

A Computational Model for Multi-Level Quadratically Constrained Quadratic Optimization under Fully Fuzzy Environment

Abstract

Fully fuzzy quadratic programming became emerge naturally in numerous real-world applications. Therefore, an effective model based on the bound and decomposition method and the separable programming method is proposed in this paper for solving Fully Fuzzy Multi-Level Quadratically Constrained Quadratic Programming (FFMLQCQP) problem, where the objective function and the constraints are quadratic, also all the coefficients and variables of both objective functions and constraints are described fuzzily as fuzzy numbers. The bound and decomposition method is recommended to decompose the given (FFMLQCQP) problem into series of crisp Quadratically Constrained Quadratic Programming (QCQP) problems with bounded variable constraints for each level. Each (QCQP) problem is then solved independently by utilizing the separable programming method, which replaces the quadratic separable functions with linear functions. At last, the fuzzy optimal solution to the given (FFMLQCQP) problem is obtained. The effectiveness of the proposed model is illustrated through an illustrative numerical example.

Keywords

Fully fuzzy programming, Multi-level programming, Quadratic programming, Bound and decomposition method, Separable programming method.

1. Introduction

The purpose of this introduction is to furnish a brief description of fundamental concepts, and literature survey in terms of Fully Fuzzy Multi-Level Quadratically Constrained Quadratic Programming (FFMLQCQP) problem. Optimization problems arise constantly in a wide range of fields, which create the requirement of viable methods for solving them. In most real-world circumstances, these optimization problems include a lot of parameters whose values are allocated by decision makers. However, those values are frequently imprecisely or ambiguously known to the decision makers. With this observation in mind, it would be more appropriate to represent these parameters as fuzzy number and dealing with them using the concepts of fuzzy set theory, which offers the possibility to construct decision models with vague data. [1-3]

A fuzzy number is a convex fuzzy subset of the arrangement of real number in the sense that it does not allude to one single value but rather to a connected set of possible values. The fuzzy programming problems in which all the parameters as well as the variables of both objective functions and constraints are represented by fuzzy numbers are called Fully Fuzzy Programming (FFP) problems. One of most common (FFP) problems is the model in which all fuzzy parameters are described by triangle numbers. [4, 5]

The conventional optimization approaches are limited by their underlying assumptions of a single decision maker (DM) and a single objective function. These approaches become troublesome as the number of decision variables increases. First methodology developed to overcome the limitations of these conventional

approaches is with the structure of Multi-Objective Programming (MOP). MOP is a mathematical programming in which multiple, conflicting objective functions can be presented, and enables the study of inconsistencies among the multiple objectives controlled by a single DM. MOP is assumed that no single solution will optimize all objectives simultaneously. Instead, there exists a set of preferred solutions.

MOP overcomes the limitation of a single objective function, but the conventional optimization approaches remain limited by the assumption of a single DM. Therefore, it has become an urgent need for developing Multi-Level Programming (MLP) techniques with various decision makers as an alternative solution methodology to overcome the single DM limitation. MLP techniques are developed to solve decentralized planning problems with multiple decision makers in hierarchical organizations, where decisions have interacted with each other.

MLP problems have recently progressively appeared in decentralized management circumstances and have become exceedingly complicated, particularly with the improvement of economic integration and in the age of big data. Multi-level decision making process is based on two key issues: Firstly, architecting a multi-level decision model to depict a hierarchical decision making process. For that, the bi-level and tri-level decision making models have been presented. Secondly, obtaining an optimal solution to the decision model. For that, multi-level decision-making techniques, which are closely related to the economic problem of Stackelberg [6] in the field of game theory, have been developed to perceive compromises between the decision makers in a hierarchical organization and make their individual decisions in sequence for optimizing their objectives. [7, 8]

In many optimization problems, the objective function is non-linear function, or one or more constraints have non-linear relationship or both. This kind of problems is known as a Non-Linear Programming (NLP) problem. Quadratic Programming (QP) problem is one of the essential (NLP) problems of both theoretical importance and application interests. Quadratically Constrained Quadratic Programming (QCQP) problem generalizes the (QP) problem in that some or all the terms in the constraints are quadratic, which include the square of a variable or the product of two variables, instead of linear, in addition to the quadratic objective function. [9-11]

Kumar et al. in [12] proposed a technique to locate the fuzzy optimal solution of Fully Fuzzy Linear Programming (FFLP) problems with equality constraints by representing all parameters as triangular fuzzy numbers. After that, the ranking function [13] was utilized to change the (FFLP) problem into proportional crisp linear programming problem for solving the problem. Ezzati et al. [14] handled an algorithm to solve the (FFLP) problem by converting it to a Multi-Objective Linear Programming (MOLP) problem with three objective functions. At that point, the lexicographic strategy was used to locate a lexicographic optimal solution of (MOLP) problem. Ren developed in [3] a method to deal with the fully fuzzy bi-level linear programming problem by applying interval programming method. The membership grade of fuzzy coefficients and variables was disintegrated into a limited number of alpha-level sets. Then, the fully fuzzy bi-level linear programming problem was transformed into an interval bi-level linear programming problem for every alpha-level set. The acquired interval bi-level linear programming problem was converted into two deterministic sub-problems which related to the lower and upper bounds of its upper level objective function, which can be solved consecutively. In [15] Emam et al. proposed an interactive approach to find the solution of fully fuzzy multi-level linear programming problem.

Emam et al. in [16] tackled a solution for multi-level large-scale quadratic programming problem with stochastic parameters in the objective functions. Taylor's series was solidified with the decomposition algorithm to transform the quadratic into linear objective function. In [17] Loganathan and Lalitha presented a fuzzy multi-objective non-linear programming problem. Every one of the coefficients of the non-linear multi-objective functions and the constraints were fuzzy numbers. Interval arithmetic based on Alpha-cut was utilized to solve the non-linear programming problem. In [18] Youness et al. proposed an algorithm to comprehend Bi-Level Multi-Objective Fractional Integer Programming Problem (BLMOFIPP) including fuzzy numbers in the right-hand side of the constraints. The fuzzy numbers was described by triangle fuzzy membership functions. The algorithm consolidated the strategy of Taylor series with the Kuhn Tucker conditions to solve Fuzzy (BLMOFIPP), and then cutting-plane technique was used to get the integer solution.

In [19] Pramy developed a method for solving fuzzy multi-objective linear fractional programming problem. All the fuzzy parameters of the problem were converted into crisp multi-objective linear fractional programming problem using the graded mean integration representation method [20]. Then, the converted problem was transformed into a single objective linear programming problem. Kassa and Tsegay [21] developed an algorithm to solve a tri-level quadratic fractional programming problems using fuzzy goal programming approach. First, a membership function was constructed to develop a fuzzy model, and reformulate the original tri-level problem into a conventional one-level optimization problem. Then, the transformed fractional one-level problem was transformed into an equivalent quadratic problem. This final optimization problem was solved to get a candidate solution for the original tri-level problem.

In the majority of the previously mentioned strategies, the (FFP) problem is first transformed into approximated single crisp programming problem and afterward the acquired crisp problem is solved to locate the optimal solution of the problem. Therefore, the obtained optimal solution is approximate and not accurate, which is not reliable solution for the decision maker.

In this paper, (FFMLQCQP) problem is addressed, where the objective function and the constraints are quadratic, also all the coefficients and variables of both objective functions and constraints are expressed as fuzzy numbers. The outline of this paper is as follows: In section 2 some important preliminaries that are useful in the given (FFMLQCQP) problem are introduced. Section 3 formulates the given (FFMLQCQP) problem. Section 4 presents the proposed model for solving the given (FFMLQCQP) problem. A procedure followed by a flowchart for the proposed model are suggested in section 5. In section 6, the efficiency of the proposed model is showed through an illustrative numerical example. Finally, section 7 finalizes the paper with its conclusion and future works.

2. Preliminaries

In what follows, some principal definitions and notions related to the fuzzy set theory and the basic arithmetic operations on fuzzy numbers, which will be used in this paper, are exhibited.

2.1 Basic Definitions [2]

Definition 1. a fuzzy set $\tilde{A}$ in universe of discourse X ($\tilde{A} \subset X$) is directly specified by the membership function $\mu_{\tilde{A}}(x)$ or indirectly by a set of ordered pairs $(x, \mu_{\tilde{A}}(x))$, where $\mu_{\tilde{A}}(x)$ represent the value of the "grade of membership" of x in $\tilde{A}$, as follows:

$$\forall x \in X \mid \tilde{A} = \{(x, \mu_{\tilde{A}}(x))\}. \quad (1)$$

Definition 2. A fuzzy set $\tilde{A}$ is convex if any point located between two other points has a membership degree higher than the minimum membership degree of these two points, as follows:

$$\forall x_1, x_2 \in X, \lambda \in [0,1] \mid \mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)). \quad (2)$$

Definition 3. A fuzzy set $\tilde{A}$ is said to be normalized if its height ($\text{Hgt}(\tilde{A})$, the maximum membership degree) is unity, i.e., at least one element of X has a membership degree equal to one, as follows:

$$\forall x \in X \mid \mu_{\tilde{A}}(x) = \text{Hgt}(\tilde{A}) = 1. \quad (3)$$

Definition 4. A fuzzy number $\tilde{A}$ is a convex fuzzy set which has a normalized and continuous membership function. The membership function of a fuzzy number $\tilde{A}$ has the following properties:

1. $\mu_{\tilde{A}}(x)$ is upper semi-continuous.
2. $\mu_{\tilde{A}}(x) = 0$ outside of the interval $[c, d]$.
3. There are real numbers a and b , such that $c \leq a \leq b \leq d$ and:
 - 3.1 $\mu_{\tilde{A}}(x)$ is monotone increasing on the interval $[c, a]$.
 - 3.2 $\mu_{\tilde{A}}(x)$ is monotone decreasing on the interval $[b, d]$.
 - 3.3 $\mu_{\tilde{A}}(x) = 1$ for each $x \in [a, b]$.

