

Valuation of Surrender Options Based of an insured with Multi-morbidity

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Abstract Embedded in Life insurance contracts are surrender options and also path dependency. Surrender option stems from many reasons. Multi-morbidity increases the rate of mortality and a variety of adverse health out-

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comes which may lead to surrendering. Poverty levels coupled with social burdens can inform a multi-morbid person to surrender a life policy contract. The study seeks determine and compare valuation of options of a multi-morbid person surrendering. In line with this objective the multi-morbid survival rate of a policy holder was incorporated in the Black-Scholes model for option pricing. The solution to the model come with its own complexities, therefore the need to resort to numerical solutions for the option valuation. Further, a comparison is made of two finite difference algorithm in solving the proposed Black-Scholes equation; the Crank-Nicolson method and the Hopscotch method. Simulations of survival were performed to compute the survival rate. Numerical solution to the Black-Scholes model and the proposed model indicates that the Crank-Nicolson method converges faster than the Hopscotch method for the Black-Scholes whiles the Hopscotch method converges faster than the Crank-Nicolson for the proposed modified Black-Scholes model. It was observed that the Hopscotch method converges faster as the multi-morbid survival rate decreases below the short rate of the Black-Scholes model.

Keywords Multimorbidity · Life insurance · Black-Scholes · Hopscotch · Crank-Nicolson · survival rates

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1 Introduction

A surrender option is an American-style put option that entitles its owner referred to as the *Policy holder* the right to sell back the contract to the issuer or the *insurer* at the *surrender value*[3,6]. The fair valuation of such an option as well as an accurate assessment of the surrender values, are clearly keen topics in the management of a life insurance company, both on the solvency and on the competitiveness side [10]. Life insurance contracts and pension plans are complex financial securities that come in many variations as stated by Broadie and Detemple [5]. This is due to the surrender options which are integrated in life insurance policies and are also path dependent. Contracts with guaranteed rate of returns every year before maturity are common throughout the developing countries. The contracts are sometimes equipped with the right to be terminated prior to maturity.

Existing models do not take into account the likelihood that someone surrendering is multi-morbid. Anwar [2] and Li [8] considered the numerical solution of the Black-Scholes model for the European call option however did not consider the situation where the insured is co-morbid and need to surrender.

The paper seeks to address the issue concerning the fact that policy holders can terminate the contract prior to maturity due to the co-existence of two or more diseases in their lives which calls for some forms of financial burden, and since there is an option for them to surrender prior to maturity then surrendering becomes very paramount. Consequently, what will happen to value of the insurance portfolio, also the surrender value that is due a policy holder

when such an action is taken by him or her. Analytical solution to the valuation problems cannot be found because of the complexities and the presence of path-dependence derivatives in the life insurance contract. Hence the need to resort to numerical methods in the valuation of insurance contract.

Although González-Gaxiola [12] and [13] demonstrated how the Adomian Decomposition Method could be used to approximate the solution of Black-Scholes equation, they did not take into account the survival rate of co-morbid person. Also, Company [14] simulated and analysed nonlinear Black-Scholes equations to model illiquid markets whereas this paper looks at Black-Scholes equation that incorporates the survival rate of the insured. Furthermore, this paper employs the method of Crank-Nicholson and Hopscotch to solve the Black-Scholes (modified) equation and a comparison is done.

2 The Black-Scholes Model

The Black-Scholes model is used to calculate the theoretical price of European put and call options, ignoring any dividends paid during the options lifetime [2]. Consider A as the price of the Asset, a random variable. Let $V(A, t)$ be the value of an option as a function of time and asset price, K be the strike price, r be the risk-free interest rate, σ be the Volatility or the standard deviation of the asset return, and t be the time in years. Then the Black-Scholes equation is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + rA \frac{\partial V}{\partial A} - rV = 0 \quad (1)$$

The model is a second order partial differential equation and is based on the following assumptions [2, 4]:

1. There is a constant interest rate on the riskless asset.
2. The stock is a non-dividend paying stock
3. The market is frictionless
4. Short selling is possible
5. There are no arbitrage opportunities
6. The instantaneous log returns of the stock price is an infinitesimal random walk with drift.
7. Borrowing and lending of stock and cash at a riskless rate and at a fractional amount are allowed.

The solution to Black-Scholes PDE can either have infinite sets of solutions or no solution, hence the need to consider the boundary and initial conditions for a contract like the European-style whose value (payoff) is given by

$$\max(K - A_T, 0) \quad (2)$$

When an asset loses its value, a put is worth its strike price K . This is $V_{i,0} = K$, for $i = 0, 1, \dots, N$. The value of the contract approaches zero (0) as the price of the asset increases. Hence $A_{max} = A_M$ and this means

$$V_{i,M} = 0, \text{ for } i = 0, 1, \dots, N \quad (3)$$

Since the value of the contract is known at time T , we can find the initial condition

$$V_{N,j} = \max(K - j\Delta A, 0), \text{ for } j = 0, 1, \dots, M \quad (4)$$

The initial condition results in the value of the contract V at the end of the period of the condition and not the beginning, implying a backward move from maturity to time zero. The American style is also handled almost the same way,

