

IMPLICIT HYBRID POINTS APPROACH FOR SOLVING GENERAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH INITIAL VALUES

ABSTRACT

This article focuses on the development of implicit hybrid method for the general solution of second order ordinary differential equations with initial values (IVPs). The method of collocation and interpolation of power series was used to derive the method, while Taylor series is used to develop and analyze the predictors y_{n+i} and y'_{n+i} $i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$. The method is found to be consistent and zero-stable. The method shows to be more efficient and accurate when compared with existing work by other authors.

Key words: Power Series, Taylor's series, implicit method, initial values problems (IVPs), first order ordinary differential equations (ODEs).

1. INTRODUCTION

Differential equations are often being encountered in sciences, social sciences and engineering. This article focuses on general second order ordinary differential equation with initial value problems of this form:

$$y'' = f(x, y, y'), y(a) = y_0, y'(a) = \beta \quad (1)$$

Customarily, second order ordinary differential equations are solved by reducing it to a system of first order ODEs and then appropriate numerical method for first order is used to solve the resulting system.

The reduction process has been discussed by various authors such as Lambert (1973), Fatunla (1988), Brugnano and Trigiante (1998) and Awoyemi (1999). The approach was very successful but with some drawbacks as discussed by Onumanyi et al (1994), Awoyemi (1999), Awoyemi and Kayode (2005), Adesanya *et al* (2008) and Badmus and Yahaya

(2009). These drawbacks are computer programs associated with the methods which are mostly complicated when incorporating subroutines to supply the starting values for the methods which invariably resulted into more computer time and more computational burden. Many authors have developed methods for the direct solution of (1) without reducing it to systems of first order ordinary differential equations. Linear multistep method (LMM) for the direct solution of (1) have been considered by Brown (1977), Lambert (1973), Awoyemi (1999, 2001, 2003,), Adesanya et al. (2008). They independently proposed linear multistep methods with continuous coefficients to solve (1) in the predictor-corrector and block mode based on collocation and interpolation method and used Taylor's series expansion to supply starting values. Hybrid block method for the solution of third order ordinary differential equations was carried out by Ogunware et al (2015). Bolarinwa *et al*, (2013) proposed Taylor series approximation method to improve on the setback usually faced with Predictor-Corrector and Block methods. Olanegan *et al*, (2015), developed continuous hybrid linear multistep method (CHLMM) of one-step for the generalized solution of second order ordinary differential equations. The work extended the results generated to solve second order ordinary differential equation by multistep collocation and interpolation technique using Taylor series for implementation. This method helps to investigate the impact of the interpolation point which on substitution and evaluation obtained the direct integration of (1) without reduction to systems of first order differential equations.

In this article, we developed a two-point continuous hybrid method of better accuracy to approximate (1) directly with the use of Taylor series for implementation and evaluation. We tested our method on some application questions of second order IVPs ordinary differential equation and compared our result with existing methods.

2. The Method

In this section, we apply the interpolation and collocation procedures and we choose our interpolation at the first two point of the method and our collocation point at both grid and off-grid points.

We consider a power series in the form:

$$y(x) = \sum_{j=0}^{(i+c)} a_j x^j \quad (2)$$

Where i and c are the number of the interpolation and collocation points respectively.

The first and second derivatives are

$$y'(x) = \sum_{j=1}^{(i+1)-1} j a_j x^{j-1} \quad (3)$$

and

$$y'' = \sum_{j=2}^{(i+c)-1} j(j-1) a_j x^{j-2} \quad (4)$$

Combining (2) and (3) generates the differential system

$$y'' = \sum_{j=0}^{(i+c)-1} j(j-1) a_j x^{j-2} = f(x, y, y') \quad (5)$$

Collocating (5) at $x = x_{n+i}$, $i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ and interpolating (2) at $x = x_{n+i}$, $i = 0, \frac{1}{2}$ gives a system of non-linear equation of the form

$$AX = U \quad (6)$$

where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6]^T$$

$$U = \left[f_n, f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, y_n, y_{n+\frac{1}{2}} \right]^T \text{ and}$$

$$X = \begin{bmatrix} 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x_{n+\frac{1}{2}}^2 & 20x_{n+\frac{1}{2}}^3 & 30x_{n+\frac{1}{2}}^4 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 \\ 0 & 0 & 2 & 6x_{n+\frac{3}{2}} & 12x_{n+\frac{3}{2}}^2 & 20x_{n+\frac{3}{2}}^3 & 30x_{n+\frac{3}{2}}^4 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 \\ 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 & x_{n+\frac{1}{2}}^6 \end{bmatrix}$$

