)^{\frac{1}{2}} \\ |L(v)| &\leq \|g\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} + \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \end{aligned}$$

Or

$$\begin{aligned} f &\in L^2(\Omega) \Rightarrow \|f\|_{L^2(\Omega)} \leq k_1 \\ g &\in L^2(\partial\Omega) \Rightarrow \|g\|_{L^2(\partial\Omega)} \leq k_2 \end{aligned}$$

So

$$|L(v)| \leq k_2 \|v\|_{L^2(\partial\Omega)} + k_1 \|v\|_{L^2(\Omega)}$$

According to the continuity of the trace function on $H^1(\Omega)$: $u \mapsto \gamma u|_{\partial\Omega} = u$ such as

$$\|\gamma u\|_{L^2(\partial\Omega)} \leq c \|u\|_{H^1(\Omega)}$$

We have

$$|L(v)| \leq c k_2 \|v\|_{H^1(\Omega)} + k_1 \|v\|_{L^2(\Omega)}$$

And using Poincaré inequality then

$$\begin{aligned} |L(v)| &\leq c k_2 \|v\|_{H^1(\Omega)} + \alpha k_1 \|v\|_{H^1(\Omega)} \\ |L(v)| &\leq (c k_2 + \alpha k_1) \|v\|_{H^1(\Omega)} \\ |L(v)| &\leq k \|v\|_{H^1(\Omega)} \end{aligned}$$

with

$$k = c k_2 + \alpha k_1$$

Hence L is continuous.

Let us check if a is $H^1(\Omega)$ -elliptical

By setting $u = v$

$$a(u, u) = \int_{\Omega} \nabla u^2 d\Omega + \int_{\Omega} u^2 d\Omega$$

We have

$$a(u, u) = \|u\|_{H^1(\Omega)}^2$$

Hence a is coercive.

Properties being verified according to the Lax-Milgram theorem there exists $u \in H^1(\Omega)$ unique such as $a(u, v) = L(v)$ and $\|u\|_{H^1(\Omega)} \leq \frac{\|L\|}{\alpha}$.

1.2 Numerical resolution of the problem

We will use the finite element method of Lagrange type $\mathbb{P}_1$ to solve the following one-dimensional pure Neumann problem for the Poisson equation.

$$\begin{cases} -u'' = f \text{ in }]0, 1[\\ u'(0) = \alpha, u'(1) = \beta, \quad \alpha, \beta \in \mathbb{R} \end{cases} \quad (1.11)$$

with the weak form

$$\int_{\Omega} u'v' dx = \int_{\Omega} f.v dx - \alpha v(0) + \beta v(1), \quad \forall v \in H^1(\Omega)$$

Let $(x_j)_{j=0, \dots, N+1}$ be a subdivision of $]0, 1[$ such that

$$0 = x_0 < x_1 < \dots < x_{N+1} = 1.$$

Suppose the uniform step be given by $h = x_{j+1} - x_j$.

We define the approximation space $H^1(0, 1)$ by

$$V_h = \{v \in C(0, 1) | v|_{[x_j, x_{j+1}]} \in \mathbb{P}_1, \quad \forall j = 0, \dots, N\}.$$

The approximation of the variational formulation (1.11) is to find $u_h \in V_h$ such that

$$\forall v_h \in V_h, \quad \int_{\Omega} u_h' v_h' dx = \int_{\Omega} f v_h dx + \beta v_h(1) - \alpha v_h(0) \quad (1.12)$$

V_h being a vector subspace of $H^1(0, 1)$ we can therefore define a canonical basis $(\phi_0, \dots, \phi_{N+1})$ such that:

$$u_h \in V_h \Rightarrow u_h(x) = \sum_{i=0}^{N+1} u_i \phi_i(x)$$

$$v_h \in V_h \Rightarrow v_h(x) = \sum_{j=0}^{N+1} v_j \phi_j(x)$$

So

$$u'_h(x) = \sum_{i=0}^{N+1} u_i \phi'_i(x)$$

And

$$v'_h(x) = \sum_{j=0}^{N+1} v_j \phi'_j(x)$$

Which amounts to finding $u_h(x_0), \dots, u_h(x_{N+1})$ such as: $\forall i = 0, \dots, N+1$

$$\int_0^1 \left(\sum_{i=0}^{N+1} u_i \phi'_i(x) \right) \left(\sum_{j=0}^{N+1} v_j \phi'_j(x) \right) dx = \int_0^1 f \left(\sum_{j=0}^{N+1} v_j \phi_j(x) \right) dx + \beta \sum_{j=0}^{N+1} v_j \phi_j(1) - \alpha \sum_{j=0}^{N+1} v_j \phi_j(0)$$