$$V_{N,j} = \max(j\Delta A - K, 0), \text{ for } j = 0, 1, \dots, M \quad (5)$$

When $A \rightarrow \infty$ it is more likely that the option will be exercised and the value will be $A - K$. As $A \rightarrow \infty$ the strike price has not any impact on the option value, thus the option value is equivalent to

$$V(A, t) = A - Ke^{-r(T-t)}, \text{ as } A \rightarrow \infty \quad (6)$$

Thus for call and put option, the boundary conditions are

$$\text{Call Option} \begin{cases} V(A, T) \max(A_T - K, 0) & \forall 0 \leq A \leq \infty \\ V(0, t) = 0 & \forall 0 \leq t \leq T \\ V(A, t) = A - Ke^{-r(T-t)}, & \text{as } A \rightarrow \infty \end{cases} \quad (7)$$

$$\text{Put Option} \begin{cases} V(A, T) \max(K - A_T, 0) \\ V(A, t) = 0 \\ V(0, t) = Ke^{-r(T-t)}, \end{cases} \quad (8)$$

3 The Modified Black-Scholes Model

The approach used here involves a Black-Scholes dividend-paying asset price, the short rate and the survival rate ($\mathcal{S}$) from a hazard function of a non-monotonic hazard. The survival rate from a hazard function of a non-monotonic hazard actually determines the rate of an insured being multi-morbid. Since integrated in life insurance contracts is an American option, which is an American put option that gives the holder the right to sell the contract back to the insurer. Also being multi-morbid has associated forms of financial burden or obligation, thus surrendering is done based on the policy holders rate of being multi-morbid.

The assumptions of the rate of being multi-morbid include the following:

1. the rate of being multi-morbid ($\mathcal{S}$) is a probability between 0 and 1.
2. The multi-morbid rate is the median rate of being multi-morbid for all rates as time changes or while the contract is still in place

Hence incorporating the survival rate ($\mathcal{S}$) into the Black-Scholes model gives

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + (r - \mathcal{S})A \frac{\partial V}{\partial A} - rV = 0 \quad (9)$$

Without loss of generality, let $\tau = r - \mathcal{S}$ where τ is the net rate of multi-morbidity, then the model becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 A^2 \frac{\partial^2 V}{\partial A^2} + \tau A \frac{\partial V}{\partial A} - rV = 0 \quad (10)$$

There are three cases to consider:

1. The net rate of multi-morbidity is positive if the survival rate ($\mathcal{S}$) is less than risk-free interest rate i.e. $\tau > 0$ if $\mathcal{S} < r$
2. When the survival rate is the same as the risk-free interest rate, the net rate of multi-morbidity is zero i.e. $\tau = 0$ if $\mathcal{S} = r$
3. The net rate of multi-morbidity is negative if the survival rate ($\mathcal{S}$) is greater than risk-free interest rate i.e. $\tau < 0$ if $\mathcal{S} > r$

This paper is not looking at the case (2) where $\tau = 0$.

Considering the modified model in the case (1) where the net rate of multi-morbidity (τ) is positive, the modified model reduces to the normal Black-Scholes model and is solved numerically as can be seen in [2,8]. Considering case 3 where the net rate of multi-morbidity is negative ($\tau < 0$), Mastro [9] in treating American option pricing in the presence of an effective dividend-paying rate considered a similar case where the dividend-paying rate is greater than the risk-free interest and discussed its effect which is similar to the survival rate ($\mathcal{S}$) being greater than the risk-free rate rendering the net rate of multi-morbidity negative.

4 Numerical Solution

The major way a market participant will be able to obtain a price is by using an appropriate numerical method. Some of these numerical methods include Monte-Carlos Simulation, Binomial tree methods, finite difference method and Risk-neutral valuation methods. In this paper, we will compare the Crank-Nicolson and Hopscotch methods as the explicit method is conditionally stable in the valuation of life insurance contracts embedded with surrender option[2].

4.1 Numerical approximation of derivatives to Black-Scholes PDE

The Black-Scholes equation from equation (1) can be solved by setting up a discrete grid: Suppose that T is the time of expiration of an option, and $A_{\max}$ is a suitable large asset price, then $A(t)$ cannot yield $A_{\max}$ within the time horizon under consideration. With regard to asset price, though the domain for the PDE is not bounded, boundaries must be set for computational purposes,

so we need $A_{\max}$. Pertaining to time and asset prices, the grid consists of points (A, t) :

$$\begin{aligned} A &= 0, \Delta A, 2\Delta A, \dots, M\Delta A = A_{\max} \\ t &= 0, \Delta t, 2\Delta t, \dots, N\Delta t = T \\ V_{i,j} &= V(i\Delta A, j\Delta t) \longrightarrow \text{grid notation} \end{aligned} \quad (11)$$

The derivatives approximation can then be written using grid notation since the discrete grid (11) has been set:

– **Forward Difference (FD):**

$$\frac{\partial V}{\partial A} = \frac{V_{i+1,j} - V_{i,j}}{\Delta A}, \quad \frac{\partial V}{\partial t} = \frac{V_{i,j+1} - V_{i,j}}{\Delta t} \quad (12)$$