Using Gaussian elimination method to solve for a_j 's in (6) gives a continuous multistep method in the form:

$$y(t) = \alpha_\theta(t) y_{n+\theta} + h \left[\sum_{\eta=0}^{\kappa} \beta_\eta(t) f_{n+\eta} + \beta_\kappa(t) f_{n+\kappa} \right]$$

Where $\theta = 0, \frac{1}{2}$, $\eta = 0, \frac{1}{2} \left(\frac{1}{2} \right) 2$, $f_{n+\kappa} = f(x_n + \kappa h)$

Then, using the transformation $t = \frac{x - x_n}{h}$ we have $\frac{dt}{dx} = \frac{1}{h}$ where $y_{n+\theta}$, $y_{n+\kappa}$, $f_{n+\eta}$ and $f_{n+\kappa}$

give a continuous scheme and the coefficients are put as follows:

$$\alpha_0 = -2t - 1 = (-1 - 2t)$$

$$\alpha_{1/2} = 2t + 2$$

$$\beta_0 = \frac{h^2}{2880} [57 + 97t + 80t^3 - 40t^4 - 96t^5 + 64t^6]$$

$$79 \quad \beta_{1/2} = \frac{h^2}{720} [153 + 361t - 160t^3 + 160t^4 + 48t^5 - 64t^6] \quad (7)$$

$$\beta_1 = \frac{h^2}{480} [7 + 111t + 240t^2 - 200t^4 + 64t^6]$$

$$\beta_{3/2} = \frac{h^2}{2880} [3 - 13t + 160t^3 + 160t^4 - 48t^5 + 64t^6]$$

$$\beta_2 = \frac{h^2}{2880} [-3 + 5t - 80t^3 - 40t^4 + 96t^5 + 64t^6]$$

80 Differentiating (7) we have:

$$\alpha'_0 = -\frac{2}{h}$$

$$\alpha'_{1/2} = \frac{2}{h}$$

$$\beta'_0 = \frac{h}{2880} [97 + 240t^2 - 160t^3 - 480t^4 + 384t^5]$$

$$81 \quad \beta'_{1/2} = \frac{h}{720} [361 - 480t^2 + 640t^3 + 240t^4 + 384t^5] \quad (8)$$

$$\beta'_1 = \frac{h}{480} [111 + 480t - 800t^3 + 384t^5]$$

$$\beta'_{3/2} = \frac{h}{720} [-13 + 480t^2 + 640t^3 - 240t^4 + 384t^5]$$

$$\beta'_2 = \frac{h}{2880} [5 - 240t^2 - 160t^3 + 480t^4 + 384t^5]$$

82 Evaluating (7) and (8) at $t = 1$ which implies that $x = x_{n+2}$ gives

$$83 \quad y_{n+2} - 4y_{n+1/2} + 3y_n = \frac{h^2}{480} [7f_{n+2} + 132f_{n+3/2} + 222f_{n+1} + 332f_{n+1/2} + 27f_n]. \quad (9)$$

84

85 with order $p = 5$, Error constant, $c_7 = \frac{108507}{1666666667}$ or 6.51×10^{-5}

86 The first derivative is given as:

$$87 \quad y'_{n+2} = \frac{1}{h} (-2y_n + 2y_{n+1/2}) + \frac{h}{2880} [81f_n + 1508f_{n+1/2} + 1050f_{n+1} + 5004f_{n+3/2} - 469f_{n+2}]$$

88 (10)

89 3. Taylor's Series Algorithm for the Implementation of the Method

90 To generate y values for the approximate solution, the scheme and its first derivative are
 91 expanded term by term, up to the order of the scheme, by Taylor series gives:

$$92 \quad y_{n+i} = y(x_n + ih) = y_n + ih y'(x_n) + \frac{(ih)^2}{2!} f_n'' + \frac{(ih)^3}{3!} f_n''' + \frac{(ih)^4}{4!} f_n^{iv} + \frac{(ih)^5}{5!} f_n^v + \dots$$

$$93 \quad y'_{n+i} = y'(x_n) + ih f_n' + \frac{(ih)^2}{2!} f_n'' + \dots$$

94 and

$$95 \quad f_{n+i} = y''(x_n + ih), y''(x_n + ih) = f_n'' + ih f_n''' + \frac{(ih)^2}{2!} f_n^{iv} + \frac{(ih)^3}{3!} f_n^{iv} + \frac{(ih)^4}{4!} f_n^{iv} + \frac{(ih)^5}{5!} f_n^v + \dots$$

