

Modeling Exponential Growth and Exponential Decay Real Phenomena by ψ –Caputo Fractional Derivative

Abstract. The concept of " ψ –Caputo" fractional derivative is discussed in this article. This method is based on the fractional derivative in Caputo sense of a function with respect to another function ψ , called kernel. The kernel function ψ , is any increasing function such that $\psi'(x) \neq 0, \forall x \in [a, b]$. Experimental studies are used to support the fact that fractional approach of solving differential equations is often better than the classical ordinary approach. The solution to two exponential decay models and one exponential growth model are built using the classical approach and the kernel approach. Several kernel functions are considered and their performances evaluated.

Key words: Exponential decay, Exponential growth, ψ –Caputo Fractional derivative, Optimization, Initial value problems.

1 Introduction

Fractional calculus was discovered more than three centuries ago. Earlier works on Fractional calculus focused only on theories. Hence, existence and uniqueness of solution to problem were studied, without further materialization. [8-12] are some known references where the theory of existence and the uniqueness of fractional differential equations and their solutions are found.

The classical approach involving system of integer-order of differentiation may sometime provide solution with high magnitude error term. However, fractional order of differential equation has been proven efficient, and may produce lower magnitude error term in many situations where the classical approaches fail to do so. The efficiency of the fractional differentiation approach has been proven in various fields of sciences such as physics, chemistry, epidemiology, finance and biology sciences [1-7], these are few of them just to mention.

It is sometime difficult to find the analytical solution of some differential equations. Difficulty arises usually in nonlinear cases. To overcome such difficulty, numerical approaches are recommended. The solution of an equation using numerical approach is obtained through iterations. In [23-28], the authors discussed some numerical approaches to solve fractional differential equations and to optimize the fractional order of derivative.

In this work, we added our contribution to what have been done so far by many researchers such as Almeida [21] who introduced fractional derivative of a function with respect to another function, called the ψ –Caputo. Almeida et al [7,21] have proven further that the fractional order of differential equation is better than the classical in modeling the population growth, a country's GDP, the Newtown law of cooling and many others real world phenomena. We proved that, the ψ –Caputo fractional differential equation is better

than their integer-order counterpart for modeling the radioactivity decay and the RC circuit charging and discharging processes, provided that a suitable kernel ψ is selected [22].

2 Preliminaries

Definition 2.1 [22] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $g : [0, +\infty] \rightarrow \mathbb{R}$ is defined as

$$({}_{RL}I_{0^+}^\alpha g)(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (t-s)^{\alpha-1} g(s) ds.$$

Provided that the right hand side of the integral is pointwise defined on $(0, +\infty)$ and Γ is the gamma function $\Gamma(v) = \int_0^\infty e^{-t} t^{v-1} dt, \forall v > 0$.

Definition 2.2 [22] The Caputo derivative of order $\alpha > 0$ for a function $g : [0, +\infty] \rightarrow \mathbb{R}$ is defined as

$$({}_c D_{0^+}^\alpha g)(t) = \begin{cases} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} g^{(n)}(s) ds, & n-1 < \alpha < n \in \mathbb{N}, \\ g^{(n)}(t), & \alpha \in \mathbb{N}. \end{cases}$$

Where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α .

Definition 2.3 [21] Let $\alpha > 0$, $g \in L^1[a, b]$ and $\psi \in C^1[a, b]$ be an increasing function with $\psi'(x) \neq 0, \forall x \in [a, b]$ then $I_{0^+}^{\alpha, \psi} g(t)$ denotes the fractional integral of g w.r.t ψ and it is given by

$$I_{0^+}^{\alpha, \psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} g(s) ds.$$

Definition 2.4 [21] Let $\alpha > 0$, $g, \psi \in C^n[a, b]$ with ψ be an increasing function and $\psi'(x) \neq 0, \forall x \in [a, b]$ then ${}_c D_{0^+}^{\alpha, \psi} g(t)$ denotes the fractional derivative of g w.r.t ψ and it is given by

$${}_c D_{0^+}^{\alpha, \psi} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{n-\alpha-1} \left(\frac{1}{\psi'(s)} \frac{d}{dt} \right)^n g(s) ds.$$