– **Backward Difference (BD):**

$$\frac{\partial V}{\partial A} = \frac{V_{i,j} - V_{i-1,j}}{\Delta A}, \quad \frac{\partial V}{\partial t} = \frac{V_{i,j} - V_{i,j-1}}{\Delta t} \quad (13)$$

– **Central Difference (CD):**

$$\frac{\partial V}{\partial A} = \frac{V_{i+1,j} - V_{i-1,j}}{2\Delta A}, \quad \frac{\partial V}{\partial t} = \frac{V_{i,j+1} - V_{i,j-1}}{2\Delta t} \quad (14)$$

– **second derivative:**

$$\frac{\partial^2 V}{\partial A^2} = \frac{V_{i+1,j} - 2V_{i,j} + V_{i-1,j}}{2\Delta A^2} \quad (15)$$

4.2 Crank-Nicolson Method

This method is defined as the **hybrid** between the implicit finite difference method and explicit finite difference method. When the Crank-Nicolson idea applied to the Black-Scholes model i.e. finding average of Implicit and Explicit schemes which is discussed in the Hopscotch method, the following grid equation pattern is obtained [8]:

$$\begin{aligned} & \frac{V_{i+1,j} - V_{i,j}}{\Delta t} + \frac{rj\Delta A}{2\Delta A} [V_{i+1,j+1} - V_{i+1,j-1} + V_{i,j+1} - V_{i,j-1}] \\ & + \\ & \frac{\sigma^2 j^2 \Delta A_2}{4\Delta A^2} [V_{i,j-1} - 2V_{i,j} + V_{i,j+1} + V_{i+1,j-1} - 2V_{i+1,j} + V_{i+1,j+1}] \\ & = \frac{1}{2} [rV_{i,j} + rV_{i+1,j}] \end{aligned} \quad (16)$$

From above, re-arranging will give

$$_j V_{i,j-1} + \beta_j V_{i,j} + \gamma_j V_{i,j+1} = x_j V_{i+1,j-1} + y_j V_{i+1,j} + z_j V_{i+1,j+1} \quad (17)$$

4.2.1 Accuracy of Crank-Nicolson Method

The Crank-Nicolson method with the accuracy $0(\Delta t^2, \Delta A^2)$, making it more accurate than the explicit and implicit method. From equating the central and symmetric central differences and expand $V_{i+1,j}$ by Tailor series at $V_{i+\frac{1}{2},j}$ we have:

$$\begin{aligned} V_{i+\frac{1}{2},j} &= V(t + \frac{\Delta t}{2}, A). \\ \Rightarrow V_{i+1,j} &= V_{i+\frac{1}{2},j} + \frac{\partial V}{\partial t} \Delta t + 0(\Delta t^2) \end{aligned} \quad (18)$$

Expanding $V_{i,j}$ at $V_{i+\frac{1}{2},j}$ gives

$$V_{i,j} = V_{i+\frac{1}{2},j} - \frac{\partial V}{\partial t} \Delta t + 0(\Delta t^2) \quad (19)$$

Adding equations (18 and 19) and finding the average, we have

$$\frac{V_{i,j} + V_{i+1,j}}{2} = V_{i+\frac{1}{2},j} + 0(\Delta t^2) \quad (20)$$

This implies that

$$\begin{aligned} V_{i+\frac{1}{2},j-1} - 2V_{i+\frac{1}{2},j} + V_{i+\frac{1}{2},j+1} &= \frac{1}{2}[V_{i,j-1} - 2V_{i,j} + V_{i,j+1}] \\ &\quad + \frac{1}{2}[V_{i+1,j-1} - 2V_{i+1,j} + V_{i+1,j+1}] \\ &\quad + 0(\Delta t^2) \end{aligned} \quad (21)$$

From equation (21), the right-hand represents the difference central at i and $i + 1$. Dividing by ΔA^2 we obtain

$$\frac{\partial^2 V(t + \frac{\Delta t}{2}, \Delta T)}{\partial A^2} = \frac{1}{2} \left[\frac{\partial^2 V(t, A)}{\partial A^2} + \frac{\partial^2 V(t + \Delta t, A)}{\partial A^2} \right] + 0(\Delta t^2, \Delta A^2) \quad (22)$$

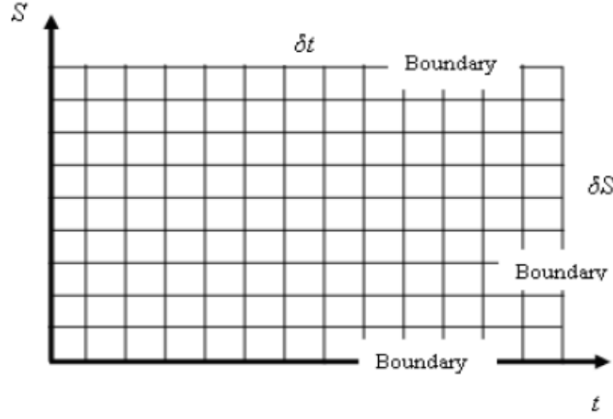
This is the symmetric central difference approximation. The subscript j is arbitrary and we deduce the central difference approximation as:

$$V_{i+\frac{1}{2},j+1} - V_{i+\frac{1}{2},j-1} = \frac{1}{2}[V_{i,j+1} - V_{i,j-1}] + \frac{1}{2}[V_{i+1,j+1} - V_{i+1,j-1}] + 0(\Delta t^2) \quad (23)$$