Which implies

$$\sum_{i=0}^{N+1} \sum_{j=0}^{N+1} \int_0^1 (\phi'_i(x) \phi'_j(x)) u_i v_j dx = \sum_{j=0}^{N+1} \int_0^1 (f \phi_j(x)) v_j dx + \beta \sum_{j=0}^{N+1} v_j \phi_j(1) - \alpha \sum_{j=0}^{N+1} v_j \phi_j(0)$$

$$\sum_{i=0}^{N+1} \left(\int_0^1 \phi'_i(x) \phi'_j(x) dx \right) u_i = \int_0^1 f \phi_j(x) dx + \beta \phi_j(1) - \alpha \phi_j(0), \quad \forall 0 \leq j \leq N+1$$

$$\sum_{i=0}^{N+1} a_{ij} u_i = \int_0^1 f \phi_j(x) dx + \beta \phi_j(1) - \alpha \phi_j(0), \quad \forall 0 \leq j \leq N+1$$

With

$$a_{ij} = \int_0^1 \phi'_i(x) \phi'_j(x) dx$$

$\forall i = 0, \dots, N+1$ we define the ϕ_i functions by:

$$\phi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h} & \text{if } x \in [x_{i-1}, x_i] \\ 0 & \text{else} \\ \frac{x_{i+1}-x}{h} & \text{if } x \in [x_i, x_{i+1}] \end{cases}$$

We notice that we have

$$\phi_i(1) = \phi_i(x_{N+1}) = \begin{cases} 1 & \text{if } i = N+1 \\ 0 & \text{else} \end{cases}$$

and

$$\phi_i(0) = \phi_i(x_0) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{else} \end{cases}$$

We then define $b_h \in \mathbb{R}^{N+2}$ by

$$(b_h)_i = \begin{cases} \int_{x_{i-1}}^{x_{i+1}} f\phi_i(x)dx & \text{if } 1 \leq i \leq N \\ \int_0^{x_1} f\phi_0(x)dx - \alpha & \text{if } i = 0 \\ \int_{x_N}^{x_{N+1}} f\phi_{N+1}(x)dx + \beta & \text{if } i = N+1 \end{cases}$$

By taking $U_h = (u_h(x_0), \dots, u_h(x_{N+1}))^T \in \mathbb{R}^{N+2}$, we get that U_h is a solution of $a_{ij}U_h = b_h$, where $a_{ij} \in \mathbb{R}^{(N+2) \times (N+2)}$ is the rigidity matrix

We have

$$\begin{aligned} a_{00} &= \int_0^1 \phi'_0(x)\phi'_0(x)dx = \int_{x_0}^{x_1} \frac{1}{h^2}dx = \frac{1}{h} \\ a_{(N+1)(N+1)} &= \int_0^1 \phi'_{N+1}(x)\phi'_{N+1}(x)dx = \int_{x_N}^{x_{N+1}} \frac{1}{h^2}dx = \frac{1}{h} \\ a_{ii} &= \int_0^1 \phi'_i(x)\phi'_i(x)dx = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right) \left(\frac{1}{h}\right) dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right) \left(-\frac{1}{h}\right) dx = \frac{2}{h} \quad (\text{if } i \neq 0 \text{ and } i \neq N+1) \\ a_{i-1i} &= \int_0^1 \phi'_{i-1}(x)\phi'_i(x)dx = \int_{x_{i-1}}^{x_i} \left(-\frac{1}{h}\right) \left(\frac{1}{h}\right) dx = -\frac{1}{h^2} (x_i - x_{i-1}) = -\frac{1}{h} \\ a_{ii+1} &= \int_0^1 \phi'_i(x)\phi'_{i+1}(x)dx = \int_{x_i}^{x_{i+1}} \phi'_i(x)\phi'_{i+1}(x)dx + \int_{x_{i+1}}^{x_{i+2}} \phi'_i(x)\phi'_{i+1}(x)dx \\ &= \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right) \left(\frac{1}{h}\right) dx = -\frac{1}{h} \end{aligned}$$