Definition 5. A fuzzy set $\tilde{A}$ over X , is called triangular fuzzy number with a unique maximizing point a , left width $\alpha > 0$ and right width $\beta > 0$, if its membership function has the following form:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \frac{a-x}{\alpha} & \text{if } a - \alpha \leq x < a \\ 1 & \text{if } x = a \\ 1 - \frac{x-a}{\beta} & \text{if } a < x \leq a + \beta \\ 0 & \text{otherwise} \end{cases}. \quad (4)$$

2.2 Fuzzy Arithmetic Operations [22]

Let $\tilde{A} = (a_1, a_2, a_3)$ and $\tilde{B} = (b_1, b_2, b_3)$ are two triangular fuzzy numbers, and k is any real number, then the arithmetic operations between these two triangular fuzzy numbers are presented as follows:

Definition 6. $\tilde{A}$ and $\tilde{B}$ are said to be equal if and only if $a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$.

Definition 7. $k\tilde{A} = k(a_1, a_2, a_3) = (ka_1, ka_2, ka_3)$, for $k \geq 0$.

Definition 8. $k\tilde{A} = k(a_1, a_2, a_3) = (ka_3, ka_2, ka_1)$, for $k < 0$.

Definition 9. $\tilde{A} \oplus \tilde{B} = (a_1, a_2, a_3) \oplus (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

Definition 10. $\tilde{A} \ominus \tilde{B} = (a_1, a_2, a_3) \ominus (b_1, b_2, b_3) = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$.

Definition 11. $\tilde{A} \otimes \tilde{B} = \begin{cases} (a_1 b_1, a_2 b_2, a_3 b_3) & \text{if } a_1 \geq 0 \\ (a_1 b_3, a_2 b_2, a_3 b_3) & \text{if } a_1 < 0, a_3 \geq 0 \\ (a_1 b_3, a_2 b_2, a_3 b_1) & \text{if } a_3 < 0 \end{cases}$.

3. Fully Fuzzy Multi-Level Quadratically Constrained Quadratic Programming (FFMLQCQP) Problem

Let $\tilde{F}_i: R^m \rightarrow R$, ($i = 1, 2, \dots, n$) are the fully fuzzy first level objective function, second level objective function and n^{th} level objective function, respectively. $\tilde{x}$ are fuzzy decision variables indicating the control of each level decision maker. In this manner, the first level decision maker (FLDM) has control over the variable $\tilde{x}_1$, the second level decision maker (SLDM) has control over the variable $\tilde{x}_2$, and so on. $(\tilde{C}_{ij}, \tilde{A}_{hj})$ where ($i = 1, 2, \dots, n$), ($j = 1, 2, \dots, m$), ($h = 1, 2, \dots, o$), are fuzzy real numbers, which describing the coefficients of the linear terms. $(\tilde{L}_{ikj}, \tilde{Q}_{hkj})$ where ($i = 1, 2, \dots, n$), ($k = 1, 2, \dots, m$), ($j = 1, 2, \dots, m$), ($h = 1, 2, \dots, o$), are fuzzy real matrices, which describing the coefficients of the quadratic terms. $\tilde{G}$ is a fully fuzzy quadratically constraint set, where $\tilde{b}_h$ ($h = 1, 2, \dots, o$) is a fuzzy vector of right hand sides of constraints. Accordingly, the formulation of the (FFMLQCQP) problem can be considered as follows:

[FLDM]

$$\begin{aligned} \text{Max}_{\tilde{x}_1} \quad & \tilde{F}_1(\tilde{x}) = \sum_{j=1}^m \tilde{C}_{1j} \otimes \tilde{x}_j \oplus \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m \tilde{x}_k \otimes \tilde{L}_{1kj} \otimes \tilde{x}_j, \\ & \text{where } \tilde{x}_2, \tilde{x}_3, \dots, \tilde{x}_m \text{ solves;} \end{aligned} \quad (5.a)$$

[SLDM]

$$\begin{aligned} \text{Max}_{\tilde{x}_2} \quad & \tilde{F}_2(\tilde{x}) = \sum_{j=1}^m \tilde{C}_{2j} \otimes \tilde{x}_j \oplus \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m \tilde{x}_k \otimes \tilde{L}_{2kj} \otimes \tilde{x}_j, \\ & \text{where } \tilde{x}_3, \dots, \tilde{x}_m \text{ solves;} \end{aligned} \quad (5.b)$$

...

[n^{th} LDM]

$$\begin{aligned} \text{Max}_{\tilde{x}_n} \quad & \tilde{F}_n(\tilde{x}) = \sum_{j=1}^m \tilde{C}_{nj} \otimes \tilde{x}_j \oplus \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m \tilde{x}_k \otimes \tilde{L}_{nkj} \otimes \tilde{x}_j, \\ & \text{where } \tilde{x}_{n+1}, \dots, \tilde{x}_m \text{ solves;} \end{aligned} \quad (5.c)$$

Subject to:

$$\tilde{x}_j \in \tilde{G} \quad (j = 1, 2, \dots, m), \quad (5.d)$$

Where

$$\begin{aligned} \text{(QCQP)} \quad & \tilde{G} = \{ \sum_{j=1}^m \tilde{A}_{hj} \otimes \tilde{x}_j \oplus \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m \tilde{x}_k \otimes \tilde{Q}_{hkj} \otimes \tilde{x}_j \leq \tilde{b}_h \quad (h = 1, 2, \dots, o), \\ & \tilde{x}_j \geq 0 \quad (j = 1, 2, \dots, m). \} \end{aligned}$$

Definition 12. If $\tilde{x}^* \in R^m$ is a feasible solution for the (FFMLQCQP) problem (5.a) - (5.d); no other feasible solution $\tilde{x}_j \in \tilde{G}$ exists, where ($j = 1, 2, \dots, m$), such that $\tilde{F}_i(\tilde{x}^*) \leq \tilde{F}_i(\tilde{x})$; so $\tilde{x}^*$ is the optimal solution for the (FFMLQCQP) problem.

4. A Proposed Model for Finding the Fuzzy Optimal Solution of (FFMLQCQP) Problem

In what follows, a practical model is proposed for overcoming the intricacy of the (FFMLQCQP) problem using the bound and decomposition method [23, 24], and afterwards locating the fuzzy optimal solution for the given problem using the separable programming method [25, 26].

4.1 Bound and Decomposition Method

In the proposed model, the bound and decomposition method [23, 24] is used to decompose the (FFMLQCQP) problem into three crisps (QCQP) issues with bounded variable constraints for each level.

Firstly, all the fuzzy parameters and the fuzzy variables of both objective functions and constraints for each level ($\tilde{F}_i, \tilde{x}_j, \tilde{x}_k, \tilde{C}_{ij}, \tilde{A}_{hj}, \tilde{L}_{ikj}, \tilde{Q}_{hjk}$ and $\tilde{b}_h$) are represented by the following triangular fuzzy numbers $(Z_{1i}, Z_{2i}, Z_{3i}), (x_j, y_j, t_j), (x_k, y_k, t_k), (\varsigma_{ij}^1, \varsigma_{ij}^2, \varsigma_{ij}^3), (\gamma_{hj}^1, \gamma_{hj}^2, \gamma_{hj}^3), (\zeta_{ikj}^1, \zeta_{ikj}^2, \zeta_{ikj}^3), (\varphi_{hjk}^1, \varphi_{hjk}^2, \varphi_{hjk}^3)$ and $(\beta_h^1, \beta_h^2, \beta_h^3)$. Hence, the given (FFMLQCQP) problem can be reformulated as follows:

[ith LDM] where (i = 1, 2, ..., n)

$$\begin{aligned} & \text{Max}_{(x_i, y_i, t_i)} (Z_{1i}, Z_{2i}, Z_{3i})(x, y, t) \\ & = \sum_{j=1}^m (\varsigma_{ij}^1, \varsigma_{ij}^2, \varsigma_{ij}^3) \otimes (x_j, y_j, t_j) \oplus \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m (x_k, y_k, t_k) \otimes (\zeta_{ikj}^1, \zeta_{ikj}^2, \zeta_{ikj}^3) \otimes (x_j, y_j, t_j), \end{aligned} \quad (6.a)$$

where (x_j, y_j, t_j) solves, $(j = i + 1, \dots, m)$;