Dividing by $2\Delta A$, we get

$$\frac{\partial V(t + \frac{\Delta t}{2}, \Delta T)}{\partial A} = \frac{1}{2} \left[\frac{\partial V(t, A)}{\partial A} + \frac{\partial V(t + \Delta t, A)}{\partial A} \right] + 0(\Delta t^2, \Delta A) \quad (24)$$

and this is referred to as the first order central difference approximation. Subtracting equation (23) from (22), we obtain the approximation of $\frac{\partial V}{\partial t}$ at $(t + \frac{1}{2}\Delta t, A)$, that is:



Source: [8]

Fig. 1 Partial Difference in grid form when solved

$$\frac{\partial V(t + \frac{\Delta t}{2}, \Delta T)}{\partial A} = \frac{V_{i+1,j} - V_{i,j}}{\Delta t} + 0(\Delta t^2) \quad (25)$$

Therefore, the Black-Scholes formula centred at $(t + \frac{1}{2}\Delta t, A)$ has a Finite Difference Approximation (FDA) as

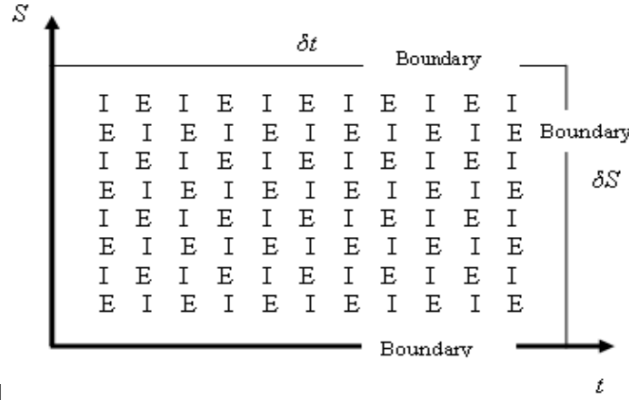
$$\begin{aligned} & \frac{V_{i+1,j} - V_{i,j}}{\Delta t} + \frac{(r - \lambda)j\Delta A}{4\Delta A} [V_{i,j+1} - V_{i,j-1} + V_{i+1,j+1} - V_{i+1,j-1}] \\ & + \frac{\sigma^2 j^2 \Delta A^2}{4\Delta A^2} [V_{i,j-1} - 2V_{i,j} + V_{i,j+1} + V_{i+1,j-1} + 2V_{i+1,j} + V_{i+1,j+1}] \\ & = rV_{i,j} \end{aligned} \quad (26)$$

[7], the Crank-Nicolson scheme has a leading order $0(\Delta t^2, \Delta A^2)$.

4.3 The Hopscotch Method

After solving the PDE, we then create mesh (or grid) as in Fig. 1. If we combine the forward- and backwards difference and place the nodes as in the Fig. 2, calculations of explicit and implicit are done as we move from one node to the other but we alternate these calculations as we move from node to node. Making sure that at each time, we first of all do the calculations at the 'explicit nodes' in the usual way. We then do calculations at the 'implicit nodes' without solving a set of simultaneous equations because the values at the adjacent nodes would have been calculated. Furthermore, mixing the nodes in this way, we get almost the same accuracy as in the Crank-Nicolson scheme. That is the Hopscotch method, as well as the Crank-Nicolson method, can avoid the numerical instability.

Hopscotch method can be used in finding solutions to parabolic and EPDEs in two or more state variables [11]. Financial applications regarding the utility



Source: [8]

Fig. 2 Combined and Backward difference placed in the nodes

of Hopscotch has not been realised yet. The idea is to divide the mesh points in the two-dimensional x-y mesh (ih, jh) as follows:

$i + j$ odd

$i + j$ even

The Hopscotch consists of two 'sweeps'. In the first sweep (and subsequent odd-numbered sweeps) the mesh points $i + j$ is odd, are calculated based on current values (time level n) at the neighbouring points. This is defined as:

$$\frac{V_{ij}^{n+1} - V_{ij}^n}{K} = \Delta_x^2 V_{ij}^n + \Delta_y^2 V_{ij}^n \quad \text{for } (i + j) \text{ odd.} \quad (27)$$

For the second sweep at the same time $n + 1$ the same calculation is based at the next node(even). This second sweep is fully implicit, the scheme is:

$$\frac{V_{ij}^{n+1} - V_{ij}^n}{K} = \Delta_x^2 V_{ij}^{n+1} + \Delta_y^2 V_{ij}^n \quad \text{for } (i + j) \text{ even.} \quad (28)$$

In the second and subsequent even-numbered time steps, the roles of the implicit(I) and explicit (E) are interchanged.