96

97 where

$$98 \quad f_n = f(x_n, y_n, y'_n), f_n^{(i)} = f^{(i)}(x_n, y_n, y'_n), i = 1, 2, 3 \dots$$

99 f', f'' and f'''

100 by partial derivatives are:

$$101 \quad f' = \frac{df}{dx} = \frac{\partial f}{\partial x} + y'_n \frac{\partial f}{\partial y} + f \frac{\partial f}{\partial y'}$$

$$102 \quad f'' = \frac{d^2 f}{dx^2} = 2(Ay' + Bf) + Cfy' + D + E$$

$$103 \quad f''' = \frac{d^3 f}{dx^3} = 2G + 3(Hy' + If) + Jfy' + K + L + M$$

$$104 \quad f^{iv} = \frac{d^4 f}{dx^4} = N + 4fO + Pf' + Q(y')^2 + R + S + T + U + V + W$$

$$105 \quad f^v = \frac{d^5 f}{dx^5} = X + Y + Z + a + b + c + d + 6e + g + h + i + j + k$$

106 where

$$107 \quad A = \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial^2 f}{\partial y \partial y'}, B = \frac{\partial^2 f}{\partial x \partial y'}, C = \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + f \frac{\partial f}{\partial y'}, D = \frac{\partial^2 f}{\partial x^2} + (y')^2 \frac{\partial^2 f}{\partial y^2} + f^2 \frac{\partial^2 f}{(\partial y')^2}$$

$$108 \quad E = f \frac{\partial y}{\partial y}, G = y' f' \frac{\partial^2 f}{\partial y \partial y'} + f' \frac{\partial^2 f}{(\partial y')^2} + y' f f y' \frac{\partial^2 f}{\partial y \partial y'} + f' \frac{\partial^2 f}{\partial x \partial y}$$

$$109 \quad H = \frac{\partial^3 f}{\partial x \partial y} + y' \frac{\partial^3 f}{\partial x \partial y^2} + f \frac{\partial^3 f}{\partial y^2} + y' f \frac{\partial^3 f}{\partial y^2 \partial y'} + f^2 \frac{\partial^3 f}{\partial y (\partial y')^2} + 2 \frac{\partial^3 f}{\partial x \partial y \partial y'}$$

$$110 \quad I = \frac{\partial^3 f}{\partial x^2 \partial y'} + \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial^3 f}{\partial y (\partial y')^2} + f \frac{\partial^2 f}{\partial y \partial y'}, \quad J = f \frac{\partial f}{\partial y} + \partial y' \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial^3 f}{\partial y (\partial y')^2} + f \frac{\partial^2 f}{\partial y \partial y'}$$

$$111 \quad K = \left(\frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + f \frac{\partial f}{\partial y'} \right) \left[\frac{\partial^2 f}{\partial x \partial y'} + y' \frac{\partial^2 f}{\partial y \partial y'} + f \frac{\partial^2 f}{(\partial y')^2} \right] \quad L = \frac{\partial^3 f}{\partial x^3} + f^3 \frac{\partial^3 f}{(\partial y')^3} + (y')^3 \frac{\partial^3 f}{\partial y^3}$$

$$112 \quad M = f' \frac{\partial f}{\partial y}$$

$$N = (y')^3 \frac{\partial^4 f}{\partial y^4} + f^5 \frac{\partial^4 f}{(\partial y')^4}, \quad O = 4f \left[\frac{\partial^4 f}{\partial x^2 \partial y} + \frac{\partial^4 f}{\partial x^3 \partial y'} \right]$$