Subject to:

$$(x_j, y_j, t_j) \in G \quad (j = 1, 2, \dots, m), \quad (6.b)$$

Where

(QCQP)

$$\begin{aligned} G = \{ & \sum_{j=1}^m (\gamma_{hj}^1, \gamma_{hj}^2, \gamma_{hj}^3) \otimes (x_j, y_j, t_j) \oplus \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m (x_k, y_k, t_k) \otimes (\varphi_{hjk}^1, \varphi_{hjk}^2, \varphi_{hjk}^3) \otimes (x_j, y_j, t_j) \\ & \leq (\beta_h^1, \beta_h^2, \beta_h^3) \quad (h = 1, 2, \dots, o), \\ & (x_j, y_j, t_j) \geq 0 \quad (j = 1, 2, \dots, m). \} \end{aligned}$$

Since (x_j, y_j, t_j) is a triangular fuzzy number, then $x_j \leq y_j \leq t_j$ ($j = 1, 2, \dots, m$). The above relation is called bounded variable constraints. From the definitions of the fuzzy arithmetic operations, which are presented in sub-section 2.2, the above problem (6.a), (6.b) can be converted into three crisp (QCQP) problems with bounded variable constraints for each level, called Middle, Upper and Lower Multi-Level (QCQP) ((MMLQCQP), (UMLQCQP) and (LMLQCQP) respectively), as follows:

[ith LDM] where (i = 1, 2, ..., n)

(MMLQCQP) Problem:

$$\begin{aligned} & \text{Max}_{y_i} Z_{2i}(y) = \sum_{j=1}^m \varsigma_{ij}^2 y_j + \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m y_k \zeta_{ikj}^2 y_j, \end{aligned} \quad (7.a)$$

where y_j solves, $(j = i + 1, \dots, m)$;

Subject to:

$$y_j \in G_M \quad (j = 1, 2, \dots, m), \quad (7.b)$$

Where

$$\begin{aligned} \text{(QCQP)} \quad G_M = \{ & \sum_{j=1}^m \gamma_{hj}^2 y_j + \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m y_k \varphi_{hjk}^2 y_j \leq \beta_h^2 \quad (h = 1, 2, \dots, o), \\ & y_j \geq 0 \quad (j = 1, 2, \dots, m). \} \end{aligned}$$

(UMLQCQP) Problem:

$$\begin{aligned} & \text{Max}_{t_i} Z_{3i}(t) = \sum_{j=1}^m \varsigma_{ij}^3 t_j + \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m t_k \zeta_{ikj}^3 t_j, \end{aligned} \quad (8.a)$$

where t_j solves, $(j = i + 1, \dots, m)$;

Subject to:

$$t_j \in G_U \quad (j = 1, 2, \dots, m), \quad (8.b)$$

Where

$$\begin{aligned} (QCQP) \quad G_U = \{ & \sum_{j=1}^m \gamma_{hj}^3 t_j + \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m t_k \varphi_{hjk}^3 t_j \leq \beta_h^3 \quad (h = 1, 2, \dots, o), \\ & \sum_{j=1}^m \varsigma_{ij}^3 t_j + \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m t_k \zeta_{ikj}^3 t_j \geq Z_{2i}^*, \\ & t_j \geq y_j \quad (j = 1, 2, \dots, m), \\ & t_j \geq 0 \quad (j = 1, 2, \dots, m). \} \end{aligned}$$

(LMLQCQP) Problem:

$$\begin{aligned} \text{Max } Z_{1i}(x) = & \sum_{j=1}^m \varsigma_{ij}^1 x_j + \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m x_k \zeta_{ikj}^1 x_j, \\ & x_i \quad \text{where } x_j \text{ solves, } (j = i + 1, \dots, m); \end{aligned} \quad (9.a)$$

Subject to:

$$x_j \in G_L \quad (j = 1, 2, \dots, m), \quad (9.b)$$

Where

$$\begin{aligned} (QCQP) \quad G_L = \{ & \sum_{j=1}^m \gamma_{hj}^1 x_j + \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m x_k \varphi_{hjk}^1 x_j \leq \beta_h^1 \quad (h = 1, 2, \dots, o), \\ & \sum_{j=1}^m \varsigma_{ij}^1 x_j + \frac{1}{2} \sum_{j=1}^m \sum_{k=1}^m x_k \zeta_{ikj}^1 x_j \leq Z_{2i}^*, \\ & x_j \leq y_j \quad (j = 1, 2, \dots, m), \\ & x_j \geq 0 \quad (j = 1, 2, \dots, m). \} \end{aligned}$$

4.2 Separable Programming Method

Separable programming method [25, 26] deals with (QCQP) problems in which both the quadratic objective functions and the quadratic constraints are separable functions. A separable function is a function where each term includes just a single variable. Some quadratic functions are not directly separable but can be made detachable by appropriate substitutions. Therefore, each quadratic objective function and quadratic constraint can be expressed as a combination of separable tasks with individual variables. After that, each of these separable functions with individual variables can be approximated as nearly as possible to a linear function with a more substantial number of variables. At last, the optimal solution to the (QCQP) problem can be obtained.

Consider the constructed three crisp (QCQP) problems in sub-section 4.1, which are called (MMLQCQP) (7.a), (7.b), (UMLQCQP) (8.a), (8.b) and (LMLQCQP) (9.a), (9.b) problems. At that point, each quadratic objective function ($Z_{1i}(x)$, $Z_{2i}(y)$ and $Z_{3i}(t)$) is expressed as a combination of separable functions with individual variables ($f_{1iw}(x_w)$, $f_{2iw}(y_w)$ and $f_{3iw}(t_w)$) where ($w = 1, 2, \dots, \theta$), and each quadratic constraint $G(x, y, t)$ as a combination of separable constraints ($g_{1w}^d(x_w)$, $g_{2w}^d(y_w)$ and $g_{3w}^d(t_w)$) where ($d = 1, 2, \dots, D$).

Assume that (a_{isw}^1 , a_{isw}^2 and a_{isw}^3) where ($s = 0, 1, 2, \dots, S_w$) be the breaking points of each separable function ($f_{1iw}(x_w)$, $f_{2iw}(y_w)$ and $f_{3iw}(t_w)$). (x_{sw} , y_{sw} and t_{sw}) are a new variables, which representing the increment of each variable (x_w , y_w and t_w) in the range ($(a_{i(s-1,w)}^1, a_{isw}^1), (a_{i(s-1,w)}^2, a_{isw}^2), (a_{i(s-1,w)}^3, a_{isw}^3)$). (ρ_{isw}^1 , ρ_{isw}^2 and ρ_{isw}^3) is the slope of the line segment corresponding to the separable functions ($f_{1iw}(x_w)$, $f_{2iw}(y_w)$ and $f_{3iw}(t_w)$) in the same range. (ρ_{sw}^{1d} , ρ_{sw}^{2d} and ρ_{sw}^{3d}) is the slope of the line segment corresponding to the separable constraints ($g_{1w}^d(x_w)$, $g_{2w}^d(y_w)$ and $g_{3w}^d(t_w)$). As a result, the approximated linear forms of the constructed three crisp (QCQP) problems (MMLQCQP), (UMLQCQP) and (LMLQCQP) problems can be stated as:

[ith LDM] where (i = 1, 2, ..., n)

(MMLQCQP) Problem:

$$\text{Max}_{y_i} Z_{2i}(y) = \sum_{w=1}^{\theta} f_{2iw}(y_w), \quad (10.a)$$

where y_j solves, (j = i + 1, ..., m);

Subject to: (10.b)

$$\sum_{w=1}^{\theta} g_{2w}^d(y_w) \leq \beta_h^2 \quad (d = 1, 2, \dots, D) \quad (h = 1, 2, \dots, o),$$

$$y_w \geq 0 \quad (w = 1, 2, \dots, \theta).$$

Where:

$$f_{2iw}(y_w) = \sum_{s=1}^{S_w} \rho_{isw}^2 y_{sw} \quad (w = 1, 2, \dots, \theta),$$

$$\rho_{isw}^2 = \frac{f_{2iw}(a_{isw}^2) - f_{2iw}(a_{i(s-1,w)}^2)}{a_{isw}^2 - a_{i(s-1,w)}^2} \quad (s = 0, 1, 2, \dots, S_w) \quad (w = 1, 2, \dots, \theta),$$

$$g_{2w}^d(y_w) = \sum_{s=1}^{S_w} \rho_{sw}^{2d} y_{sw} \quad (w = 1, 2, \dots, \theta) \quad (d = 1, 2, \dots, D),$$

$$\rho_{sw}^{2d} = \frac{g_{2w}^d(a_{isw}^2) - g_{2w}^d(a_{i(s-1,w)}^2)}{a_{isw}^2 - a_{i(s-1,w)}^2} \quad (s = 0, 1, 2, \dots, S_w) \quad (w = 1, 2, \dots, \theta) \quad (d = 1, 2, \dots, D),$$

$$0 \leq y_{sw} \leq a_{isw}^2 - a_{i(s-1,w)}^2 \quad (s = 0, 1, 2, \dots, S_w) \quad (w = 1, 2, \dots, \theta),$$

$$y_w = \sum_{s=1}^{S_w} y_{sw} \quad (w = 1, 2, \dots, \theta).$$

(UMLQCQP) Problem:

$$\text{Max}_{t_i} Z_{3i}(t) = \sum_{w=1}^{\theta} f_{3iw}(t_w), \quad (11.a)$$

where t_j solves, (j = i + 1, ..., m);