Lets consider the grid form of the Hopscotch Scheme in two steps:

1. An **Explicit Scheme:** We can have an expression giving the subsequent value next value $V_{i,j}$ explicitly in terms of $V_{i+1,j-1}$ and $V_{i+1,j+1}$. since we know the value of the contract at maturity time. We therefore discretize Black-Scholes partial differential equation (PDE) by denoting the forward difference for time and central difference for the asset price discretization. We have;

$$\begin{aligned} & \frac{V_{i+1,j} - V_{i,j}}{\Delta t} + \frac{rj\Delta A}{2\Delta A} [V_{i+1,j+1} - V_{i+1,j-1}] \\ & + \frac{\sigma^2 j^2 A^2}{2\Delta A^2} [V_{i+1,j-1} - 2V_{i+1,j} + V_{i+1,j+1}] \\ & = rV_{i,j} \end{aligned} \quad (29)$$

Now making $V_{i,j}$ the subject, we obtain

$$V_{i,j} = \frac{1}{1+r\Delta t} [\alpha_j V_{i+1,j-1} + \beta_j V_{i+1,j} + \gamma_j V_{i+1,j+1}] \quad (30)$$

for $i = 0, 1, \dots, N$ and $j = 1, 2, \dots, M$

Where the weights α_j, β_j and γ_j are given by

$$\begin{cases} \alpha_j = \frac{\sigma^2 j^2 \Delta t}{2} - \frac{(r-S)j\Delta t}{2} \\ \beta_j = 1 - \sigma^2 j^2 \Delta t \\ \gamma_j = \frac{(r-S)j\Delta t}{2} + \frac{\sigma^2 j^2 \Delta t}{2} \end{cases} \quad (31)$$

This system is represented in the matrix form as

$$\begin{bmatrix} \beta_0 & \gamma_0 & 0 & \dots & 0 & 0 & 0 \\ \alpha_1 & \beta_1 & \gamma_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{M-1} & \beta_{M-1} & \gamma_{M-1} \\ 0 & 0 & 0 & \dots & 0 & \alpha_M & \beta_M \end{bmatrix} \begin{bmatrix} V_{i+1,0} \\ V_{i+1,1} \\ \vdots \\ V_{i+1,M-1} \\ V_{i+1,M} \end{bmatrix} = \begin{bmatrix} V_{i,0} - \alpha_0 \\ V_{i,1} \\ \vdots \\ V_{i,M-1} \\ V_{i,M} - C_M \end{bmatrix} \quad (32)$$

2. An **Implicit Scheme:** We discretize Black-Scholes PDE using Finite Difference (FD) for time and central difference for the asset price. We have

$$\begin{aligned} \frac{V_{i+1,j} - V_{i,j}}{\Delta t} + \frac{rj\Delta A}{2\Delta A} [V_{i,j+1} - V_{i,j-1}] + \\ \frac{\sigma^2 j^2 \Delta A^2}{2\Delta A^2} [V_{i,j+1} - 2V_{i,j} + V_{i,j-1}] \\ = rV_{i+1,j} \end{aligned} \quad (33)$$

making $V_{i+1,j}$ the subject in equation (33)

$$V_{i+1,j} = \frac{1}{1-r\Delta t} [x_j V_{i,j-1} + y_j V_{i,j} + z_j V_{i,j+1}] \quad (34)$$

for $i = 0, 1, \dots, N$ and $j = 1, 2, \dots, M-1$.

Similarly to the explicit method, the implicit method is accurate to $O(\Delta t, \Delta A^2)$.

The weights x, y and z are given by

$$\begin{cases} x_j = \frac{(r-S)j\Delta t}{2} - \frac{\sigma^2 j^2 \Delta t}{2} \\ y_j = \frac{1 + \sigma^2 j^2 \Delta t}{1 - r\Delta t} \\ z_j = -\frac{\sigma^2 j^2 \Delta t}{2} - \frac{(r-S)j\Delta t}{2} \end{cases} \quad (35)$$

The system of equations in tridiagonal matrix form is

$$\begin{bmatrix} V_{i+1,0} - x_0 \\ V_{i+1,1} \\ \vdots \\ V_{i+1,M-1} \\ V_{i+1,M} - z_M \end{bmatrix} = \frac{1}{1 - r\Delta t} \begin{bmatrix} y_0 & z_0 & 0 & \dots & 0 & 0 & 0 \\ x_1 & y_1 & z_1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & x_{M-1} & y_{M-1} & y_{M-1} \\ 0 & 0 & 0 & \dots & 0 & x_M & y_M \end{bmatrix} \begin{bmatrix} V_{i,0} \\ V_{i,1} \\ \vdots \\ V_{i,M-1} \\ V_{i,M} \end{bmatrix} \quad (36)$$

5 Analysis and Results

Here we look at the application of the modified Black-Scholes partial differential equation, Crank-Nicolson and Hopscotch difference schemes in the valuation of life insurance contract and also compare and contrast between the convergence of the modified model and Black-Scholes based on the assumptions of the models.

5.1 Matlab Implementation

The matrices that are obtained by using the Crank-Nicolson and Hopscotch finite difference schemes are generally large tridiagonal matrices and requires more computational time. For this reason, R-studio and Matlab were used to find the solutions to the systems. R-studio codes were implemented for the survival function and Matlab codes were also implemented for Crank-Nicolson and Hopscotch method.