$$113 \quad P = f' \left[\begin{aligned} & 2 \frac{\partial^2 f}{(\partial y')^2} + (f') \frac{\partial^2 f}{(\partial y')^2} + 2 (f') \frac{\partial^2 f}{\partial x \partial y} + 9 y' \\ & \frac{\partial^3 f}{\partial x \partial y \partial y'} + 6 \frac{\partial^3 f}{\partial x^2 \partial y'} + \\ & 6 (y')^2 \frac{\partial^3 f}{\partial y^2 \partial y'} + 2 y' \frac{\partial^2 f}{\partial y} + \\ & 2 \frac{\partial^2 f}{\partial x \partial y} + 4 f \frac{\partial^2 f}{\partial y \partial y'} + \\ & 4 f y' \frac{\partial^3 f}{\partial x (\partial y')^2} + 2 f \frac{\partial^3 f}{\partial x^2 \partial y} \end{aligned} \right]$$

$$114 \quad Q = (y')^2 \left[5 \frac{\partial^3 f}{\partial y^3} + 4 (y') \frac{\partial^4 f}{\partial x \partial y^3} + 4 f (y') \frac{\partial^4 f}{\partial y^3 \partial y'} + 6 \frac{\partial^4 f}{\partial x^2 \partial y^2} \right]$$

$$R = 2 y' \left[\begin{aligned} & 6 f \frac{\partial^4 f}{\partial x^2 \partial y \partial y'} + 4 \frac{\partial^3 f}{\partial x \partial y^2} + 4 f \frac{\partial^3 f}{\partial y^2 \partial y'} + \\ & 6 f (y') \frac{\partial^4 f}{\partial x \partial y^2 \partial y} + 3 f (y') \frac{\partial^4 f}{\partial y^2 (\partial y')^2} + \\ & 4 f \frac{\partial^3 f}{\partial y (\partial y')^2} + 6 f^2 \frac{\partial^4 f}{\partial x \partial y (\partial y')^2} + \frac{\partial^3}{\partial x \partial y^2} + \\ & f f' \frac{\partial^3 f}{\partial x \partial y} + f (y') \frac{\partial^3 f}{\partial y^3} + 4 f^2 \frac{\partial^3 f}{\partial y^2 \partial y'} \end{aligned} \right]$$

$$115 \quad S = 2 f^2 \left[\begin{aligned} & 5 \frac{\partial^3 f}{\partial x \partial y \partial y'} + 2 f \frac{\partial^4 f}{\partial x (\partial y')^3} + 2 f y' \frac{\partial^4 f}{\partial y (\partial y')^3} + \\ & 3 \frac{\partial^4 f}{\partial x^2 (\partial y')^2} + \frac{\partial^2 f}{\partial y^2} + f' \frac{\partial^3 f}{(\partial y')^3} + \\ & f' \frac{\partial^3 f}{\partial x \partial y \partial y'} + 3 f \frac{\partial^4 f}{\partial y (\partial y')^2} \end{aligned} \right]$$

116

$$T = 2f \left[3 \frac{\partial^2 f}{\partial y \partial y'} + \frac{\partial^2 f}{\partial y^2} + 4 \frac{\partial^3 f}{\partial x (\partial y')^2} \right]$$

$$117 \quad U = f'' \left[\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial x \partial y'} + 4f' \frac{\partial^2 f}{\partial y \partial y'} + f \frac{\partial^2 f}{(\partial y')^2} + 2f \frac{\partial^2 f}{\partial x^2 \partial y} \right]$$

$$118 \quad V = 4y' \frac{\partial^4 f}{\partial x^3 \partial y}, W = f''' \frac{\partial f}{\partial y'}, \quad X = (y')^4 \frac{\partial^5 f}{\partial y^5} + 5(y')^3 \frac{\partial^4 f}{\partial y^4} + 3(y')^2 \frac{\partial^4 f}{\partial y^4}$$

$$119 \quad Y = \left[\begin{aligned} & f^6 \frac{\partial^5 f}{(\partial y')^5} + 5f^4 \frac{\partial^4 f}{(\partial y')^4} + 5f^3 \frac{\partial^4 f}{(\partial y')^4} + \\ & f^3 f' \frac{\partial^4 f}{(\partial y')^4} + 10f \frac{\partial^3 f}{(\partial y')^3} + 4ff' \frac{\partial^3 f}{(\partial y')^3} + \\ & 2f^2 f'' \frac{\partial^3 f}{(\partial y')^3} + f(y')^3 \frac{\partial^4 f}{\partial y^4} + 10y' \frac{\partial^3 f}{\partial y^3} + \\ & 4fy' \frac{\partial^3 f}{\partial y^3} + 3f'(y')^2 \frac{\partial^3 f}{\partial y^3} + f^2 y' \frac{\partial^3 f}{\partial y^3} \end{aligned} \right]$$