Subject to: (11.b)

$$\sum_{w=1}^{\theta} g_{3w}^d(t_w) \leq \beta_h^3 \quad (d = 1, 2, \dots, D) \quad (h = 1, 2, \dots, o),$$

$$\sum_{w=1}^{\theta} f_{3iw}(t_w) \geq Z_{2i}^*,$$

$$t_w \geq y_w \quad (w = 1, 2, \dots, \theta),$$

$$t_w \geq 0 \quad (w = 1, 2, \dots, \theta).$$

Where:

$$f_{3iw}(t_w) = \sum_{s=1}^{S_w} \rho_{isw}^3 t_{sw} \quad (w = 1, 2, \dots, \theta),$$

$$\rho_{isw}^3 = \frac{f_{3iw}(a_{isw}^3) - f_{3iw}(a_{i(s-1,w)}^3)}{a_{isw}^3 - a_{i(s-1,w)}^3} \quad (s = 0, 1, 2, \dots, S_w) \quad (w = 1, 2, \dots, \theta),$$

$$g_{3w}^d(t_w) = \sum_{s=1}^{S_w} \rho_{sw}^{3d} t_{sw} \quad (w = 1, 2, \dots, \theta) \quad (d = 1, 2, \dots, D),$$

$$\rho_{sw}^{3d} = \frac{g_{3w}^d(a_{isw}^3) - g_{3w}^d(a_{i(s-1,w)}^3)}{a_{isw}^3 - a_{i(s-1,w)}^3} \quad (s = 0, 1, 2, \dots, S_w) \quad (w = 1, 2, \dots, \theta) \quad (d = 1, 2, \dots, D),$$

$$0 \leq t_{sw} \leq a_{isw}^3 - a_{i(s-1,w)}^3 \quad (s = 0, 1, 2, \dots, S_w) \quad (w = 1, 2, \dots, \theta),$$

$$t_w = \sum_{s=1}^{S_w} t_{sw} \quad (w = 1, 2, \dots, \theta),$$

$$y_w = \sum_{s=1}^{S_w} y_{sw} \quad (w = 1, 2, \dots, \theta).$$

(LMLQCQP) Problem:

$$\text{Max}_{x_i} Z_{1i}(x) = \sum_{w=1}^{\theta} f_{1iw}(x_w), \quad (12.a)$$

where x_j solves, (j = i + 1, ..., m);

Subject to: (12.b)

$$\sum_{w=1}^{\theta} g_{1w}^d(x_w) \leq \beta_h^1 \quad (d = 1, 2, \dots, D) \quad (h = 1, 2, \dots, o),$$

$$\sum_{w=1}^{\theta} f_{1iw}(x_w) \leq Z_{2i}^*,$$

$$x_w \leq y_w \quad (w = 1, 2, \dots, \theta),$$

$$x_w \geq 0 \quad (w = 1, 2, \dots, \theta).$$

Where:

$$\begin{aligned}
f_{1iw}(x_w) &= \sum_{s=1}^{S_w} \rho_{1sw}^1 x_{sw} & (w = 1, 2, \dots, \theta), \\
\rho_{1sw}^1 &= \frac{f_{1iw}(a_{1sw}^1) - f_{1iw}(a_{1(s-1,w)}^1)}{a_{1sw}^1 - a_{1(s-1,w)}^1} & (s = 0, 1, 2, \dots, S_w) \quad (w = 1, 2, \dots, \theta), \\
g_{1w}^d(x_w) &= \sum_{s=1}^{S_w} \rho_{1sw}^{1d} x_{sw} & (w = 1, 2, \dots, \theta) \quad (d = 1, 2, \dots, D), \\
\rho_{1sw}^{1d} &= \frac{g_{1w}^d(a_{1sw}^1) - g_{1w}^d(a_{1(s-1,w)}^1)}{a_{1sw}^1 - a_{1(s-1,w)}^1} & (s = 0, 1, 2, \dots, S_w) \quad (w = 1, 2, \dots, \theta) \quad (d = 1, 2, \dots, D), \\
0 \leq x_{sw} &\leq a_{1sw}^1 - a_{1(s-1,w)}^1 & (s = 0, 1, 2, \dots, S_w) \quad (w = 1, 2, \dots, \theta), \\
x_w &= \sum_{s=1}^{S_w} x_{sw} & (w = 1, 2, \dots, \theta), \\
y_w &= \sum_{s=1}^{S_w} y_{sw} & (w = 1, 2, \dots, \theta).
\end{aligned}$$

Accordingly, if the previously mentioned approximated linear forms of the constructed three crisp (QCQP) problems are solvable, then the given (FFMLQCQP) problem is solvable, and the fuzzy optimal solution to the given problem can be obtained.

5. A Procedure for Finding the Fuzzy Optimal Solution of (FFMLQCQP) Problem

The solution procedure of the proposed model for solving the given (FFMLQCQP) problem can be summarised in the following steps:

- Step 1.** Formulate the (FFMLQCQP) problem (Problem (5.a) - (5.d)).
- Step 2.** Set $i=1$, which indicating the i^{th} level decision maker (w^{th} LDM).
- Step 3.** Formulate the i^{th} LDM programming problem.
- Step 4.** If the i^{th} LDM obtains the optimal solution then go to Step 20, otherwise, go to Step 5.
- Step 5.** All the fuzzy parameters and variables of both objective functions and constraints for i^{th} LDM are represented by the triangular fuzzy numbers, as problem (6.a), (6.b).
- Step 6.** Based on the fuzzy arithmetic operations and the bound and decomposition method [23, 24], the fully fuzzy (QCQP) problem for i^{th} LDM is decomposed into three crisps (QCQP) difficulties for i^{th} LDM, called Middle (7.a), (7.b), Upper (8.a), (8.b) and Lower (9.a), (9.b) w^{th} Level (QCQP) problems.
- Step 7.** Formulate the Middle i^{th} Level (QCQP) problem (Problem (7.a), (7.b)).
- Step 8.** Convert the non-separable quadratic objective functions and quadratic constraints into separable forms using appropriate arithmetic substitutions.
- Step 9.** Express the quadratic objective function and each quadratic constraint as a combination of separable tasks with individual variables.
- Step 10.** Define the breaking points of each separable function estimated from the bounded variable constraints, and the increment of each variable based on the breaking points, which introducing new variables.
- Step 11.** Compute the slope of the line segment corresponding to each separable function.

- Step 12.** Each of these separable functions is approximated to a linear function with a combination of larger number of variables, as illustrated in problems (10.a) - (10.c), (11.a) - (11.c) and (12.a) - (12.c).
- Step 13.** Solve the resulting linear programming problem for obtaining the individual optimal solution of the decomposed w_i^{th} Level (QCQP) problem.
- Step 14.** If the individual optimal solution of the Upper w_i^{th} Level (QCQP) problem is obtained, then go to Step 17, otherwise go to Step 15.
- Step 15.** All variables in the objective function and constraints of the Upper w_i^{th} Level (QCQP) problem must satisfy the bounded variable constraints, a problem (8.a), (8.b).
- Step 16.** Formulate the Upper w_i^{th} Level (QCQP) problem, then go to Step 8.
- Step 17.** If the individual optimal solution of the Lower i^{th} Level (QCQP) problem is obtained, then go to Step 4, otherwise go to Step 18.
- Step 18.** All variables in the objective function and constraints of the Lower i^{th} Level (QCQP) problem must satisfy the bounded variable constraints, as problem (9.a), (9.b).
- Step 19.** Formulate the Lower i^{th} Level (QCQP) problem, then go to Step 8.
- Step 20.** If $i=n$, then go to Step 23, otherwise go to step 21.
- Step 21.** Set $i=i+1$.
- Step 22.** The i^{th} LDM defines his/her problem in point of view of the $(i-1)^{\text{th}}$ LDMs (upper-level decision makers) by setting the controlled variables of the $(i-1)^{\text{th}}$ LDMs to the i^{th} LDM constraints, then go to Step 4.
- Step 23.** The fuzzy optimal solution of the (FFMLQCQP) problem is obtained.
- Step 24.** Stop.

Figure (1) illustrates a flowchart to describe the procedure of the proposed model for solving the proposed (FFMLQCQP) problem.