5.2 Stability of Crank-Nicolson Method and Hopscotch

Table 1 shows the eigenvalues of the matrix of the scheme as $N \rightarrow \infty$ [1] and indicates that as $N \rightarrow \infty$, the eigenvalues approaches one showing the stability of the Crank-Nicolson's method. Also this method is with an accuracy of $O(\Delta t^2, \Delta A^2)$ and that also indicates how accurate the results is to the actual value. The stability of Hopscotch is done in the same manner.

5.3 Comparing the convergence of the Crank-Nicolson and Hopscotch methods

The data from a company used are as follows: Asset price, $A = 50$, strike price, $K = 52$, risk-free interest rate, $r = 0.05$, surrender period $t = 2$ years, maturity period, $T = 30$ years, and $\sigma = 0.02231$. The surrender value of the life insurance contract is 5.4650 with the value at maturity being 8.220 for non-dividend paying asset (see Table 2).

In Table 2, the values in the bracket are the difference between the actual values and values obtained from the various numerical methods.

Table 1 The eigenvalues of the Crank-Nicolson method as $N \rightarrow \infty$

N = 100		N = 500		N = 1000		N = 2000		N = 4000	
j	λ_j	j	λ	j	λ	j	λ	j	λ
93	1.0572	493	1.0119	993	1.0060	1993	1.0030	3993	1.0015
94	1.0398	494	1.0088	994	1.0044	1994	1.0022	3994	1.0011
95	1.0284	495	1.0062	995	1.0016	1995	1.0016	3995	1.0008
96	1.0189	496	1.0040	996	1.0020	1996	1.0010	3996	1.0005
97	1.0111	497	1.0023	997	1.0012	1997	1.0006	3997	1.0003
98	1.0055	498	1.0011	998	1.0006	1998	1.0003	3998	1.0001
99	1.0020	499	1.0004	999	1.0002	1999	1.0001	3999	1.0000

5.4 Simulation of survival rate ($\mathcal{S}$) using R-studio package under exponential distribution.

The rate of being multi-morbid is obtained from simulation of the survival rate with sample size of 60. The simulation is done 10,000 times with the help of the R software from which the values of lower and upper confidence intervals were obtained. The lower and upper values for maximum, minimum, mean and median were obtained, from which a random value was picked to be fixed into the Black-Scholes model. The last values of both lower and confidence intervals are picked and the following were calculated from it;

For Exponential Distribution

Random values were picked between 0.234057 and 0.04544183 for minimum survival rate, 0.6914513 and 0.4209266 for maximum survival rate, 0.4539849 and 0.2148305 for mean survival rate, and finally 0.4574076 and 0.2158757 for median survival rate.

For Weibull Distribution

Random values were picked between 0.3387227 and 0.5783236 for minimum survival rate, 0.8723033 and 0.9792022 for maximum survival rate, 0.5981787 and 0.8293827 for mean survival rate, and finally 0.5931416 and 0.83586830 for median survival rate.

One should note that survival rate is defined as the rate of developing multi-morbidity condition or being multi-morbid.

6 Discussion

From the analysis, it is realised that, asset price discretization and time discretization are two fundamental sources of error. Checking for consistency, stability and convergence which are the fundamental factors that characterized a numerical scheme, Lax Equivalence theorem was used.

Table 2 Comparison of the two methods in the valuation of surrender option with no survival rate (S). Surrender value at $t = 2$ years. Expected value = 5.4650

No. of steps	Crank-Nicolson	Hopscotch
30	5.4204(.0446)	5.4387(.0302)
90	5.4465(.0185)	5.4483(.0167)
150	5.4503(.0147)	5.4512(.0138)
270	5.4541(.0109)	5.4534(.0116)
330	5.4546(.0104)	5.4545(.0105)
450	5.4552(.0098)	5.4549(.0101)
570	5.4556(.0094)	5.4550(.0100)
630	5.4557(.0093)	5.4552(.0098)
720	5.4559(.0091)	5.4543(.0107)
780	5.4559(.0091)	5.4543(.0107)
810	5.4560(.0090)	5.4554(.0096)
870	5.4560(.0090)	5.4554(.0096)

Table 3 Comparison of the two methods in the valuation of surrender option with minimum rate (S) = 0.04 of being multimorbid. Where S is the minimum of the simulated survival rate of the upper and lower confidence intervals (under exponential distribution) Surrender value at $t = 2$ years.

No. of steps	Crank-Nicolson	Hopscotch
30	6.9215	7.7545
90	6.9305	7.7762
150	6.9317	7.7807
270	6.9335	7.7838
330	6.9337	7.7845
450	6.9337	7.7853
570	6.9339	7.7858
630	6.9339	7.7860
720	6.9340	7.7862
780	6.9340	7.7863
810	6.9340	7.7863
870	6.9340	7.7864

The study used the eigenvalue to check how stable the two finite difference methods would be. The results from Table 1 showed that the Crank-Nicolson and Hopscotch methods were unconditionally stable.

The results from Table 2 showed that values of Crank-Nicolson were closer to the expected values than that of Hopscotch values. Hence, making Crank-Nicolson method better than Hopscotch method as far as faster convergence to the expected value is concerned.

The simulated survival rates that were deducted from the rate of returns also showed that, the higher the rate of developing the co-morbidity condition (the higher the S value) the greater the payoff value to the insured.

