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On the geometry of Complex-Contact Riemannian Submersion

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Abstract

In this paper we give the structure equations of a complex-contact Riemannian submersion and show that the fibers of this submersion is a contact metric submanifold of the total space. We investigate the distributions and obtain a necessary and sufficient conditions for such submersion to be a totally geodesic map.

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1 Introduction

Immersion and submersion are special tools in differential geometry, they play important roles in Riemannian geometry and in other aspects of the differential geometry. The Riemannian submersions are of great interest both in mathematics and physics, owing to their applications in the Yang-Mills theory, Kaluza-Klein theory, supergravity and superstring theories.

The theory of Riemannian submersion was initiated by O'Neill (1966) and Gray Gray (1967). In Watson (1976), the Riemannian submersions were considered between almost Hermitian manifold by Watson under the name almost Hermitian submersion in which the Riemannian submersion is an almost complex mapping and as a result the vertical and horizontal distributions are invariant with respect to the almost complex structure of the total space of the submersion. In fact almost Hermitian submersion have been actively studied between different kinds of subclasses of almost Hermitian

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manifolds by Maria et al (2004). This has also been extended to different kinds of subclasses of almost contact manifolds. See Chinea (1985) for details. Most of the studies related to Riemannian or almost Hermitian submanifolds can be found in Maria et al (2004) and Shahid et al (2013). Sahin constructed Gauss-Weingarten like formulas and introduced the O'Neill's type tensor fields for Riemannian maps and obtained new conditions for Riemannian maps to be totally geodesic in terms of these notion. He introduced semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds as a generalization of holomorphic submersions and ant-invariant submersions and studied the geometry of such maps. Furthermore, he introduced the notion of slant submersion from almost Hermitian manifold onto an arbitrary Riemannian manifold. See Sahin (2012) and Sahin (2011). Almost contact metric submersions constitute a class of Riemannian submersions whose total space is an almost contact metric manifold. In fact, different kinds of Riemannian submersions between manifolds endowed with different structures have been studied. Examples are the following; Contact-Complex submersions, quaternionic submersions, geometry of contact and cosymplectic manifold, Semi-Riemannian submersion and Lorentzian submersion. See Blair (2002), Maria et al (2004), Ianus et al (2011) and Ianus et al (2008) for details.

From all these studies by these different authors, we surveyed the Riemannian submersion in which the total space is an almost Hermitian manifold and the base space is a contact metric manifold, which is called Complex-Contact Riemannian submersion. We recall and studied the geometric properties of this type of submersion. This forms the bases of our work in this paper.

We organize the paper in the following format; After the introduction in the first section, we recall the definitions of almost Hermitian and almost contact metric manifolds and some of their associated notions and identities. We also recall the definition and fundamental equations of Riemannian submersion, where we introduce the O'Neill or fundamental tensors and their important properties in the second section.

In the third section we introduce the complex-contact Riemannian submersion which is another form of semi-invariant submersion with some of its important properties. In this section we show that the fibers of this submersion is a contact metric submanifold.

In the last section, we investigate the geodesicity of the vertical and the horizontal distributions and obtain a necessary and sufficient conditions for the distributions to be totally geodesic.

2 Preliminaries

In this section we shall recall several notions and results that will be needed throughout the paper. As usual, all manifolds and maps involved are assumed to be of class C^∞. Furthermore, we denote by $\Gamma(TM)$, the space of the sections of the tangent bundle of a differentiable manifold M.

An almost complex manifold is a manifold M with an almost complex structure. The almost complex structure is a tensor field J of type $(1,1)$ such that $J^2 = -I$, where I stands for the identity endomorphism of the tangent bundle $\Gamma(TM)$. Such a manifold is orientable and has even dimension, say $2m$, m is an integer.

An almost Hermitian manifold (M, J, g) is an almost complex manifold (M, J) with a $J-$invariant Riemannian metric g. The $J-$invariance of g means that

$$g(X, Y) = g(JX, JY), \tag{2.1}$$

for any sections $X, Y \in \Gamma(TM)$. The Kahler form of the almost Hermitian manifold (M, J, g) is the differential form of bidegree $(1,1)$ defined by $\Omega(X, Y) = g(X, JY)$ for any vectors $X, Y \in \Gamma(TM)$. For more details on almost Hermitian manifolds, See Maria et al (2004).