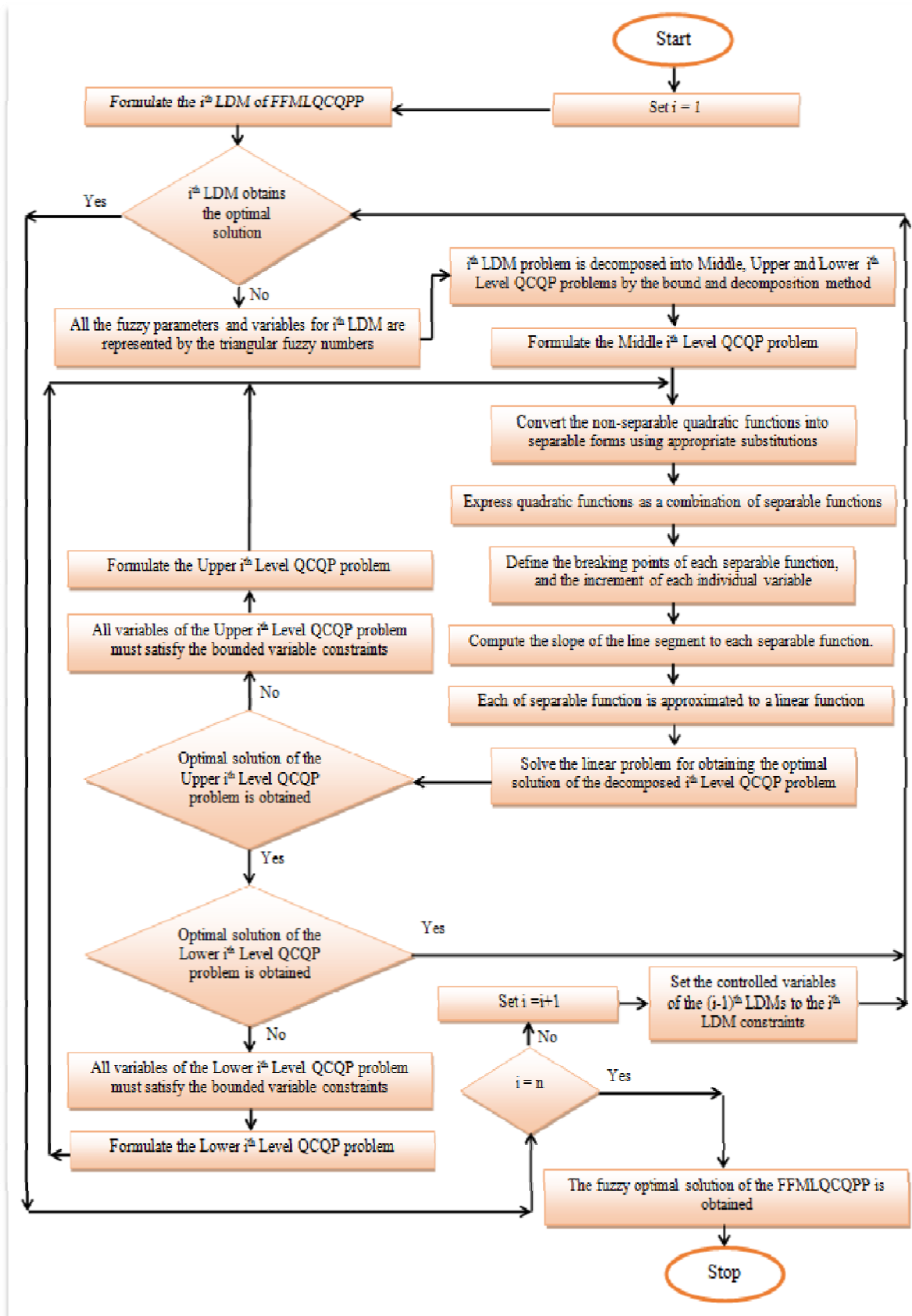


Fig 1: A flowchart for the proposed procedure

6. Numerical Example

In this section, the proficiency of the proposed model is illustrated through the following illustrative numerical example for solving the given (FFMLQCQP) problem.

[FLDM]

$$\begin{aligned} \text{Max } \tilde{F}_1(\tilde{x}) = & ((7,10,15) \otimes \tilde{x}_1 \oplus (2,4,6) \otimes \tilde{x}_2 \oplus (3,5,9) \otimes \tilde{x}_3 \oplus (\tilde{x}_1, \tilde{x}_2) \otimes \begin{pmatrix} (8,12,17) & (2,6,7) \\ (1,1,3) & (1,2,4) \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}), \\ \tilde{x}_1 \quad & \text{where } \tilde{x}_2, \tilde{x}_3 \text{ solves;} \end{aligned}$$

[SLDM]

$$\begin{aligned} \text{Max } \tilde{F}_2(\tilde{x}) = & ((4,5,8) \otimes \tilde{x}_1 \oplus (12,16,19) \otimes \tilde{x}_2 \oplus (7,9,11) \otimes \tilde{x}_3 \oplus (\tilde{x}_2, \tilde{x}_3) \otimes \begin{pmatrix} (8,9,11) & (0,4,5) \\ (0,2,3) & (0,0,0) \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix}), \\ \tilde{x}_2 \quad & \text{where } \tilde{x}_3 \text{ solves;} \end{aligned}$$

[TLDM]

$$\begin{aligned} \text{Max } \tilde{F}_3(\tilde{x}) = & ((7,8,14) \otimes \tilde{x}_1 \oplus (4,9,9) \otimes \tilde{x}_2 \oplus (13,18,20) \otimes \tilde{x}_3 \oplus (\tilde{x}_1, \tilde{x}_3) \otimes \begin{pmatrix} (1,2,6) & (0,1,3) \\ (3,4,4) & (9,10,14) \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_3 \end{pmatrix}), \\ \tilde{x}_3 \end{aligned}$$

Subject to:

$$\begin{aligned} \text{(QCQP)} \quad & (1,3,6) \otimes \tilde{x}_1 \oplus (5,7,10) \otimes \tilde{x}_2 \oplus (2,5,9) \otimes \tilde{x}_3 \otimes \tilde{x}_3 \leq (35,60,85), \\ & (2,10,12) \otimes \tilde{x}_1 \oplus (2,3,5) \otimes \tilde{x}_2 \oplus (6,6,11) \otimes \tilde{x}_3 \leq (10,18,39), \\ & (3,4,7) \otimes \tilde{x}_1 \oplus (\tilde{x}_1, \tilde{x}_2) \otimes \begin{pmatrix} (5,6,7) & (0,2,4) \\ (0,1,2) & (1,3,3) \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} \leq (40,70,115), \\ & \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \geq 0. \end{aligned}$$

1. [FLDM]

Assume $\tilde{x}_1 = (x_1, y_1, t_1)$, $\tilde{x}_2 = (x_2, y_2, t_2)$, $\tilde{x}_3 = (x_3, y_3, t_3)$, $\tilde{F}_1 = (Z_{11}, Z_{21}, Z_{31})$. Therefore, the Fully Fuzzy (QCQP) problem of the FLDM is reformulated as follows:

$$\begin{aligned} \text{Max } (Z_{11}, Z_{21}, Z_{31})(x, y, t) = & ((7,10,15) \otimes (x_1, y_1, t_1) \oplus (2,4,6) \otimes (x_2, y_2, t_2) \oplus (3,5,9) \otimes (x_3, y_3, t_3) \oplus \\ & ((x_1, y_1, t_1), (x_2, y_2, t_2)) \otimes \begin{pmatrix} (8,12,17) & (2,6,7) \\ (1,1,3) & (1,2,4) \end{pmatrix} \otimes \begin{pmatrix} (x_1, y_1, t_1) \\ (x_2, y_2, t_2) \end{pmatrix}), \\ \text{where } x_2, x_3, y_2, y_3, t_2, t_3 \text{ solves;} \end{aligned}$$

Subject to:

$$\begin{aligned} \text{(QCQP)} \quad & (1,3,6) \otimes (x_1, y_1, t_1) \oplus (5,7,10) \otimes (x_2, y_2, t_2) \oplus (2,5,9) \otimes (x_3, y_3, t_3) \otimes (x_3, y_3, t_3) \leq (35,60,85), \\ & (2,10,12) \otimes (x_1, y_1, t_1) \oplus (2,3,5) \otimes (x_2, y_2, t_2) \oplus (6,6,11) \otimes (x_3, y_3, t_3) \leq (10,18,39), \\ & (3,4,7) \otimes (x_1, y_1, t_1) \oplus ((x_1, y_1, t_1), (x_2, y_2, t_2)) \otimes \begin{pmatrix} (5,6,7) & (0,2,4) \\ (0,1,2) & (1,3,3) \end{pmatrix} \otimes \begin{pmatrix} (x_1, y_1, t_1) \\ (x_2, y_2, t_2) \end{pmatrix} \\ & \leq (40,70,115), \\ & x_1, x_2, x_3, y_1, y_2, y_3, t_1, t_2, t_3 \geq 0. \end{aligned}$$