Lemma 2.1 [21] Let $\alpha > 0$, n a natural number such that $\alpha \in (n-1, n)$ if $g, \psi \in C^n[a, b]$ then

$$I_{0^+}^{\alpha, \psi} ({}_c D_{0^+}^{\alpha, \psi} g)(t) = g(t) - \sum_{i=0}^{n-1} \frac{\left(\frac{1}{\psi'(s)} \frac{d}{dt} \right)^i g(0)}{i!} (\psi(t) - \psi(0))^i.$$

3 Methodology

To investigate the power of the ψ -Caputo fractional differential equation over the classical differential equation, we considered some linear model differential equations whose classical solutions are given either in the form of exponential decay or exponential growth curve model. For each model, the true data obtained from experimental study is considered, and then the model based on classical approach of differential equation is defined alongside its counterpart derived from the ψ -Caputo fractional differential equation. Both models are used to fit the original data set. The percentage of the root square deviation error is used to compare the performance of each of the methods applied.

Consider an experimental data vector of size n , obtained from a time dependent process such that the i^{th} value is denoted by $y_i = y(t_i)$. If the original data is predicted by a function $\hat{y}$, in a way that the i^{th} fitted data value is denoted by $\hat{y} = \hat{y}(t_i)$, the root-mean-squared deviation is computed (see [19]) by

$$RMSD = \sqrt{\frac{\sum_{t=0}^n (y(t) - \hat{y}(t))^2}{n}}. \quad (1)$$

The quantity given by (1) can be used to evaluate the percentage of error that occurs while using predicted data instead of original data. The ratio below is used for the purpose

$$R = \sqrt{\frac{\sum_{t=0}^n (y(t) - \hat{y}(t))^2}{n}} \bigg/ \sqrt{\frac{\sum_{t=0}^n (y(t))^2}{n}}. \quad (2)$$

Equation (2) gives the magnitude of the error that occurs when the estimated values are used in place of the original values.

4 Examples

In this section, we proved by the mean of examples that the ψ -Caputo fractional differential equation can fit the data better than the classical differential equation, provided that a suitable kernel ψ is selected.

4.1 Radioactivity Experiment

Atoms present in the matter are characterized by their nuclear state. The nuclear state of an element can change due to some chemical reaction or even when it is heated to some extent. Radioactivity studies focus on nuclear state change. In general, the nuclear state change of an atom produces either electromagnetic radiation or some particles. In the process of nuclear state change, the number of atoms involve in the process may change as well. Hence one can observe for instance a fusion of two neutrons with two protons or fission of nucleus. The known forms of radiation are Alpha, Beta and Gamma. An experiment found in [18], studies

the radioactivity of Silver and Aluminum (^{110}Ag , ^{108}Ag and ^{28}Al) , which represents a decay model, since particles are emitted over the time. The experimental data of ^{28}Al is found in [1,page 6]. A scatter plot of the data shows that an exponential decay model is suitable to fit the data set. The analytic form of the model is given by

$$\frac{dn(t)}{dt} = -\lambda n(t), \quad t \geq 0, \quad (3)$$

The classical solution to the differential equation (3) is given by

$$n(t) = n_0 \exp\left(-\frac{t}{\tau}\right). \quad (4)$$

Where $\lambda = \frac{1}{\tau}$, t is the time elapsed, n is the number of particles yet to disintegrate at the time t . The quantity n_0 is the initial number of particles at $t=0$, and τ is the mean life-time. The purpose of the experiment is to estimate the value of τ that would lead to a best fit of the experimental data using formula (4). Experimental data and *Matlab* optimization routine *lsqcurvefit* [20], returned $\lambda = 0.0121 \leftrightarrow \tau = 82.6446$ as the best value that would help in minimizing the error while fitting the original data using (4).

Let us now admit that the Radioactivity model defined above is ruled by the ψ –*Caputo* fractional differential equation. This equation is derived from the classical equation (3), with the assumption that the order of derivative is not integer. The fractional counterpart of (3) using ψ –*Caputo* derivative is given by

$${}_c D_{0^+}^{\alpha, \psi} n(t) = -\lambda n(t), \quad (5)$$

applying the fractional integration $I_{0^+}^{\alpha, \psi}$ to both sides of (5) and considering theorem (7.2) in [13], the exact solution of (5) can be written as

$$n(t) = n_0 E_{\alpha} \left(-\lambda (\psi(t) - \psi(0))^{\alpha} \right), \quad (6)$$

where $E_{\alpha}(\cdot)$ is the Mittag-Leffler function defined by $E_{\alpha}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\alpha i + 1)}$, $t \in \mathbb{R}$. $\alpha > 0$ is the fractional order of derivative and λ is the inverse of the mean life-time τ .