We denote by ∇ the Levi-Civita connection on M, we have the following formulas for the covariant derivative, the exterior differentiation and the codifferential of Ω,

$$\begin{aligned}(\nabla_X\Omega)(Y,Z) &= g(Y,(\nabla_X J)Z)\\ 3d\Omega(X,Y,Z) &= \sigma(\nabla_X\Omega)(Y,Z)\\ (\nabla_X\Omega)(Y,Z) &= (\nabla_X\Omega)(JY,JZ)\\ (\delta\Omega)(X) &= \sum_{i=1}^{m}((\nabla_{E_i}\Omega)(E_i,X)+(\nabla_{JE_i}\Omega)(JE_i,X)). \end{aligned} \tag{2.2}$$

Where σ denote the cyclic sum over X, Y, Z and $\{E_1, ..., E_m, JE_1, ..., JE_m\}$ is a local orthonormal J-frame. See Tshikunguila (2013).

Let M be a differentiable manifold of odd dimension $(2m+1)$. An almost contact structure on M is a triple (φ, ξ, η) where ξ is a characteristic vector field, η is a 1-form such that $\eta(\xi) = 1$, and φ is a tensor field of type $(1,1)$ satisfying $\varphi^2 = -I + \eta \oplus \xi$

where I is the identity transformation. If M is equipped with a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

then (g, φ, ξ, η) is called an almost contact metric structure. Therefore, the quintuple $(M^{2m+1}, g, \varphi, \xi, \eta)$ is an almost contact metric manifold. As in the case of almost Hermitian manifolds, any almost contact metric manifold admits a fundamental 2-form, Φ defined by $\Phi(Z,X) = g(Z, \varphi X)$. Further more, $\eta(Z) = g(Z, \xi)$, for any $X, Z \in \Gamma(TM)$. SeeBlair (2002) for details.

As in the case of almost Hermitian manifolds, we recall some of the important identities.

$$\begin{aligned}(\nabla_D\Phi)(E,G) = g(E,(\nabla_D\varphi)G) = -g((\nabla_D\varphi)E,G)\\ (\nabla_D\eta)E = g(E,\nabla_D\xi) = (\nabla_D\Phi)(\xi,\varphi E). \end{aligned} \tag{2.4}$$

For the almost Hermitian manifold, we have the following example;

Example 2.1. *Consider the even dimensional manifold $\Re^6$. Such that $\Re^6 = \{(x_i, y_i)\}$, $x, y \in \Re$, $i = 1,2,3$, $g = \sum_{i=1}^{3}[(dx_i)^2 + (dy_i)^2]$. If we define $J\frac{\partial}{\partial x^i} = \frac{\partial}{\partial y_i}$ and $J\frac{\partial}{\partial y^i} = -\frac{\partial}{\partial x^i}$ for $i = 1,2,3$. Then $(\Re^6, g, J)$ is an almost Hermitian manifold.*

The following is an example of an almost contact metric manifold;

Example 2.2. *Consider the odd dimensional manifold $\Re^5$ with cartesian coordinates $(x_1, x_2, x_3, x_4, x_5)$, Riemannian metric g, metric g, φ is a type $(1,1)$ tensor field, η is a one form and ξ is the characteristic vector field. If we define $\eta = dx_5$, $\xi = \frac{\partial}{\partial x_5}$ and $\varphi\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2}$, $\varphi\frac{\partial}{\partial x_2} = -\frac{\partial}{\partial x_1}$, $\varphi\frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4}$, $\varphi\frac{\partial}{\partial x_4} = -\frac{\partial}{\partial x_3}$, $\varphi\frac{\partial}{\partial x_5} = 0$, then from a straight forward calculation, $(\Re^5, g, \varphi, \eta, \xi)$ is an almost contact metric manifold.*

Definition 2.1. Tshikunguila (2013). An almost contact metric manifold $(M, g, \varphi, \xi, \eta)$ is a;

(i) Kenmotsu (KN-structure) if and only if $(\nabla_X\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$.

(ii) nearly-k-cosymplectic if $(\nabla_X\varphi)X = 0$ and $(\nabla_X\xi = 0$.

(ii) nearly-k-Sasakian if $(\nabla_X\varphi)X = g(X,X)\xi - \eta(X)X$ and $(\nabla_X\xi = -\varphi X$,

for any vector fields $X, Y \in M$.

We shall now give definition of Riemannian submersion;

Definition 2.2. O'Neill (1966) Let (M, g_m) and (N, g_n) be Riemannian manifolds. A Riemannian submersion $\pi : M \to N$ is a mapping of (M, g_m) onto $(N, g_n$ satisfying the following;

(i) Each derivative map π_* of π is onto.

(ii) π_* preserves the lengths of the horizontal vectors,

where π_* is the differential map of π.