Then matrix form is

$$\frac{1}{h} \begin{pmatrix} 1 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \\ u_{N+1} \end{pmatrix} = \begin{pmatrix} \int_0^1 f\phi_1(x)dx - \alpha \\ \int_0^1 f\phi_2(x) \\ \vdots \\ \int_0^1 f\phi_n(x)dx + \beta \end{pmatrix} \quad (1.13)$$

The matrix a_{ij} is self-adjoint and positive. Indeed, for all $(v_i) \in \mathbb{R}^{N+2}$, we have

$$\langle a_{ij}v, v \rangle = \frac{1}{h} \begin{pmatrix} v_0 - v_1 \\ -v_0 + 2v_1 - v_2 \\ \vdots \\ -v_{N-1} + 2v_N - v_{N+1} \\ -v_N + v_{N+1} \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_N \\ v_{N+1} \end{pmatrix}$$

$$\begin{aligned}
 < a_{ij}v, v > = \frac{1}{h} [(v_0 - v_1)v_0 + (-v_0 + 2v_1 - v_2)v_1 + \cdots + (-v_{N-1} + 2v_N - v_{N+1})v_N + (v_{N+1} - v_N)v_N] \\
 &= \frac{1}{h} [(v_0 - v_1)v_0 + (v_{N+1} - v_N)v_{N+1} + (-v_2 + 2v_1 - v_0) + \cdots + (-v_{N+1} + 2v_N - v_{N-1})v_N] \\
 &= \frac{1}{h} \left[(v_0 - v_1)v_0 + (v_{N+1} - v_N)v_{N+1} + \sum_{i=1}^N (-v_{i+1} + 2v_i - v_{i-1})v_i \right] \\
 &= \frac{1}{h} \left[(v_0 - v_1)v_0 + (v_{N+1} - v_N)v_{N+1} + \sum_{i=1}^N (v_i - v_{i+1})v_i + (v_i - v_{i-1})v_i \right] \\
 &= \frac{1}{h} \left[(v_0 - v_1)v_0 + (v_{N+1} - v_N)v_{N+1} + \sum_{i=1}^N (v_i - v_{i+1})v_i + \sum_{i=0}^{N-1} (v_{i+1} - v_i)v_{i+1} \right] \\
 &= \frac{1}{h} \left[(v_0 - v_1)v_0 + \sum_{i=1}^N (v_i - v_{i+1})v_i + \sum_{i=0}^{N-1} (v_{i+1} - v_i)v_{i+1} + (v_{N+1} - v_N)v_{N+1} \right] \\
 &= \frac{1}{h} \left[\sum_{i=0}^N (v_i - v_{i+1})v_i + \sum_{i=0}^N (v_{i+1} - v_i)v_{i+1} \right] \\
 &= \frac{1}{h} \sum_{i=0}^N (v_i^2 - 2v_i v_{i+1} + v_{i+1}^2) \\
 &= \frac{1}{h} \sum_{i=0}^N (v_i - v_{i+1})^2 \geq 0
 \end{aligned}$$

On the other hand, a_{ij} is not defined because $a_{ij}v.v = 0$ if and only if $v_i = v_{i+1}$, $\forall i = 0, \dots, N$.

During the numerical simulation for the calculation of the second member b_h we will use a quadrature formula including the trapezoidal formula.