Two parameters are to be optimized in equation (6). These are the fractional order of derivative α and the inverse of the mean life-time λ . Moreover, the kernel function is selected in a way to verify properties listed in definition 2.4. For each selected kernel function, corresponding optimal parameters are obtained. Table 1 is a summary of a sample of kernels, their corresponding optimal parameters and their root mean squared deviation.

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Table 1: Radioactivity error comparison.

	Classical Model	Kernel $\psi(x) = x$	Kernel $\psi(x) = \sqrt{x+1}$	Kernel $\psi(x) = \ln(x+1)$
λ	0.0121	0.0314	0.1177	0.1046
α	1	0.8252	1.0571	1.6420
SSE	$1.17 \cdot 10^5$	$2.9 \cdot 10^3$	$8.87 \cdot 10^3$	$2.65 \cdot 10^4$
RMSD	67.2560	10.5557	18.4694	31.9074
R	18.58%	2.92%	5.10%	8.81%

The optimal parameters minimize the total error term when the original data is fitted by the fractional model defined by equation (6)

Figure 1a, 1b and 1c below show the original experimental data, the fitted model using classical and ψ -Caputo fractional order of derivative approaches with different kernels as defined in table 1. It is observable that in general fitting the data with the fractional approach provides less sum of squared error than the classical approach. The performance is evaluated in table 1.

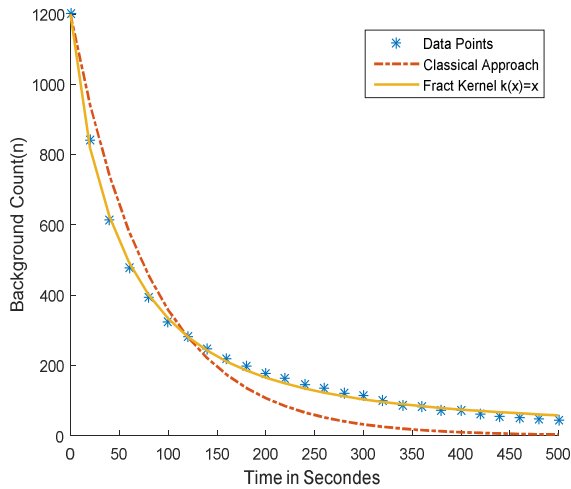


Figure 1 a

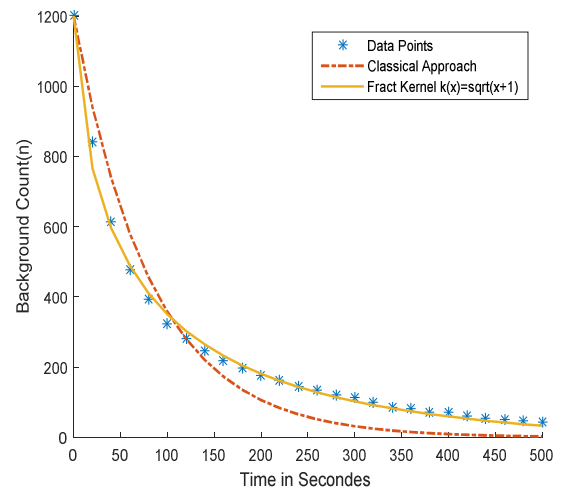


Figure 1 b

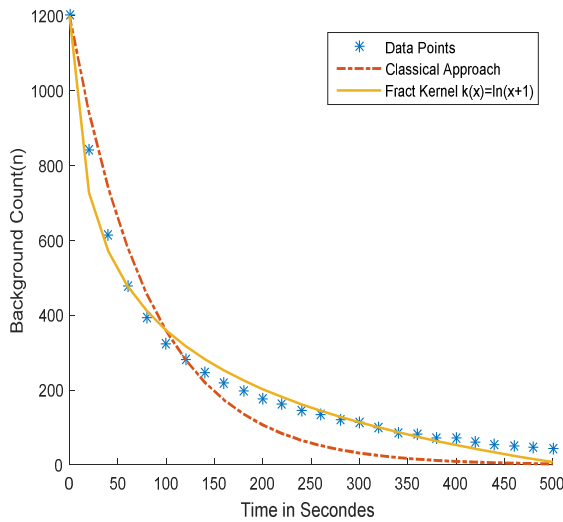


Figure 1 c

153 From both table 1 and figure 1. It is observable that the ψ –Caputo fractional differential
 154 equation has an error rate smaller than the classical method of differential equations.