From the fuzzy arithmetic operations, which are presented in subsection 2.2, and based on the bound and decomposition method [23, 24], the Fully Fuzzy (QCQP) problem of the FLDM reduces to the

following three crisp (QCQP) problems, called Lower, Middle and Upper Level (QCQP) ((LLQCQP), (MLQCQP) and (ULQCQP) respectively) problems of the FLDM, as follows:

$$\begin{aligned}
 \text{(LLQCQP)} \quad & \text{Max}_{x_1} Z_{11}(x) = (8x_1^2 + 3x_1x_2 + 7x_1 + x_2^2 + 2x_2 + 3x_3) \\
 \text{(MLQCQP)} \quad & \text{Max}_{y_1} Z_{21}(y) = (12y_1^2 + 7y_1y_2 + 10y_1 + 2y_2^2 + 4y_2 + 5y_3) \\
 \text{(ULQCQP)} \quad & \text{Max}_{t_1} Z_{31}(t) = (17t_1^2 + 10t_1t_2 + 15t_1 + 4t_2^2 + 6t_2 + 9t_3) \\
 & \text{where } x_2, x_3, y_2, y_3, t_2, t_3 \text{ solves;}
 \end{aligned}$$

Subject to:

$$\begin{aligned}
 \text{(QCQP)} \quad & x_1 + 5x_2 + 2x_3^2 \leq 35, \\
 & 2x_1 + 2x_2 + 6x_3 \leq 10, \\
 & 5x_1^2 + 3x_1 + x_2^2 \leq 40, \\
 & 3y_1 + 7y_2 + 5y_3^2 \leq 60, \\
 & 10y_1 + 3y_2 + 6y_3 \leq 18, \\
 & 6y_1^2 + 3y_1y_2 + 4y_1 + 3y_2^2 \leq 70, \\
 & 6t_1 + 10t_2 + 9t_3^2 \leq 85, \\
 & 12t_1 + 5t_2 + 11t_3 \leq 39, \\
 & 7t_1^2 + 6t_1t_2 + 7t_1 + 3t_2^2 \leq 115, \\
 & x_1, x_2, x_3, y_1, y_2, y_3, t_1, t_2, t_3 \geq 0.
 \end{aligned}$$

1.1. The (MLQCQP) problem of the FLDM

$$\begin{aligned}
 \text{Max}_{y_1} Z_{21}(y) &= (12y_1^2 + 7y_1y_2 + 10y_1 + 2y_2^2 + 4y_2 + 5y_3) \\
 &\text{where } y_2, y_3 \text{ solves;}
 \end{aligned}$$

Subject to:

$$\begin{aligned}
 \text{(QCQP)} \quad & 3y_1 + 7y_2 + 5y_3^2 \leq 60, \\
 & 10y_1 + 3y_2 + 6y_3 \leq 18, \\
 & 6y_1^2 + 3y_1y_2 + 4y_1 + 3y_2^2 \leq 70, \\
 & y_1, y_2, y_3 \geq 0.
 \end{aligned}$$

Now, the separable programming method [25, 26] is applied on the (MLQCQP) problem of the FLDM to obtain the individual optimal solution. First, the y_1y_2 term (which is the non-separable function with two variables) is needed to convert into separable form by using the following arithmetic substitution:

$$y_1y_2 = \left[\left(\frac{y_1 + y_2}{2} \right)^2 - \left(\frac{y_1 - y_2}{2} \right)^2 \right].$$

Assume two new decision variables m_1, m_2 , where: $m_1 = \frac{y_1 + y_2}{2}$, $m_2 = \frac{y_1 - y_2}{2}$.

$$\therefore y_1y_2 = (m_1^2 - m_2^2), \text{ with the two additional constraints: } \frac{1}{2}y_1 + \frac{1}{2}y_2 - m_1 = 0, \quad \frac{1}{2}y_1 - \frac{1}{2}y_2 - m_2 = 0.$$

So, the (MLQCQP) problem of the FLDM is rewritten as follows:

$$\begin{aligned}
 \text{Max}_{y_1} Z_{21}(y) &= (12y_1^2 + 7m_1^2 - 7m_2^2 + 10y_1 + 2y_2^2 + 4y_2 + 5y_3) \\
 &\text{where } y_2, y_3 \text{ solves;}
 \end{aligned}$$

Subject to:

$$\begin{aligned}
 \text{(QCQP)} \quad & 3y_1 + 7y_2 + 5y_3^2 \leq 60, \\
 & 10y_1 + 3y_2 + 6y_3 \leq 18, \\
 & 6y_1^2 + 3m_1^2 - 3m_2^2 + 4y_1 + 3y_2^2 \leq 70,
 \end{aligned}$$

$$\begin{aligned}\frac{1}{2}y_1 + \frac{1}{2}y_2 - m_1 &= 0, \\ \frac{1}{2}y_1 - \frac{1}{2}y_2 - m_2 &= 0, \\ y_1, y_2, y_3 &\geq 0.\end{aligned}$$

The separable functions with individual variables of this (MLQCQP) problem for the FLDM are:

$$\begin{aligned}f_1(y_1) &= 12y_1^2 + 10y_1, \quad f_2(y_2) = 2y_2^2 + 4y_2, \quad f_3(y_3) = 5y_3, \quad f_4(m_1) = 7m_1^2, \quad f_5(m_2) = -7m_2^2. \\ g_1^1(y_1) &= 3y_1, \quad g_1^2(y_1) = 10y_1, \quad g_1^3(y_1) = 6y_1^2 + 4y_1, \quad g_1^4(y_1) = \frac{1}{2}y_1, \quad g_1^5(y_1) = \frac{1}{2}y_1. \\ g_2^1(y_2) &= 7y_2, \quad g_2^2(y_2) = 3y_2, \quad g_2^3(y_2) = 3y_2^2, \quad g_2^4(y_2) = \frac{1}{2}y_2, \quad g_2^5(y_2) = -\frac{1}{2}y_2. \\ g_3^1(y_3) &= 5y_3^2, \quad g_3^2(y_3) = 6y_3. \\ g_4^1(m_1) &= 3m_1^2, \quad g_4^2(m_1) = -m_1. \\ g_5^1(m_2) &= -3m_2^2, \quad g_5^2(m_2) = -m_2.\end{aligned}$$

The ranges of the variables y_1, y_2, y_3, m_1, m_2 (estimated from the constraints) are:

$$0 \leq y_1 \leq 3.42, \quad 0 \leq y_2 \leq 4.83, \quad 0 \leq y_3 \leq 3.47, \quad 0 \leq m_1 \leq 4.125, \quad -2.415 \leq m_2 \leq 1.71.$$

Let k_i be the breaking points of the separable functions, where:

$$k_1 = 4 \text{ for } y_1, \quad k_2 = 5 \text{ for } y_2, \quad k_3 = 4 \text{ for } y_3, \quad k_4 = 5 \text{ for } m_1, \quad k_5 = 5 \text{ for } m_2.$$

Based on the equations, which are illustrated in problem (10.c), the slopes of the line segment corresponding to the defined separable functions are computed. **The complete resulting linear programming problem with a larger number of variables then becomes:**

$$\begin{aligned}\text{Max } Z_{21}(y) &= (22y_{11} + 46y_{21} + 70y_{31} + 87.04y_{41} + 7y_{14} + 21y_{24} + 35y_{34} + 49y_{44} + 56.875y_{54} \\ &\quad + 30.91y_{15} + 21y_{25} + 7y_{35} - 7y_{45} - 18.97y_{55} + 6y_{12} + 10y_{22} + 14y_{32} + 18y_{42} \\ &\quad + 21.66y_{52} + 5y_{13} + 5y_{23} + 5y_{33} + 5y_{43})\end{aligned}$$

Subject to:

$$3y_{11} + 3y_{21} + 3y_{31} + 3y_{41} + 7y_{12} + 7y_{22} + 7y_{32} + 7y_{42} + 7y_{52} + 5y_{13} + 15y_{23} + 25y_{33} + 32.35y_{43} \leq 60,$$

$$10y_{11} + 10y_{21} + 10y_{31} + 10y_{41} + 3y_{12} + 3y_{22} + 3y_{32} + 3y_{42} + 3y_{52} + 6y_{13} + 6y_{23} + 6y_{33} + 6y_{43} \leq 18,$$

$$10y_{11} + 22y_{21} + 34y_{31} + 42.52y_{41} + 3y_{14} + 9y_{24} + 15y_{34} + 21y_{44} + 24.375y_{54} + 13.246y_{15} + 9y_{25} + 3y_{35} - 3y_{45} - 8.13y_{55} + 3y_{12} + 9y_{22} + 15y_{32} + 21y_{42} + 26.49y_{52} \leq 70,$$

$$\frac{1}{2}y_{11} + \frac{1}{2}y_{21} + \frac{1}{2}y_{31} + \frac{1}{2}y_{41} + \frac{1}{2}y_{12} + \frac{1}{2}y_{22} + \frac{1}{2}y_{32} + \frac{1}{2}y_{42} + \frac{1}{2}y_{52} - y_{14} - y_{24} - y_{34} - y_{44} - y_{54} = 0,$$

$$\frac{1}{2}y_{11} + \frac{1}{2}y_{21} + \frac{1}{2}y_{31} + \frac{1}{2}y_{41} - \frac{1}{2}y_{12} - \frac{1}{2}y_{22} - \frac{1}{2}y_{32} - \frac{1}{2}y_{42} - \frac{1}{2}y_{52} - y_{15} - y_{25} - y_{35} - y_{45} - y_{55} = 0,$$

$$y_{k1} \geq 0, \quad k = 1, 2, 3, 4,$$

$$y_{k2} \geq 0, \quad k = 1, 2, 3, 4, 5,$$

$$y_{k3} \geq 0, \quad k = 1, 2, 3, 4,$$

$$y_{k4} \geq 0, \quad k = 1, 2, 3, 4, 5,$$

$$y_{k5} \text{ can be positive or negative.}$$

The optimal solution for the resulting linear programming problem is:

$$(y_{11}, y_{21}, y_{31}, y_{41}, y_{12}, y_{22}, y_{32}, y_{42}, y_{52}, y_{13}, y_{23}, y_{33}, y_{43}) = (0, 0, 0, 1.141357, 0, 0, 0, 0, 0, 1.097738, 0, 0, 0).$$