1.3 Analytical resolution of the problem

We will try to solve the one-dimensional Neumann problem by using the classical method of resolution. To avoid the trivial difficulty of determining the constants in the Modified Neumann problem for the Poisson equation, we will add the condition of Dirichlet $u(1) = \beta$. Hence we have the following problem

$$\begin{cases} -u''(x) = f(x) \\ u'(0) = \alpha, u(1) = \beta, \quad \alpha, \beta \in \mathbb{R}. \end{cases} \quad (1.14)$$

Consider $f(x) = 5\pi^2 \cos(\pi x)$

We have

$$\frac{d}{dx} \left(\frac{du(x)}{dx} \right) = -f(x) \Rightarrow \frac{dY}{dx} = -5\pi^2 \cos(\pi x);$$

$$Y = -5\pi \sin(\pi x) + c_1;$$

Then

$$\frac{du(x)}{dx} = -5\pi \sin \pi x + c_1$$

Hence

$$u(x) = 5 \cos(\pi x) + c_1 x + c_2.$$

Determine c_1 and c_2 . After manipulation we obtain

$$\begin{cases} c_1 = \alpha \\ c_2 = 5 + \beta - \alpha \end{cases}$$

Hence the exact solution of problem (1.14) is

$$u(x) = 5 \cos(\pi x) + (x - 1) \alpha + 5 + \beta, \quad \alpha, \beta \in \mathbb{R}. \quad (1.15)$$



1.4 Numerical simulations

The aim here is to represent on the same graph the solutions (1.15) and (1.13) exact and numerical, respectively, taking into account the number of points N and of step h of the finite element method in order to converge the two solutions. This simulation will be implemented in *Scilab*.

Figure 1 illustrates the exact solution (1.15) for $\alpha = \beta = 1$.

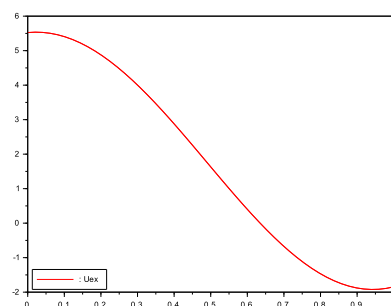
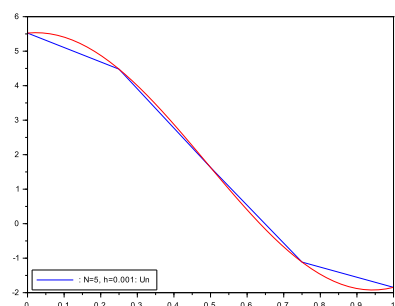


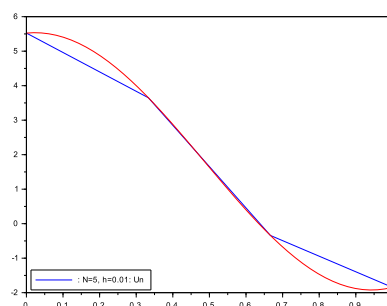
Figure 1: Representation of the exact solution

By fixing $N = 5$ in (1.13), we have attempted to vary the step h of the method to verify the numerical convergence of numerical solution of (1.13) on the exact solution of (1.15). (See Figure 2)

- Taking $h = 0.001$, we notice that the two solutions exact and numerical respectively converge very fast numerically (See Figure 2 a).
- Taking $h = 0.01$, we notice that at the beginning both exact and numerical solutions respectively tend to distance themselves and then converge and finally move away (See Figure 2 b), which is explained by slow numerical convergence.



(a)



(b)

Figure 2: Representation of the exact and numerical solutions for $N = 5$

In Figure 3, we fixed $N = 10$.

- Taking $h = 0.001$, we notice that the convergence between the exact and the numerical solution is the almost numerically the same, which is explained by the strong numerical convergence (Figure 3 a).
- Taking $h = 0.01$, we notice that the exact and numerical solutions, respectively converge almost everywhere numerically. (See Figure 3 b).