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156 4.2 RC circuit.

157 The RC circuit is a well known experimental setting that is used to measure the charging and
 158 discharging time of a capacitor, C. Figure 2 shows the experimental setting of an RC circuit.

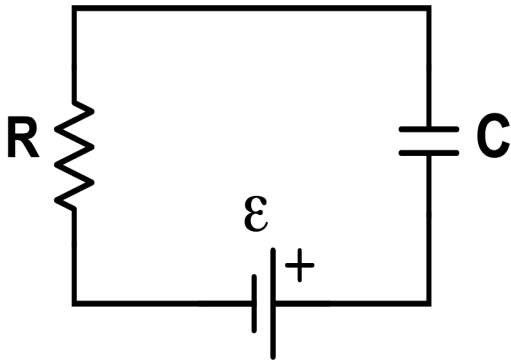


Figure 2a : RC in charging mode

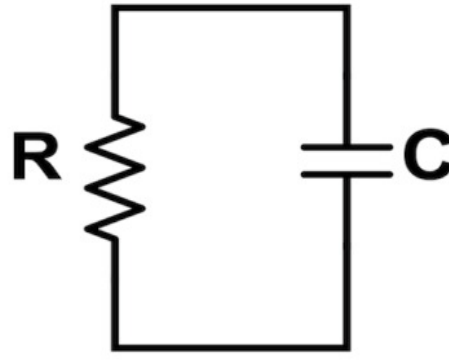


Figure 2b : RC in discharging mode

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160 In the charging mode , see figure 2a, The capacitor C , the resistor R , and the DC/voltage
 161 source ε are connected . This leads to an accumulation of electric charges in the capacitor C
 162 over the time. In the discharging mode, the potential source is removed from the circuit. Only
 163 a capacitor initially full of electric charges Q is directly connected to a resistor R . Such
 164 setting leads to a loss of electric charges over the time. An experimental data set of both
 165 charging and discharging of a RC circuit is found at [14] .

166 Any electric component that resists to electric charges flow in an electric circuit is called
 167 resistor. Such component usually produces heat as a results of it electric resistance. That is

168 the principle used in electric heater, iron for instance. A resistor of resistance R in a circuit
 169 obeys to the Ohm's law

$$170 \quad V = I \times R , \quad (7)$$

171 where V is the voltage applied to the resistor, I is the rate at which electric charges flow
 172 through the resistor, and R is a constant called the resistance.

173 A capacitor is any electric circuit component with the ability to store from a circuit
 174 connected to a voltage source and to supply electric charge Q , in a circuit without voltage
 175 source. When it is attached to a resistor, the capacitor will push the electric charges through
 176 the resistor, creating electric current. Capacitors obey the equation

$$177 \quad V = \frac{Q}{C} , \quad (8)$$

178 where Q is the charge stored in the capacitor, C is a constant called the capacitance of the
 179 capacitor, and V is the resulting voltage.

180 Setting the formulas (7) and (8) for V equal to each other gives

$$181 \quad \frac{Q}{C} = IR . \quad (9)$$

182 Moreover, the rate I at which charges flow through the resistor is the same as the rate at
 183 which charges flow out of the capacitor, so

$$184 \quad -\frac{dQ}{dt} = I . \quad (10)$$

185 A combination of (9) and (10) introduces the RC circuit model which can be written as

$$186 \quad \frac{dQ}{dt} = -\frac{1}{RC} Q . \quad (11)$$

187 It is familiar that the solution for the differential equation (11) is given by

$$188 \quad Q = Q_0 e^{\frac{-1}{RC}t} , \quad (12)$$

189 where Q_0 is the amount held in the capacitor at $t=0$, using the fact that $Q=CV$, (12)
 190 becomes

$$191 \quad V = V_0 e^{\frac{-1}{RC}t} . \quad (13)$$

192 It is common in RC circuit to estimate the capacitance C . Experimental data retrieved from
 193 [14], has the following initial parameters. The resistance $R=2200\Omega$, the potential of the
 194 voltage source $V_0=4.897$. The purpose of the experiment is to estimate the value of the
 195 capacitance C found in (13). Based on experimental data and optimization routine, the value
 196 of $C=0.0057$ appears to be the best that would help minimizing the total error if equation (13)
 197 is used to fit the original data.