Let $\pi : (M, g_m) \to (N, g_n)$ be a Riemannian submersion, then for each $b \in N$, $\pi^{-1}(b)$ is a submanifold of M of $dimM - dimN$. The submanifolds $\pi^{-1}(b)$ are called fibers, and a vector field on M is called a vertical vector if it is always tangent to the fibers, and the vector field is called a horizontal vector if it is always orthogonal to the fibers. Throughout this paper we shall denote the projections of the tangent spaces of M onto the subspaces of vertical and horizontal vectors by $\mathcal{V}$, $\mathcal{H}$ respectively.

A Riemannian submersion $\pi : (M, g_m) \to (N, g_n)$ determines $(1, 2)-$tensor fields T and A on M . As in O'Neill (1966) they are called the fundamental tensors defined for arbitrary vector fields $E, F \in \Gamma(TM)$ by

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F. \tag{2.5}$$

$$A_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F. \tag{2.6}$$

where ∇ is the covariant derivative of M with the following properties;

(i) At each point, T_E is a skew symmetric linear operator on the tangent space of M,

(ii) T is vertical , that is , $T_E = T_{\mathcal{V}E}$

(iii) For vertical vector fields V and W, T has the symmetric property $T_V W = T_W V$.

(iv) At each point, A_E is a skew symmetric linear operator on the tangent space of M.

(v) A is horizontal, that is , $A_E = A_{\mathcal{H}E}$.

(vi) For horizontal vector fields X and Y, A has the alternation property $A_X Y = -A_Y X$.

If a horizontal vector field (a section in the horizontal distribution) is π related (projectable) to a vector field on the base manifold N , one calls it basic vector. It follows from the definition of the Riemannian submersion that

$$g_m(X, Y) = g_n(X_* Y_*) \circ \pi \tag{2.7}$$

if X and Y are basic vector fields, $\pi-$related to $X_*, Y_* \in \Gamma(TN)$, respectively.That is every vector field X_* on N has a unique horizontal lift X to M and X is basic. Thus $X \rightleftharpoons X_*$ is a one to one correspondence between the horizontal vectors on M and vector fields on N. This correspondence preserves brackets, inner products and covariant derivatives. Let ∇ and ∇^* be the Levi-Civita connections of the Riemannian metrics on M and N respectively, then we have the following O'Neill (1966)

(i) $g_m(X, Y) = g_n(X_*, Y_*) \circ \pi$.

(ii) $\mathcal{H}[X, Y] = [X_*, Y_*] \circ \pi$.

(iii) $\mathcal{H}\nabla_X Y = \nabla^*_{X_*} Y_* \circ \pi$.

We want to summarize the relation of T and A to the various covariant derivatives as follows

Proposition 2.1. *O'Neill (1966).If X and Y are horizontal vector fields,and V and W are vertical vector fields. Then*

(i) $\nabla_V W = T_V W + \mathcal{V}\nabla_V W$.

(ii) $\nabla_V X = T_V X + \mathcal{H}\nabla_V X$.

(iii) $\nabla_X V = A_X V + \mathcal{V}\nabla_X V$.

(iv) $\nabla_X Y = A_X Y + \mathcal{H}\nabla_X Y$.

If X is a basic vector field , then $\mathcal{H}\nabla_V X = A_X V$.

We know that T_E and A_E are, at each point skew-symmetric linear operators, it follows that $(\nabla_F T)_E$ and $(\nabla_F A)_E$ are also skew-symmetric and linear.

Here we recall the notion of the second fundamental form of a map between Riemannian manifolds. Let (M, g_m) and (N, g_n) be Riemannian manifolds and that $\pi : (M, g_m) \longrightarrow (N, g_n)$ is a smooth map between them. Then the differential π_* can be viewed as a section of the bundle $Hom(TM, \pi^{-1}TN) \longrightarrow M$, where $\pi^{-1}TN$ is the pullback bundle which has fibers $(\pi^{-1}TN)_u = T_{\pi(u)}N$, $u \in M$. The $Hom(TM, \pi_1 TN)$ has a connection ∇' induced from the Levi-Civita connection ∇ of M and the pullback connection. We define the second fundamental form of π by the equation

$$(\nabla'\pi_*)(X, Y) = \nabla^{\pi}_X \pi_* Y - \pi_*(\nabla_X Y) \tag{2.8}$$

for $X, Y \in \Gamma(TM)$, where ∇^{π} is the pullback connection. It is known that the second fundamental form is symmetric. See Blair (2002)

3 Complex-Contact Riemannian Submersion

In this section we attempt to consider a Riemannian submersion π from an almost Hermitian manifold $(M, g, J,)$ to an almost contact metric manifold $(M', g', \xi', \varphi', \eta')$ such that the projection map commutes with the fundamental tensors. We refer to the work done by for some details Chinea (1988).