This optimal solution can translate to:

$$y_1 = y_{11} + y_{21} + y_{31} + y_{41} = 1.141357.$$

$$y_2 = y_{12} + y_{22} + y_{32} + y_{42} + y_{52} = 0.$$

$$y_3 = y_{13} + y_{23} + y_{33} + y_{43} = 1.097738.$$

Hence, the individual optimal solution to the (MLQCQP) problem of the FLDM is:

$$(y_1, y_2, y_3) = (1.141357, 0, 1.097738) \quad , \quad \text{and} \quad Z_{21} = 32.5346.$$

1.2. The (ULQCQP) problem of the FLDM with the satisfied bounded variable constraints

$$\text{Max}_{t_1} Z_{31}(t) = (17t_1^2 + 10t_1t_2 + 15t_1 + 4t_2^2 + 6t_2 + 9t_3)$$

where t_2, t_3 solves;

Subject to:

$$\begin{aligned} \text{(QCQP)} \quad & 6t_1 + 10t_2 + 9t_3^2 \leq 85, \\ & 12t_1 + 5t_2 + 11t_3 \leq 39, \\ & 7t_1^2 + 6t_1t_2 + 7t_1 + 3t_2^2 \leq 115, \\ & 17t_1^2 + 10t_1t_2 + 15t_1 + 4t_2^2 + 6t_2 + 9t_3 \geq 32.5346, \\ & t_1 \geq 1.141357, \\ & t_3 \geq 1.097738, \\ & t_1, t_2, t_3 \geq 0. \end{aligned}$$

Repeat the same action as the (MLQCQP) problem by applying the separable programming method on the (ULQCQP) problem of the FLDM. The (ULQCQP) problem after converting the non-separable function t_1t_2 into separable form is as follows:

$$\text{Max}_{t_1} Z_{31}(t) = (17t_1^2 + 10m_1^2 - 10m_2^2 + 15t_1 + 4t_2^2 + 6t_2 + 9t_3)$$

where t_2, t_3 solves;

Subject to:

$$\begin{aligned} \text{(QCQP)} \quad & 6t_1 + 10t_2 + 9t_3^2 \leq 85, \\ & 12t_1 + 5t_2 + 11t_3 \leq 39, \\ & 7t_1^2 + 6m_1^2 - 6m_2^2 + 7t_1 + 3t_2^2 \leq 115, \\ & 17t_1^2 + 10m_1^2 - 10m_2^2 + 15t_1 + 4t_2^2 + 6t_2 + 9t_3 \geq 32.5346, \\ & \frac{1}{2}t_1 + \frac{1}{2}t_2 - m_1 = 0, \\ & \frac{1}{2}t_1 - \frac{1}{2}t_2 - m_2 = 0, \\ & t_1 \geq 1.141357, \\ & t_3 \geq 1.097738, \\ & t_1, t_2, t_3 \geq 0. \end{aligned}$$

The ranges of the variables t_1, t_2, t_3, m_1, m_2 (estimated from the constraints) are:

$$\begin{aligned} 1.141357 \leq t_1 \leq 4.06 \quad , \quad 0 \leq t_2 \leq 6.2 \quad , \quad 1.097738 \leq t_3 \leq 3.1 \quad , \quad 0.571 \leq m_1 \leq 5.13 \quad , \\ -2.53 \leq m_2 \leq 2.03. \end{aligned}$$

The slopes of the line segment corresponding to the separable functions are computed based on the equations, which are illustrated in problem (11.c). **Hence, the individual optimal solution to the (ULQCQP) problem of the FLDM is:**

$$(t_1, t_2, t_3) = (2.052032, 0, 1.306874) \quad , \quad \text{and} \quad Z_{31} = 114.1265.$$

1.3. The (LLQCQP) problem of the FLDM with the satisfied bounded variable constraints

$$\begin{aligned} \text{Max}_{x_1} \quad & Z_{11}(x) = (8x_1^2 + 3x_1x_2 + 7x_1 + x_2^2 + 2x_2 + 3x_3) \\ \text{where } & x_2, x_3 \text{ solves;} \end{aligned}$$

Subject to:

$$\begin{aligned} \text{(QCQP)} \quad & x_1 + 5x_2 + 2x_3^2 \leq 35, \\ & 2x_1 + 2x_2 + 6x_3 \leq 10, \\ & 5x_1^2 + 3x_1 + x_2^2 \leq 40, \\ & 8x_1^2 + 3x_1x_2 + 7x_1 + x_2^2 + 2x_2 + 3x_3 \leq 32.5346, \\ & x_1 \leq 1.141357, \\ & x_3 \leq 1.097738, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Repeat the same action as the (MLQCQP) and the (ULQCQP) problems by applying the separable programming method on the (LLQCQP) problem of the FLDM. The problem after converting the non-separable function x_1x_2 into separable form is as follows:

$$\begin{aligned} \text{Max}_{x_1} \quad & Z_{11}(x) = (8x_1^2 + 3m_1^2 - 3m_2^2 + 7x_1 + x_2^2 + 2x_2 + 3x_3) \\ \text{where } & x_2, x_3 \text{ solves;} \end{aligned}$$

Subject to:

$$\begin{aligned} \text{(QCQP)} \quad & x_1 + 5x_2 + 2x_3^2 \leq 35, \\ & 2x_1 + 2x_2 + 6x_3 \leq 10, \\ & 5x_1^2 + 3x_1 + x_2^2 \leq 40, \\ & 8x_1^2 + 3m_1^2 - 3m_2^2 + 7x_1 + x_2^2 + 2x_2 + 3x_3 \leq 32.5346, \\ & \frac{1}{2}x_1 + \frac{1}{2}x_2 - m_1 = 0, \\ & \frac{1}{2}x_1 - \frac{1}{2}x_2 - m_2 = 0, \\ & x_1 \leq 1.141357, \\ & x_2 \leq 0, \\ & x_3 \leq 1.097738, \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

The ranges of the variables x_1, x_2, x_3, m_1, m_2 (estimated from the constraints) are:

$$0 \leq x_1 \leq 1.141357 \quad , \quad x_2 \leq 0 \quad , \quad 0 \leq x_3 \leq 1.097738 \quad , \quad 0 \leq m_1 \leq 0.571 \quad , \quad 0 \leq m_2 \leq 0.571.$$

The slopes of the line segment corresponding to the separable functions are computed based on the equations, which are illustrated in problem (12.c). **Hence, the individual optimal solution to the (LLQCQP) problem of the FLDM is:**

$$(x_1, x_2, x_3) = (1.141357, 0, 1.097738) \quad , \quad \text{and} \quad Z_{11} = 21.7043.$$

Therefore, the optimal solution of the FLDM is obtained as:

$$\tilde{x}_1^F = (1.141357, 1.141357, 2.052032) ,$$

$$\tilde{x}_2^F = (0, 0, 0) \quad , \quad \tilde{x}_3^F = (1.097738, 1.097738, 1.306874).$$

2. [SLDM]

The SLDM defines his/her problem in view of the FLDM by setting $(\tilde{x}_1^F) = (1.141357, 1.141357, 2.052032)$ to the SLDM constraints. Assume $\tilde{x}_1 = (x_1, y_1, t_1)$, $\tilde{x}_2 = (x_2, y_2, t_2)$, $\tilde{x}_3 = (x_3, y_3, t_3)$, $\tilde{F}_2 = (Z_{12}, Z_{22}, Z_{32})$. Therefore, the Fully Fuzzy (QCQP) problem of the SLDM is rewritten as follows:

$$\text{Max}_{x_2, y_2, t_2} (Z_{12}, Z_{22}, Z_{32})(x, y, t) = ((4,5,8) \otimes (x_1, y_1, t_1) \oplus (12,16,19) \otimes (x_2, y_2, t_2) \oplus (7,9,11) \otimes (x_3, y_3, t_3) \oplus$$

$$((x_2, y_2, t_2), (x_3, y_3, t_3)) \otimes \begin{pmatrix} (8,9,11) & (0,4,5) \\ (0,2,3) & (0,0,0) \end{pmatrix} \otimes \begin{pmatrix} (x_2, y_2, t_2) \\ (x_3, y_3, t_3) \end{pmatrix}),$$

where x_3, y_3, t_3 solves;