Let us assume now that the RC circuit model (11) is built based on a fractional differential equation. Such equation can be written using the ψ -Caputo derivative as

$$\begin{cases} {}^c D_{0^+}^{\alpha, \psi} Q(t) = -\frac{1}{RC} Q(t) \\ Q(0) = Q_0 \end{cases} \quad (14)$$

Applying $I_{0^+}^{\alpha, \psi}$ to both sides of (14) and using theorem (7.2) from [13], the exact solution of (14) can be written as

$$Q(t) = Q_0 E_\alpha \left(-\frac{1}{RC} (\psi(t) - \psi(0))^\alpha \right), \quad (15)$$

using the relation $Q = CV$, equation (15) becomes,

$$V(t) = V_0 E_\alpha \left(-\frac{1}{RC} (\psi(t) - \psi(0))^\alpha \right). \quad (16)$$

Different kernel functions ψ are considered for experiment. Table 2 summarizes the experiments results providing optimal values of the parameters involved alongside the error rate obtained when fitting the original data by a model. The estimated parameters in this experiment are the capacitance value C and the fractional order of derivative α .

Table 2: RC discharging performances.

	Classical Model	Kernel $\psi(x) = x$	Kernel $\psi(x) = \sqrt{x+1}$	Kernel $\psi(x) = \ln(x+1)$
C	0.0057	0.0051	0.0014	0.0018
α	1	0.9871	1.1798	1.5855
SSE	0.8503	0.0098	0.7431	1.5906
RMSD	0.0922	0.0099	0.0862	0.1261
R	5.43%	0.58%	5.08%	7.43%

It is observable from table 2 that the kernel function $\psi(x) = x$ performs better than others. Indeed it requires an optimal capacitance which is almost equal to the capacitance value of the classical approach, a fractional order of derivative close to 1, and a 0% error rate. The kernel $\psi(x) = \ln(x+1)$, produces an error rate slightly greater than the rate of the classical approach. The other kernel namely $\psi(x) = \sqrt{x+1}$ has an error rate which is almost equal to the 5% obtained in the classical approach.

Figure 3 is a set of graphs displaying the behavior of the original data and the experimental results.