Let $\pi : (M, g, J,) \to (M', g', \xi', \varphi', \eta')$ be a Riemannian submersion , such that $\pi_* \circ J = \varphi \circ \pi_*$. Let ξ denote the horizontal lift of ξ'. Then we have that

$\pi_* J\xi = \varphi\pi_*\xi = \varphi\xi' = 0$ and $-\xi' = \pi_*(J^2\xi) = \varphi(\pi_* J\xi) = 0$.

This contradict our assumption . This prompted the idea of Blair, B. Sahin on the concept of semi-invariance Riemannian submersion. (See Blair et al (1976), Sahin (2012) for more details). We now have the following definitions.

Definition 3.1. Maria et al (2004) Let (M, g, J) and $(M', g', \xi', \varphi, \eta')$ be an almost-Hermitian and an almost-Contact metric manifolds respectively. A Riemannian submersion $\pi_* : M \to M'$ is called a complex-contact Riemannian submersion if there exists a $1 - form\ \eta$ on M such that $\pi_* \circ J = \varphi \circ \pi_* - \eta \otimes \xi'$.

Here is an example of an almost complex-contact;

Example 3.1. *Let M^4 be an almost Hermitian manifold. Then the projection*

$$\pi : M^4 \times \Re^{2n+1} \times \Re \longrightarrow M^4 \times \Re$$

is a complex-contact Riemannian submersion from an almost Hermitian manifold to an almost contact metric manifold. Moreover, $M^4 \times \Re^{2n+1} \times \Re$ has even dimension and $M^4 \times \Re$ has odd dimension and we take $n \geqslant 2$.

Since the dimension of the fiber is $dimM - dimM'$ implies that the dimension of the fiber in this type of submersion is odd. Let ξ be the horizontal lift of ξ', then we can see that

$$\pi_*(J\xi) = (\pi_* J)\xi) = \varphi \circ \pi_*\xi - \eta(\xi)\xi',$$

and

$$\varphi \circ \pi_*\xi - \eta(\xi)\xi' = 0 - g(\xi, J\xi)\xi' = 0$$

if $J\xi$ is vertical vector field and η is its dual $1 - form$. The following proposition was proved;

Proposition 3.1. *Maria et al (2004)Let $\pi : M \to M'$ be a complex-contact Riemannian submersion. Then the following hold;*

(i) $J(\mathcal{V}(TM)) \subset \mathcal{V}(TM) \oplus span\{\xi\}$.

(ii) $J(\mathcal{H}(TM)) \subset \mathcal{H}(TM) \oplus span\{J\xi\}$.

(iii) *If* X *is the basic vector field on* M*,* π *related to the vector field* X' *on* M' *, then* $JX - g(X,\xi)J\xi$ *is the basic vector field* π *related to* $\varphi' X'$*.*

(iv) *For any vertical vector field* V *we write* $JV = \hat{\varphi}V - \hat{\eta}(V)\xi$*, where* $\hat{\varphi}$ *is an endomorphism on the vertical distribution and* $\hat{\eta}$ *is a* $1-form$ *restricted to the vertical vectors* $\mathcal{V}(TM)$*.*

Theorem 3.2. *Let* $\pi : M \to M'$ *be a complex-contact Riemannian submersion. Then the fiber* $(\hat{M}, \hat{\varphi}, \hat{\eta}, J\xi)$ *is a contact metric sub manifold of* M *with the restriction of* g *as an associated metric.*

Proof. Since we have that $JV = \hat{\varphi}V - \hat{\eta}(V)\xi$, means that $\hat{\varphi}V = JV + \hat{\eta}(V)\xi$ for any $V \in \mathcal{V}TM$, and so we have the following;

(i) $\hat{\varphi}(J\xi) = J(J\xi) + \hat{\eta}(J\xi)\xi = -\xi + g(J\xi, J\xi)\xi = 0.$

(ii) $\hat{\eta}(J\xi) = g(J\xi, J\xi) = 1.$

(iii) $\hat{\varphi}^2(V) = \hat{\varphi}(\hat{\varphi}(V)) = \hat{\varphi}(JV) + \hat{\eta}(V)\hat{\varphi}(\xi) = J(JV) + \hat{\eta}(JV)\xi + \hat{\eta}(V)(J\xi + \hat{\eta}(\xi)\xi) = -V + \hat{\eta}(V)J\xi.$

(iv) $g(\hat{\varphi}U, \hat{\varphi}V) = g(JU + \hat{\eta}(U)\xi, JV + \hat{\eta}(V)\xi) = g(U,V) + g(JU, \hat{\eta}(V)\xi) + g(\hat{\eta}(U)\xi, JV) + g(\hat{\eta}(U)\xi, \hat{\eta}(V)\xi) = g(U,V) - \hat{\eta}(U)\hat{\eta}(V),$

this completes the proof. □

Furthermore, $\mathcal{V}TM$ is invariant with respect to $\hat{\varphi}$.