Subject to:

$$\begin{aligned} \text{(QCQP)} \quad & (1,3,6) \otimes (x_1, y_1, t_1) \oplus (5,7,10) \otimes (x_2, y_2, t_2) \oplus (2,5,9) \otimes (x_3, y_3, t_3) \otimes (x_3, y_3, t_3) \leq (35,60,85), \\ & (2,10,12) \otimes (x_1, y_1, t_1) \oplus (2,3,5) \otimes (x_2, y_2, t_2) \oplus (6,6,11) \otimes (x_3, y_3, t_3) \leq (10,18,39), \\ & (3,4,7) \otimes (x_1, y_1, t_1) \oplus ((x_1, y_1, t_1), (x_2, y_2, t_2)) \otimes \begin{pmatrix} (5,6,7) & (0,2,4) \\ (0,1,2) & (1,3,3) \end{pmatrix} \otimes \begin{pmatrix} (x_1, y_1, t_1) \\ (x_2, y_2, t_2) \end{pmatrix} \\ & \leq (40,70,115), \\ & (x_1, y_1, t_1) = (1.141357, 1.141357, 2.052032), \\ & x_1, x_2, x_3, y_1, y_2, y_3, t_1, t_2, t_3 \geq 0. \end{aligned}$$

The Fully Fuzzy (QCQP) problem of the SLDM reduces to the following three crisp (QCQP) problems, called (LLQCQP), (MLQCQP) and (ULQCQP) problems of the SLDM, as follows:

$$\text{(LLQCQP)} \quad \text{Max}_{x_2} Z_{12}(x) = (8x_2^2 + 12x_2 + 7x_3 + 4.565428)$$

$$\text{(MLQCQP)} \quad \text{Max}_{y_2} Z_{22}(y) = (9y_2^2 + 6y_2y_3 + 16y_2 + 9y_3 + 5.706785)$$

$$\text{(ULQCQP)} \quad \text{Max}_{t_2} Z_{32}(t) = (11t_2^2 + 8t_2t_3 + 19t_2 + 11t_3 + 16.416256)$$

where x_3, y_3, t_3 solves;

Subject to:

$$\begin{aligned} \text{(QCQP)} \quad & 5x_2 + 2x_3^2 \leq 33.85643, \\ & 2x_2 + 6x_3 \leq 7.717286, \\ & x_2^2 \leq 30.06245, \\ & 7y_2 + 5y_3^2 \leq 56.575929, \\ & 3y_2 + 6y_3 \leq 6.58643, \\ & 3y_2^2 + 3.424071 y_2 \leq 57.6184, \\ & 10t_2 + 9t_3^2 \leq 72.687808, \\ & 5t_2 + 11t_3 \leq 14.375616, \\ & 3t_2^2 + 12.312192t_2 \leq 71.1599287, \\ & x_2, x_3, y_2, y_3, t_2, t_3 \geq 0. \end{aligned}$$

The SLDM repeats the same procedure as the FLDM by applying the separable programming method on the (LLQCQP), (MLQCQP) and (ULQCQP) problems of the SLDM with the satisfied bounded variable constraints till obtaining the optimal solution of the SLDM, as follows:

$$\tilde{x}_2^S = (2.1954759, 2.1954764, 2.8751236) \quad , \quad \tilde{x}_3^S = (0, 0, 0).$$

3. [TLDM]

The TLDM defines his/her problem in view of the FLDM and the SLDM by setting $(\tilde{x}_1^F) = (1.141357, 1.141357, 2.052032)$, $(\tilde{x}_2^S) = (2.1954759, 2.1954764, 2.8751236)$ to the TLDM constraints. Assume $\tilde{x}_1 = (x_1, y_1, t_1)$, $\tilde{x}_2 = (x_2, y_2, t_2)$, $\tilde{x}_3 = (x_3, y_3, t_3)$, $\tilde{F}_3 = (Z_{13}, Z_{23}, Z_{33})$, as follows:

$$\text{Max}_{x_3, y_3, t_3} (Z_{13}, Z_{23}, Z_{33})(x, y, t) = ((7,8,14) \otimes (x_1, y_1, t_1) \oplus (4,9,9) \otimes (x_2, y_2, t_2) \oplus (13,18,20) \otimes (x_3, y_3, t_3) \oplus$$

$$((x_1, y_1, t_1), (x_3, y_3, t_3)) \otimes \begin{pmatrix} (1,2,6) & (0,1,3) \\ (3,4,4) & (9,10,14) \end{pmatrix} \otimes \begin{pmatrix} (x_1, y_1, t_1) \\ (x_3, y_3, t_3) \end{pmatrix}),$$

Subject to:

$$(\text{QCQP}) \quad (1,3,6) \otimes (x_1, y_1, t_1) \oplus (5,7,10) \otimes (x_2, y_2, t_2) \oplus (2,5,9) \otimes (x_3, y_3, t_3) \otimes (x_3, y_3, t_3) \leq (35,60,85),$$

$$(2,10,12) \otimes (x_1, y_1, t_1) \oplus (2,3,5) \otimes (x_2, y_2, t_2) \oplus (6,6,11) \otimes (x_3, y_3, t_3) \leq (10,18,39),$$

$$(3,4,7) \otimes (x_1, y_1, t_1) \oplus ((x_1, y_1, t_1), (x_2, y_2, t_2)) \otimes \begin{pmatrix} (5,6,7) & (0,2,4) \\ (0,1,2) & (1,3,3) \end{pmatrix} \otimes \begin{pmatrix} (x_1, y_1, t_1) \\ (x_2, y_2, t_2) \end{pmatrix} \\ \leq (40,70,115),$$

$$(x_1, y_1, t_1) = (1.141357, 1.141357, 2.052032),$$

$$(x_2, y_2, t_2) = (2.1954759, 2.1954764, 2.8751236),$$

$$x_1, x_2, x_3, y_1, y_2, y_3, t_1, t_2, t_3 \geq 0.$$

The Fully Fuzzy (QCQP) problem of the TLDM reduces to the following three crisp (QCQP) problems, called (LLQCQP), (MLQCQP) and (ULQCQP) problems of the TLDM, as follows:

$$(\text{LLQCQP}) \quad \text{Max}_{x_3} Z_{13}(x) = (9x_3^2 + 16.424071 x_3 + 18.0741)$$

$$(\text{MLQCQP}) \quad \text{Max}_{y_3} Z_{23}(y) = (10y_3^2 + 23.706785 y_3 + 31.4955)$$

$$(\text{ULQCQP}) \quad \text{Max}_{t_3} Z_{33}(t) = (14t_3^2 + 34.364224 t_3 + 79.86956)$$

Subject to:

$$(\text{QCQP}) \quad \begin{aligned} 2x_3^2 &\leq 22.88126, \\ 6x_3 &\leq 3.32633, \\ 5y_3^2 &\leq 41.2076, \\ 6y_3 &\leq 0.0001, \\ 9t_3^2 &\leq 43.93657, \\ 11t_3 &\leq 1, \\ x_3, y_3, t_3 &\geq 0. \end{aligned}$$

The TLDM repeats the same procedure as the FLDM and the SLDM by applying the separable programming method on the (LLQCQP), (MLQCQP) and (ULQCQP) problems of the TLDM with the satisfied bounded variable constraints till obtaining the optimal solution of the TLDM, as follows:

$$\tilde{x}_3^T = (0.00001666667, 0.00001666667, 0.09090909).$$

Finally, the fuzzy optimal solution for the given (FFMLQCQP) problem is obtained as follows:

$$\tilde{x}_1^F = (1.141357, 1.141357, 2.052032),$$

$$\tilde{x}_2^S = (2.1954759, 2.1954764, 2.8751236),$$

$$\tilde{x}_3^T = (0.00001666667, 0.00001666667, 0.09090909).$$

Where:

$$\begin{aligned}\tilde{F}_1 &= (35.139647, 63.0088983, 212.4974032) , \\ \tilde{F}_2 &= (69.472171, 84.215827, 165.0642962) , \\ \tilde{F}_3 &= (18.07437, 31.49589, 83.10928).\end{aligned}$$

7. Conclusions and Future Work

An effort has been made in this paper for finding the fuzzy optimal solution of (FFMLQCQP) problem in which the objective function and the constraints were quadratic, and all the coefficients and decision variables were fuzzy numbers. To solve this problem, the (FFMLQCQP) problem was decomposed into series of crisp (QCQP) issues with bounded variable constraints. For each (QCQP) problem, the problem was solved by utilising the separable programming method, which replaced the separable quadratic functions with linear functions. At last, the fuzzy optimal solution to the given (FFMLQCQP) problem was obtained. The proposed model combined the techniques of bound and decomposition and separable programming without ignoring any part of solution area. Thus the proposed model is instrumental in the real world problems where the product is uncertain.

This paper is an attempt to establish necessary results which hopefully will help others to answer some of the inquiries in the area of (FFMLQCQP) problem. The future scope of the mentioned idea is that this model can be extended to entirely fuzzy multi-objective multi-level quadratic or fractional programming. Further, the model can also be extended to entirely fuzzy entirely rough multi-level quadratic or fractional programming. Application of the proposed method to the real world decision making situations will be required shortly.

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