The Hopscotch method gave higher values than the Crank-Nicolson method when the survival rates were incorporated into the Black-Scholes model. Hence, making the insured receive more payment with the Hopscotch method than the Crank-Nicolson method. On the part of the insurance company, more

Table 4 Comparison of the two methods in the valuation of surrender option with maximum rate (S) = 0.5 of being multimorbid. Where S is the maximum of the simulated survival rate of the upper and lower confidence intervals (under exponential distribution). Surrender value at $t = 2$ years.

No. of steps	Crank-Nicolson	Hopscotch
30	77.9123	80.9370
90	77.9100	80.8274
150	77.9098	80.8247
270	77.9097	80.8274
330	77.9097	80.8289
450	77.9097	80.8290
570	77.9097	80.8294
630	77.9097	80.8296
720	77.9097	80.8302
780	77.9097	80.8304
810	77.9097	80.8303
870	77.9097	80.8305

Table 5 Comparison of the two methods in the valuation of surrender option with minimum rate (S) = 0.8 of being multimorbid. Where S is the minimum of the simulated survival rate of the upper and lower confidence intervals (under Weibull distribution) Surrender value at $t = 2$ years.

No. of steps	Crank-Nicolson	Hopscotch
30	182.9764	184.1583
90	183.0559	183.6542
150	183.0507	183.6136
270	183.0487	183.6037
330	183.0484	183.6049
450	183.0482	183.6026
570	183.0480	183.6019
630	183.0480	183.6017
720	183.0480	183.6018
780	183.0480	183.6021
810	183.0480	183.6018
870	183.0479	183.6021

money can be lost when Hopscotch method is used to determine the surrender value as shown in Tables (4 and 5). The Crank-Nicolson method was found to give more accurate and consistent results for life insurance contract containing surrender options than the Black-Scholes partial differential equation. In the case of the modified model, the hopscotch method gave a little higher values than that of Crank-Nicolson making it converge faster.

7 Conclusion and Recommendation

The Crank-Nicolson method converges faster than the Hopscotch method when these schemes are used in solving the Black-Scholes partial differential equation. That is Crank-Nicolson method gives more accurate results than the Hopscotch method.

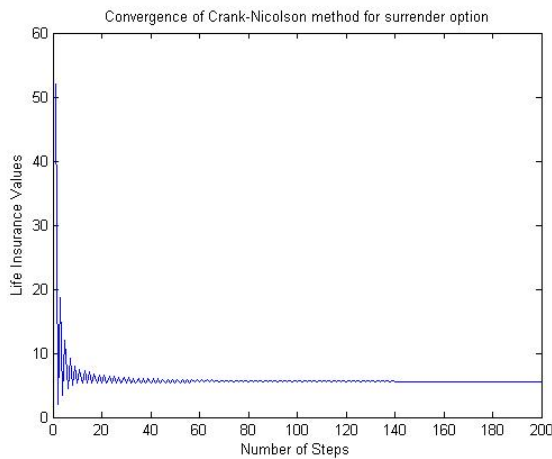


Fig. 3 Chart on Crank-Nicolson Method for valuation Surrender Option

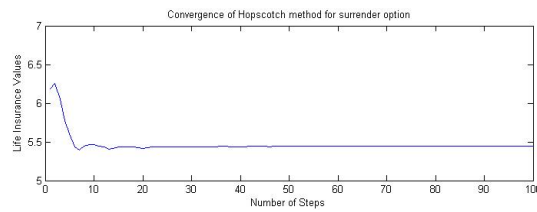


Fig. 4 Chart on Hopscotch Method for the valuation of Surrender Option

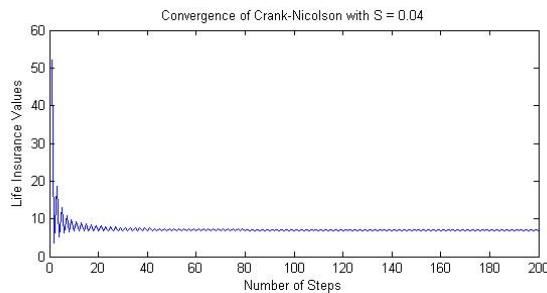


Fig. 5 Chart on Crank-Nicolson Method for the valuation of Surrender Option with Rate of being multi-morbid at $S = 0.04$

Initially, after the minimum survival rate ($S = 0.04$) was introduced, the values of Hopscotch were higher than the Crank-Nicolson method as can be seen in Table 3. As the step sizes were increased, (mesh sizes) for both methods, Crank-nicolson started converging faster than the Hopscotch method.