Theorem 3.3. *Maria et al (2004) Let* $\pi : (M, J, g) \to (M', g', \xi', \varphi', \eta')$ *be a complex-contact Riemannian submersion. Then the following hold;*

(i) $(\nabla'_{X'}\Phi)(Y', Z') \circ \pi = g(Y, (\nabla_X J)Z) - g(Z,\xi)g(Y, \nabla_X J\xi).$

(ii) $(\nabla_{J\xi}\Omega)(X,Y) = g(X, \nabla_{JY}J\xi) - g(Y,\xi)g(X, \nabla_{J\xi}J\xi) + g(JX, \nabla_Y J\xi) + g(X,\xi)g(\nabla_{J\xi}J\xi, Y).$

(iii) $(\nabla_{J\xi}\Omega)(J\xi, X) = -g(\nabla_{J\xi}J\xi, JX) + g(\nabla_\xi J\xi, X),$

where ξ *is the horizontal lift of* ξ'*,* X, Y, Z *are basic vector fields,* $\pi-$*related to* X', Y', Z' *respectively.*

Proof.

$$\begin{aligned}
&(\nabla'_{X'}\Phi)(Y', Z') \circ \pi \\
&= g'(Y', (\nabla'_{X'}\varphi')Z') \circ \pi \\
&= g'(Y', \nabla'_{X'}(\varphi' Z')) + g'(\varphi' Y', (\nabla'_{X'}Z')) \circ \pi.
\end{aligned} \tag{3.1}$$

From proposition 3.1, one has

$$\begin{aligned}
&= g(Y, \nabla_X(JZ - g(Z,\xi)J\xi)) + g((JY - g(Y,\xi)J\xi), \nabla_X Z) \\
&= g(Y, (\nabla_X JZ) - g(Y, g(Z,\xi)\nabla_X J\xi) + g(JY, \nabla_X Z) - g(g(Y,\xi)J\xi, \nabla_X Z)),
\end{aligned} \tag{3.2}$$

so from equations 2.2, one has

$$\begin{aligned}
&g(Y, (\nabla_X JZ) - g(Y, g(Z,\xi)\nabla_X J\xi) + g(JY, \nabla_X Z) - g(g(Y,\xi)J\xi, \nabla_X Z)) \\
&= (\nabla_X\Omega)(Y,Z) - g(Z,\xi)g(Y, \nabla_X J\xi) + g(Y,\xi)g(\nabla_X J\xi, Z).
\end{aligned} \tag{3.3}$$

Similarly,

$$\begin{aligned}
&(\nabla_{J\xi}\Omega)(X,Y) = g(X,(\nabla_{J\xi}J)Y) = g(X,\nabla_{J\xi}JY) - g(X,J\nabla_{J\xi}Y)\\
&= g(X,\nabla_{J\xi}JY) - g(X,\nabla_{J\xi}g(Y,\xi)J\xi) + g(JX,\nabla_{J\xi}Y)\\
&= g(X,\nabla_{J\xi}JY) - g(Y,\xi)g(X,\nabla_{J\xi}J\xi) + g(JX,\nabla_{J\xi}Y) - g(g(X,\xi)J\xi,\nabla_{J\xi}Y)\\
&= g(X,\nabla_{JY}J\xi) + g(JX,\nabla_{Y}J\xi) + g(X,\xi)g(\nabla_{J\xi}J\xi,Y). \qquad (3.4)
\end{aligned}$$

also

$$\begin{aligned}
&(\nabla_{J\xi}\Omega)(J\xi,X) = g(J\xi,(\nabla_{J\xi}J)X)\\
&= g(J\xi,\nabla_{J\xi}JX) - g(J\xi,J\nabla_{J\xi}X)\\
&= g(J\xi,\nabla_{J\xi}JX) - g(\xi,\nabla_{J\xi}X)\\
&= -g(\nabla_{J\xi}J\xi,JX) + g(\nabla_{\xi}J\xi,X), \qquad (3.5)
\end{aligned}$$

where X, Y, Z are basic vector fields on M π related to the vector field X', Y', Z' respectively on M'. □