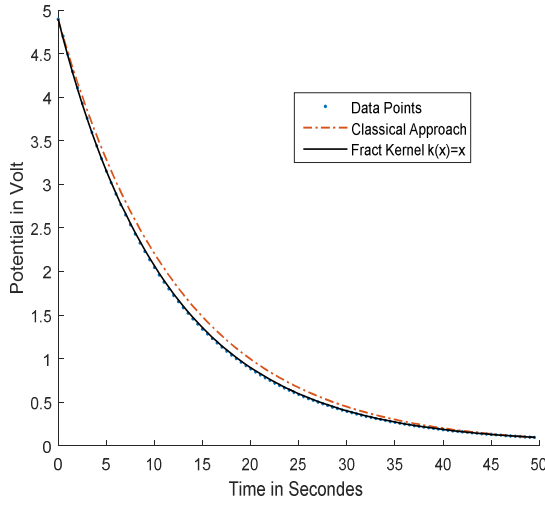


Figure 3a

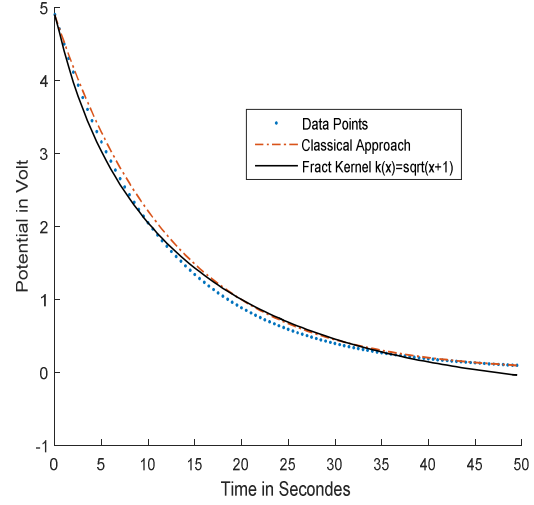


Figure 3b

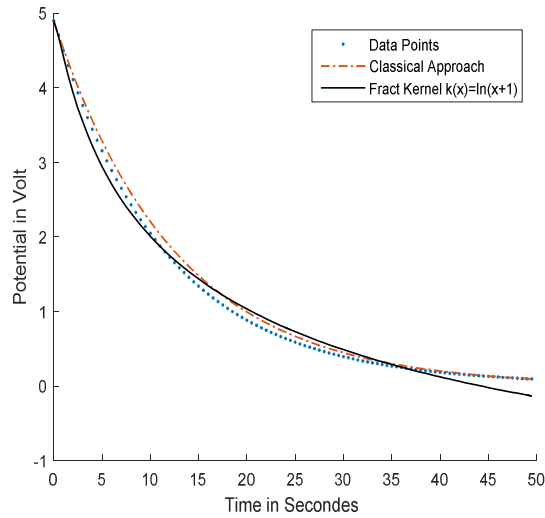


Figure 3c

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221 For the charging mode the RC circuit has the following elements, a capacitance C , the
 222 resistor R and a potential source ε , that maintains the potential leading to the accumulation
 223 of electrical charges in the circuit (see fig 2a). The Kirchhoff's current law of the circuit is
 224 written as

$$225 \quad V_0 + IR - \frac{Q}{C} = 0, \quad (17)$$

226 Since $Q'(t) = I(t)$ the charging differential equation model is derived from (17) as

$$227 \quad V_0 + \frac{dQ}{dt} R - \frac{Q}{C} = 0. \quad (18)$$

228 The exact solution of the ordinary differential equation (18) is given by

$$Q(t) = CV_0 \left(1 - e^{\frac{-1}{RC}t} \right), \quad (19)$$

which is equivalent to

$$V(t) = V_0 \left(1 - e^{\frac{-1}{RC}t} \right). \quad (20)$$

The use of experimental data retrieved from [14], gives a same value of $C = 0.0057$ as obtained in the discharging process.

To solve the fractional counterpart model equation derived from (18), let us defined (18) using the ψ -Caputo derivative as

$$\begin{cases} {}^c D_{0^+}^{\alpha, \psi} Q(t) = \frac{1}{RC} Q(t) - \frac{V_0}{R}, \\ Q(0) = Q_0 \end{cases} \quad (21)$$

Applying $I_{0^+}^{\alpha, \psi}$ to both sides of (21) and using theorem (7.2) in [13] the exact solution of (21) can be written as

$$Q(t) = CV_0 \left(1 - E_{\alpha} \left(\frac{-1}{RC} (\psi(t) - \psi(0))^{\alpha} \right) \right), \quad (22)$$

which is equivalent to

$$V(t) = V_0 \left(1 - E_{\alpha} \left(\frac{-1}{RC} (\psi(t) - \psi(0))^{\alpha} \right) \right). \quad (23)$$

Different kernel functions ψ are considered for experiment. Table 3 summarizes the experimental results providing optimal values of the parameters involved alongside the error rate obtained when fitting the original data by a model. The estimated parameters in this experiment are the capacitance value C and the fractional order of derivative α .

Table 3: RC charging performances.

	Classical Model	Kernel $\psi(x) = x$	Kernel $\psi(x) = \sqrt{x+1}$	Kernel $\psi(x) = \ln(x+1)$
C	0.0057	0.0051	0.0014	0.0018
α	1	0.9962	1.1864	1.5904
SSE	1.2646	0.0127	0.7277	1.5925
RMSD	0.1125	0.0113	0.0853	0.1262
R	2.83%	0.28%	2.15%	3.18%

It is observable from table 3 that the kernel function $\psi(x) = x$ performs better than others. Indeed it requires an optimal capacitance which is almost equal to the capacitance value of the classical approach, a fractional order of derivative close to 1, and a 0% error rate. The kernel $\psi(x) = \sqrt{x+1}$ also performs better than the classical approach. However using the kernel $\psi(x) = \ln(x+1)$, leads to a solution which is a bit weaker than the classical approach solution.

Figure 4 is a set of graphs displaying the behavior of the original data and the experimental results of the charging process.