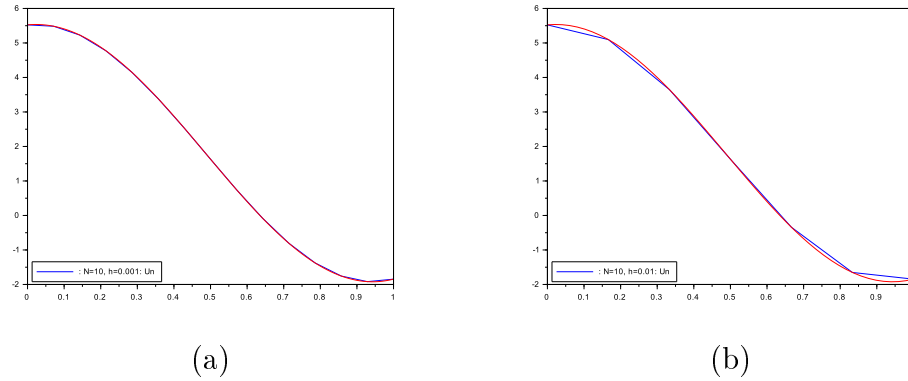


Figure 3: Representation of the exact and numerical solutions for $N = 10$

The finite element method requires a very large number of points N and a very good choice of step h of the method to ensure the convergence of the numerical solution (1.13) to the exact solution (1.15) numerically.

Conclusion et perspectives

In this work, we solved numerically the Neumann problem for the Poisson equation by presenting the three problems of Neumann.

We first showed the continuity of the solution of the Neumann problem with conduction term with respect to σ . According to what exists in the literature, we think we are the first to prove this continuity. Then we showed the existence and uniqueness of the modified Neumann problem by applying the Lax-Milgram theorem. Then we solved numerically the one-dimensional Pure Neumann problem using the finite element method and showed that its rigidity matrix is self-adjoint and positive. To solve analytically this problem we have added the Dirichlet condition to this problem because the determination of constants is difficult. Finally, we used these results to make numerical simulations to compare the numerical convergence of both exact and numerical solutions, respectively. In the future we will investigate the theoretical convergence of the Pure Neumann model and expensive to solve the problem in higher dimension (2 or 3).

References

- [1] Gabriel Acosta, Maria G. Armentano, Ricardo G. Duran and Ariel L. Lombardi, *Nonhomogeneous Neumann problem for the Poisson equation*

- in domains with an external cusp*, J. Math. Anal. Appl. 310(2015) 397-411.
- [2] John W. Barretti and Charles M. Elliott, *A finite-element Method for solving elliptic equations with Neumann Data on a curved boundary using unfitted meshes*, IMA Journal of Numerical Analysis, 4(1984), 309-325.
 - [3] Pavel Bochev and R. B. Lehoucq, *On the finite element solution of Pure Neumann problem*, SIAM, 47(2005), 50-66.
 - [4] Dorin Bucur and Nicolas Varchon, *Boundary variation of a Neumann problem*, Ann. Scuola Norm. Sup. Pisa CI, 4 (2000), 807-821.
 - [5] M. Hinze, R. Pinnau, M. Ulbrich and S. Ulbrich, *Mathematical modelling: Theory and Applications*. Springer, 2009.
 - [6] Germain Nguimbi, Pongui Ngoma Diogene V. and Likibi Pellat R., *The effect of numerical integration in mixed finite element approximation in the simulation of miscible displacement*, Int. J. of Appl. Math. Res. 6(2017), 44-48.
 - [7] Germain Nguimbi, Pongui Ngoma Diogene V. and Likibi Pellat R., *The effect of numerical integration in Galerkin methods for compressible miscible displacement in porous media*, Nonlin. Anal.and Diff. Equ. 5(2017), 17-34.
 - [8] Germain Nguimbi, Pongui Ngoma Diogene V. and Likibi Pellat R., *The effect of numerical integration in finite element methods for general non-linear hyperbolic equations*, Anal. and Diff. Equ. 5(2017), 1-15.
 - [9] Germain Nguimbi, Pongui Ngoma Diogene V. and Likibi Pellat R., *The effect of numerical integration in mixed methods for compressible miscible displacement when molecular dispersion included*, Int. J. of Num.Meth.and Appl. 15(2016), 305-332.
 - [10] Aslak Tveito and Ragnar Winther . *Introduction to partial differential equations: a computational approach*. Springer-Verlag, 1998.