Theorem 3.4. *Let π be a complex-contact Riemnnian submersion from an almost Hermitian manifold (M, g, J) to an almost contact metric manifold (M', g', ξ', η'), then*

$$(\nabla_X J)Y - g(Y,\nabla_X\xi)J\xi - g(Y,\xi)\nabla_X J\xi = g(JX,Y)\xi - g(Y,\xi)(JX - g(X,\xi)J\xi).$$

if (M', g', ξ', η') is a Kenmotsu manifold, where X, Y are basic vector fields on M π related to the vector fields X', Y', respectively on M' and ξ is the horizontal lift of ξ'.

Proof.

$$\begin{aligned}
&(\nabla'_{X'}\varphi')Y' = \nabla'_{X'}\varphi'Y' - \varphi'\nabla'_{X'}Y'\\
&= \nabla_X JY - \nabla_X g(Y,\xi)J\xi - J\nabla_X Y + g(\nabla_X Y,\xi)J\xi\\
&= (\nabla_X J)Y - Xg(Y,\xi)J\xi - g(Y,\xi)\nabla_X J\xi + g(\nabla_X Y,\xi)J\xi\\
&= (\nabla_X J)Y - g(\nabla_X Y,\xi)J\xi - g(Y,\nabla_X\xi)J\xi - g(Y,\xi)\nabla_X J\xi + g(\nabla_X Y,\xi)J\xi\\
&= (\nabla_X J)Y - g(Y,\nabla_X\xi)J\xi - g(Y,\xi)\nabla_X J\xi. \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}
&g'(\varphi'X',Y')\xi' - \eta'(Y')\varphi'X' = g'(\varphi'X',Y')\xi' - g'(Y',\xi')\varphi'X'\\
&= g(JX - g(X,\xi)J\xi,Y)\xi - g(Y,\xi)(JX - g(X,\xi)J\xi)\\
&= g(JX,Y)\xi - g(X,\xi)g(J\xi,Y)\xi - g(Y,\xi)JX + g(Y,\xi)g(X,\xi)J\xi\\
&= g(JX,Y)\xi - g(Y,\xi)JX + g(Y,\xi)g(X,\xi)J\xi. \qquad (3.7)
\end{aligned}$$

From (2.1)

$$(\nabla_X J)Y - g(Y,\nabla_X\xi)J\xi - g(Y,\xi)\nabla_X J\xi = g(JX,Y)\xi - g(Y,\xi)(JX - g(X,\xi)J\xi),$$

the result is complete. □

Corollary 3.5. *Let π be a complex-contact Riemnnian submersion from an almost Hermitian manifold (M, g, J) to an almost contact metric manifold (M', g', ξ', η'), then*

$$(\nabla_X J)X = g(X,\xi)\nabla_X J\xi$$

if (M', g', ξ', η') is a nearly-K-cosymplectic manifold, where X, Y are basic vector fields on M π related to the vector fields X', Y', respectively on M' and ξ is the horizontal lift of ξ'.

Proof. From (2.1), the definition of Kenmotsu manifold, and from the proof of theorem (3.4) we replace Y with X and obtain

$$(\nabla_X J)X - g(X,\xi)\nabla_X J\xi = o$$

and the result follows. □

Theorem 3.6. *Let π be a complex-contact Riemnnian submersion from an almost Hermitian manifold (M, g, J) to an almost contact metric manifold (M', g', ξ', η'), then*

$$(\nabla_X J)X - g(X,\xi)\nabla_X J\xi = g(X,X)\xi - g(X,\xi)X$$

and

$$\nabla_X \xi = JX - g(X,\xi)J\xi$$

if (M', g', ξ', η') is a nearly-K-Sasakian manifold, where X is a basic vector fields on M π related to the vector fields X' on M' and ξ is the horizontal lift of ξ'.

Proof. From definition (2.1) and corollary (3.5)

$$(\nabla'_{X'}\varphi')X' = (\nabla_X J)X - g(X,\xi)\nabla_X J\xi. \tag{3.8}$$

$$\begin{aligned} &g'(X',X')\xi' - \eta'(X')X' = g'(X',X')\xi' - g'(X',\xi')X' \\ &= g(X,X)\xi - g(X,\xi)X, \end{aligned} \tag{3.9}$$

this implies

$$(\nabla_X J)X - g(X,\xi)\nabla_X J\xi = g(X,X)\xi - g(X,\xi)X$$

and

$$\begin{aligned} &\nabla_X \xi = \nabla'_{X'}\xi' = -\varphi X' \\ &= JX - g(X,\xi)J\xi. \end{aligned} \tag{3.10}$$

This completes the proof. □

4 geodesic conditions of the distributions.