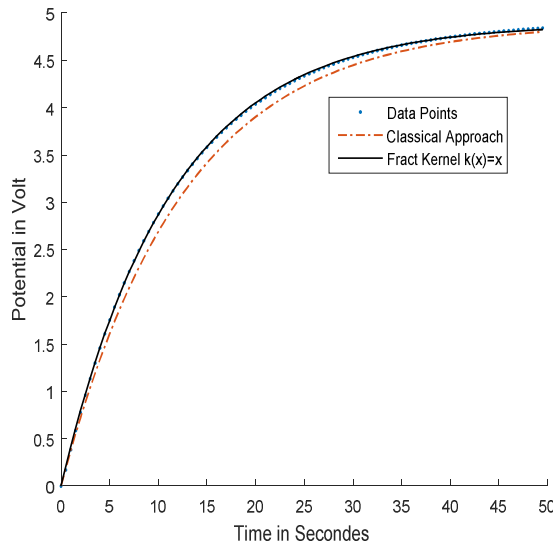


Figure 4a

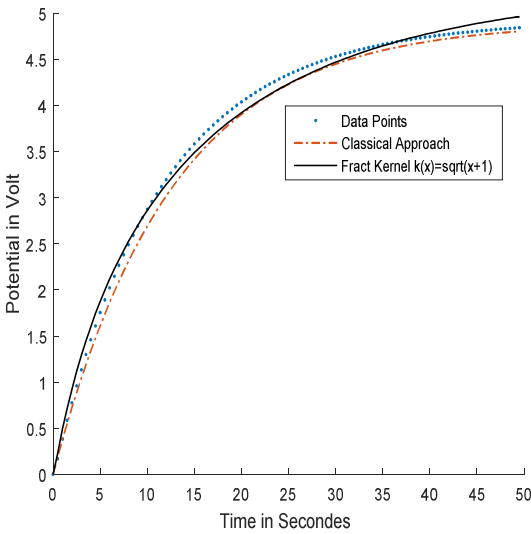


Figure 4b

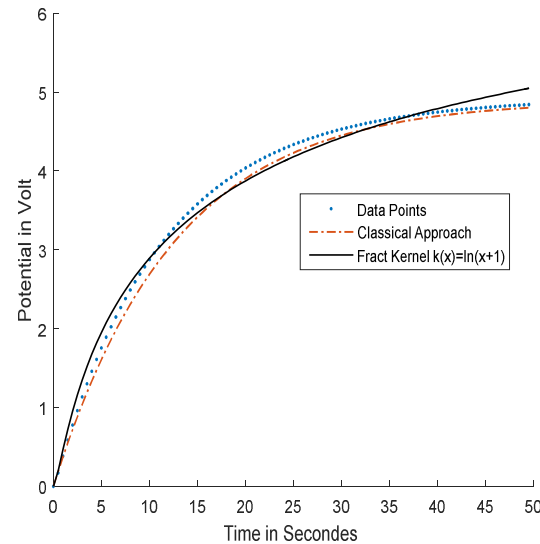


Figure 4c

5. Discussion

1-) Interpreting the meaning of a fractional derivative is often difficult. In the physical sense for instance, the first derivative found in the RC and radioactivity differential equations represent the speed at which charges flow in the circuit and the number of radioactive element decrease respectively. It is therefore important to give a meaning to the fractional derivative order α , found in the counterpart fractional differential equation of a given problem. It is not easy to provide a meaning to α , without considering the classical model of the same problem. It is worth note the fact that the fractional order derivative of the problem studied in this paper is a value approaching one. Moreover, if $\alpha = 1$, then the Mittag Leffler function coincides with the classical exponential function. That is

$$E_{\alpha}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\alpha i + 1)} = e^t, t \in \mathbb{R}^+. \text{ It follows that the solution to the classical approach}$$

coincides exactly with the solution of to the fractional approach. The statement above shows that fractional derivative is effective and meaningful.

2-) In this work, we considered differential equations whose fractional model are easily obtained from their classical counterpart. However, this should not be used as a property. The Carleman Embedding technique [15], which is applicable to classical linear systems, is not always applicable to fractional system. For instance West.B [16] proposed a function to be the exact solution to the fractional logistic equation, his allegation was later proven incorrect by Area et al. [17] because the function proposed by West was derived using the Carleman Embedding technique.

6. Conclusion

The goal of this work was to prove the efficiency of the ψ -Caputo method in solving fractional differential equations upon the classical approach. This was done through experimental study. Several kernels were used. It appeared in general that the ψ -Caputo method produced a smaller magnitude error than the classical method, provided that the right kernel is selected. Moreover there exists no rule based on which the suitable kernel is selected. The selection is done based on the criteria in definition 2.3 and the variation of experimental data.

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