When the survival rates (S) were deducted from the rate of returns in the Black-Scholes model, the Hopscotch method gave higher payoffs (values) than the Crank-Nicolson method. This means, an insurance company could lose

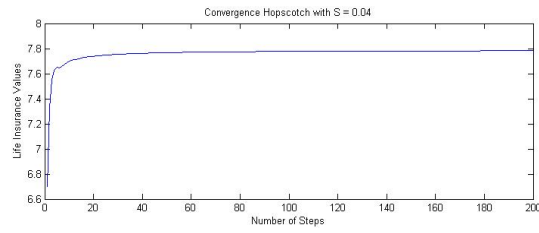


Fig. 6 Chart on Hopscotch Method for the valuation of Surrender Option with Rate of being multi-morbid at $S = 0.04$

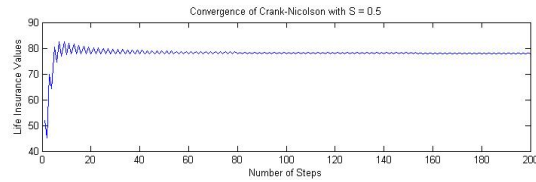


Fig. 7 Chart on Crank-Nicolson Method for the valuation of Surrender Option with Rate of being multi-morbid at ($S = 0.5$)

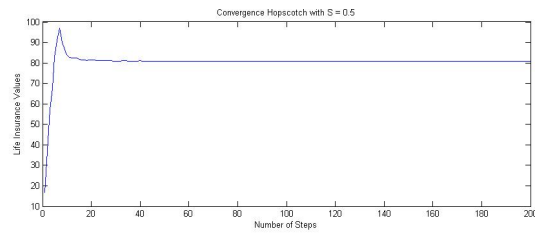


Fig. 8 Chart on Hopscotch Method for the valuation of Surrender Option with Rate of being multi-morbid at $S = 0.5$

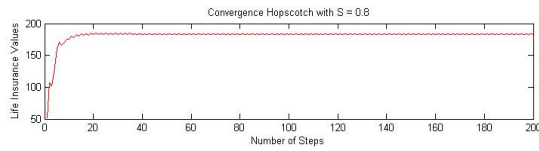


Fig. 9 Chart on Crank-Nicolson Method for the valuation of Surrender Option with Rate of being multimorbid at $S = 0.8$

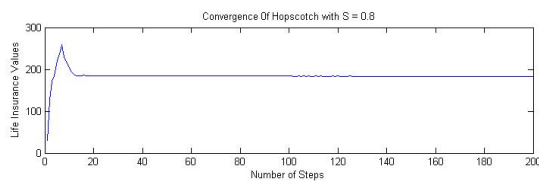


Fig. 10 Chart on Hopscotch Method for the valuation of Surrender Option with Rate of being multimorbid at $S = 0.8$

more money when Hopscotch method is used to determine the surrender value (payoff); this method favours the insured than the Crank-Nicolson method.

7.1 Recommendation

In finding the value of the American styled life insurance contracts, the Crank-Nicolson method gives more accurate results than the Hopscotch method. In the case where the modified model is going to be used, then the Hopscotch method converges faster and gives more accurate results than the Crank-Nicolson.

References

1. Ali, M.: The valuation of insurance liabilities: Numerical method approach. Masters Thesis, Kwame Nkrumah University of Science and Technology Kumasi-Ghana (2013)
2. Anwar, M.N., Andallah, L.S.: A study on numerical solution of black-scholes model. *Journal of Mathematical Finance* **8**, 72–381 (2018)
3. Bacinello, A.R.: Fair valuation of a guarantee life insurance participating contract embedding a surrender option. *Journal of Risk and Insurance* (2003)
4. Briys, E., De Varenne, F.: Life insurance in a contingent claim frame work, pricing and regulator implications. *The jенева papers on risk and insurance theory* **19**(1), 53 – 72 (2014)
5. Broadie, M., Detemple, J.B.: Anniversary article: Option pricing: Valuation models and applications. *Management science* **50**(9), 1145–1177 (2004)
6. Grossen, A., Joergensen, P.L.: Fair valuation of life insurance liabilities: The impact of interest rate guarantees, surrender options, and bonus policies. *Insurance: Mathematics and Economics* **27**, 37 – 57 (2000)
7. Kerman, J.: Numerical methods for option pricing: Binomial and finite difference approximation. Retrieved from www.stat.columbia.edu/~kerman (2002)
8. Li, W., Hong, X.: Solving black-scholes pde by crank-nicolson and hopscotch methods. Malardale university division of applied mathematics school of education, culture and communication (2010)
9. Mastro, M.: Financial Derivative and Energy Market Valuation: Theory and Implementation in Matlab. John Wiley & Sons, Inc., Hoboken, New Jersey (2013). <http://gen.lib.rus.ec/book/index.php?md5=1A0D20CB42B28688E5B5CE922AD02544>
10. Sheldon, T.J., Smith, A.D.: Market consistent valuation of life assurance business. *British Actuarial Journal* **10**(3), 543–605 (2004)
11. Zhao, J.: Convergence of compact finite difference method for second-order elliptic equations. *Applied Mathematics and Computation* (2006)
12. González-Gaxiola, O.: The Black-Scholes Equation with Variable Volatility Through the Adomian Decomposition Method, *Comm. in Math. Finance*, 5(1), 43-52 (2016)
13. González-Gaxiola, O., Ruíz de Chávez, J. and Santiago, J. A.: A Nonlinear Option Pricing Model Through the Adomian Decomposition Method *Int. J. Appl. Comput. Math.* **2**(4), 453-467 (2016). <https://doi.org/10.1007/s40819-015-0070-6>
14. Company, R., Jódar, L., Ponsoda, E., and Ballester, C.: Numerical analysis and simulation of option pricing problems modeling illiquid markets. *Comput. Math. Appl.* **59**, 29642975 (2010)