Let π be a complex-contact Riemnnian submersion from a Kahler manifold (M, g, J) onto an almost contact metric manifold (M', g', ξ', η'). Then we know from proposition (3.1), that $JV \subset \mathcal{V}TM \oplus span\{\xi\}$, $JX \subset \mathcal{H}TM \oplus span\{J\xi\}$ for any $V \in \mathcal{V}TM$ and $X \in\in \mathcal{H}TM$. Thus, we shall henceforth write

$$\begin{aligned} &JV = \phi V + \omega V \\ &JX = BX + CX. \end{aligned} \tag{4.1}$$

where $\phi V \in \mathcal{V}TM$, $\omega V \in span\{\xi\}$, $BX \in \mathcal{H}TM$ and $CX \in span\{J\xi\}$.

Definition 4.1. Maria et al (2004). An almost Hermitian manifold M is called Kahler manifold if $(\nabla_X J)Y = 0$, for all $X, Y \in \Gamma(TM)$, where ∇ is the Levi-Civita connection in M.

We recall that a differentiable map π between Riemannian manifolds (M, g_m) and (N, g_n) is called a totally geodesic map if $(\nabla\pi_*)(X,Y) = 0$ for all $X, Y \in \Gamma(TM)$.

Proposition 4.1. *Let π be a complex-contact Riemnnian submersion from a Kahler manifold (M, g, J) onto an almost contact metric manifold (N, g_n) then π is a totally geodesic map if and only if*

$$BT_U\phi V + B\mathcal{H}\nabla_U\omega V = 0$$
$$\omega\mathcal{V}\nabla_U\phi V + \omega T_U\omega V = 0.$$

Proof. Since π is a Riemannian submersion we have that $(\nabla'\pi_*)(X, Y) = 0$ for all $X, Y \in \mathcal{H}TM$.
Now let $U, V \in \mathcal{V}$ then from equation (2.8) we get

$$(\nabla'\pi_*)(U, V) = -\pi_*\nabla_U V = \pi_* J\nabla_U JV.$$

Using proposition 4.1, equations 2.1 and definition 4.1 one has

$$\begin{aligned}
&(\nabla'\pi_*)(U, V) = \pi_*[J\nabla_U JV]\\
&= \pi_*[J(\nabla_U\phi V) + J(\nabla_U\omega V)]\\
&= \pi_*[J(T_U\phi V) + J(\mathcal{V}\nabla_U\phi V) + J(T_U\omega V) + J(\mathcal{H}\nabla_U\omega V)]\\
&= \pi_*[BT_U\phi V + CT_U\phi V + \phi(\mathcal{V}\nabla_U\phi V + \omega\mathcal{V}\nabla_U\phi V\\
&+ \phi T_U\omega V + \omega T_U\omega V + B(\mathcal{H}\nabla_U\omega V + C\mathcal{H}\nabla_U\omega V]\\
&= \pi_*[BT_U\phi V + B\mathcal{H}\nabla_U\omega V + \omega\mathcal{V}\nabla_U\phi V + \omega T_U\omega V\\
&+ \phi\mathcal{V}\nabla_U\phi V + \phi T_U\omega V + CT_U\phi V + C\mathcal{H}\nabla_U\omega V]\\
&= \pi_*[BT_U\phi V + B\mathcal{H}\nabla_U\omega V + \omega\mathcal{V}\nabla_U\phi V + \omega T_U\omega V]. \qquad (4.2)
\end{aligned}$$

Since π is a linear isometry between $\mathcal{H}TM$ and (TN), then π is a totally geodesic map if and only if

$$BT_U\phi V + B\mathcal{H}\nabla_U\omega V + \omega\mathcal{V}\nabla_U\phi V + \omega T_U\omega V = 0.$$

This implies that $BT_U\phi V + B\mathcal{H}\nabla_U\omega V = 0$,
$\omega\mathcal{V}\nabla_U\phi V + \omega T_U\omega V = 0$ and the proof is complete. □

here we obtain a necessary and sufficient condition for the distribution $\mathcal{H}$ defines a totally geodesic foliation.

Proposition 4.2. *Let π be a complex-contact Riemnnian submersion from a Kahler manifold (M, g, J) onto an almost contact metric manifold (M', g', ξ', η') then the distribution $\mathcal{H}TM$ defines a totally geodesic foliation if and only if $\phi[A_X BY + \mathcal{V}\nabla_X CY] = 0$ and $C[\mathcal{H}\nabla_X BY + A_X CY] = 0$. for all $X, Y \in \mathcal{H}TM$.*

Proof. Let $X, Y \in \mathcal{H}TM$. Then from definition 4.1 we have that $\nabla_X Y = -J\nabla_X JY$. From equation 4.1 one obtains

$$\begin{aligned}
&\nabla_X Y = -J\nabla_X JY\\
&= -J[\nabla_X BY + \nabla_X CY]\\
&= -J[A_X BY + \mathcal{H}\nabla_X BY + A_X CY + \mathcal{V}\nabla_X CY]\\
&= -\phi A_X BY - \omega A_X BY - B\mathcal{H}\nabla_X BY - C\mathcal{H}\nabla_X BY\\
&- BA_X CY - CA_X CY - \phi\mathcal{V}\nabla_X CY - \omega\mathcal{V}\nabla_X CY\\
&= -\phi A_X BY - \phi\mathcal{V}\nabla_X CY - C\mathcal{H}\nabla_X BY - CA_X CY\\
&- \omega A_X BY - \omega\mathcal{V}\nabla_X CY - B\mathcal{H}\nabla_X BY - BA_X CY. \qquad (4.3)
\end{aligned}$$

This means that $\nabla_X Y \in \mathcal{H}$ if

$$-\phi A_X BY - \phi\mathcal{V}\nabla_X CY - C\mathcal{H}\nabla_X BY - CA_X CY = 0,$$

that is if $\phi[A_X BY + \mathcal{V}\nabla_X CY] = 0$ and $C[\mathcal{H}\nabla_X BY + A_X CY] = 0$. and the result follows. □

Similarly, we obtain a necessary and sufficient condition for $\mathcal{V}TM$ to be totally geodesic

Proposition 4.3. *Let π be a complex-contact Riemnnian submersion from a Kahler manifold (M, g, J) onto an almost contact metric manifold (M', g', ξ', η') then the distribution $\mathcal{V}TM$ defines a totally geodesic foliation if and only if $B[T_U\phi V + \mathcal{H}\nabla_U\omega V] = 0$ and $\omega[\mathcal{V}\nabla_U\phi V - T_U\omega V] = 0$.*

Proof. Let $U, V \in \mathcal{V}TM$. Then from definition 4.1 we have that $\nabla_U V = -J\nabla_U JV$. From equation 4.1 one obtains

$$\begin{aligned}
&\nabla_U V = -J\nabla_U JV \\
&= -J[\nabla_U\phi V + \nabla_U\omega V] \\
&= -J[T_U\phi V + \mathcal{V}\nabla_U\phi V + T_U\omega V + \mathcal{H}\nabla_U\omega V] \\
&= -BT_U\phi V - CT_U\phi V - \phi\mathcal{V}\nabla_U\phi V - \omega\mathcal{V}\nabla_U\phi V \\
&- \phi T_U\omega V - \omega T_U\omega V - B\mathcal{H}\nabla_U\omega V - C\mathcal{H}\nabla_U\omega V \\
&= -BT_U\phi V - B\mathcal{H}\nabla_U\omega V - \omega\mathcal{V}\nabla_U\phi V - \omega T_U\omega V \\
&- CT_U\phi V - C\mathcal{H}\nabla_U\omega V - \phi\mathcal{V}\nabla_U\phi V - \phi T_U\omega V. \qquad (4.4)
\end{aligned}$$

This means that $\nabla_U V \in \mathcal{V}$ if

$$-BT_U\phi V - B\mathcal{H}\nabla_U\omega V - \omega\mathcal{V}\nabla_U\phi V - \omega T_U\omega V = 0$$

that is if $BT_U\phi V + B\mathcal{H}\nabla_U\omega V = 0$ and $\omega\mathcal{V}\nabla_U\phi V - \omega T_U\omega V = 0$ which give the result. □

5 CONCLUSIONS

We have studied complex-contact Riemann submersion introduced by Maria et al (2004) and have established some of their claims as follows;

a The fibers of a complex-contact Riemannian submersion is a contact metric submanifold of the almost Hermitian manifold. Refer to theorem (3.2).

b We have also established some structure equations of the complex-contact submersion. We we refer to theorems (3.3), (3.4), (3.6) and corollary (3.5).

d The complex-contact Riemannian map is a totally geodesic map if the conditions of proposition (4.1) are satisfy.

d There exist necessary and sufficient conditions for the distributions to be totally geodesic. We refer to propositions (4.2) and (4.3).

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