$$120 \quad Z = \left[\begin{aligned} & 2f'' \frac{\partial^2 f}{(\partial y')^2} + 2ff''' \frac{\partial^2 f}{(\partial y')^2} + 2f' \frac{\partial^2 f}{(\partial y')^2} + \\ & ff'' \frac{\partial^2 f}{(\partial y')^2} + 3f'y' \frac{\partial^2 f}{\partial y^2} + 2ff' \frac{\partial^2 f}{\partial y^2} + \\ & 2f' \frac{\partial^2 f}{\partial y^2} + 2f \frac{\partial^2 f}{\partial y^2} \end{aligned} \right]$$

$$121 \quad a = \left[f^{iv} \frac{\partial f}{\partial y'} + f''' \frac{\partial f}{\partial y} \right]$$

$$122 \quad b = \left\{ \begin{aligned} & 5 \left[y' \frac{\partial^5 f}{\partial x^4 \partial y} + f \frac{\partial^5 f}{\partial x^4 \partial y'} + f(y')^2 \frac{\partial^2 f}{\partial y^3 \partial y'} \right] + \\ & (y')^3 \left[\begin{aligned} & \frac{\partial^5 f}{\partial x \partial y^4} + 16f \frac{\partial^5 f}{\partial x \partial y^3 \partial y'} + y'' \frac{\partial^5 f}{\partial y^4 \partial y'} \\ & + 10 \frac{\partial^5 f}{\partial x^2 \partial y^3} + 6f' \frac{\partial^4 f}{\partial y^3 \partial y'} + 4f' \frac{\partial^4 f}{\partial x^3 \partial y'} + \\ & 4f \frac{\partial^5 f}{\partial x \partial y^3 \partial y'} + 10f^2 \frac{\partial^5 f}{\partial y^3 (\partial y')^2} \end{aligned} \right] \end{aligned} \right\}$$

123

$$c = \left\{ (y')^2 \left[\begin{aligned} &20f \frac{\partial^4 f}{\partial y^3 \partial y'} + 10f^3 \frac{\partial^5 f}{\partial y^2 (\partial y')^3} + 6 \frac{\partial^5 f}{\partial x^3 \partial y^2} + \\ &17 \frac{\partial^4 f}{\partial x \partial y^3} + 4 \frac{\partial^2 f}{\partial x^3 \partial y^2} + 6ff' \frac{\partial^4 f}{\partial y (\partial y')^2} + \\ &20f \frac{\partial^4 f}{\partial y^2 (\partial y')^2} + 9 \frac{\partial^4 f}{\partial x \partial y^3} \end{aligned} \right] + f^5 \left[\frac{\partial^5 f}{\partial x (\partial y')^4} + y' \frac{\partial^5 f}{\partial y (\partial y')^4} \right] \right\}$$

124

$$d = \left\{ f^3 \left[\begin{aligned} &6 \frac{\partial^5 f}{\partial x^3 (\partial y')^2} + 25y' \frac{\partial^4 f}{\partial x (\partial y')^4} + 4y' \frac{\partial^5 f}{\partial x (\partial y')^4} + \\ &10 \frac{\partial^4 f}{\partial y (\partial y')^3} + 4 \frac{\partial^5 f}{\partial x (\partial y')^4} + f' \frac{\partial^4 f}{\partial x \partial y (\partial y')^2} \end{aligned} \right] + f^2 \left[\begin{aligned} &4 \frac{\partial^5 f}{\partial x^3 (\partial y')^2} + 25y' \frac{\partial^4 f}{\partial y (\partial y')^3} + 25 \frac{\partial^4 f}{\partial x (\partial y')^3} \\ &+ 8f \frac{\partial^4 f}{\partial x^3 \partial y} + 8ff' \frac{\partial^5 f}{\partial x^3 \partial y \partial y'} + 4(y')^4 \left[\frac{\partial^5 f}{\partial x \partial y^4} + f \frac{\partial^5 f}{\partial y^4 \partial y'} \right] + 10f^2 \left[\frac{\partial^4 f}{\partial x^2 \partial y \partial y'} + y' \frac{\partial^2 f}{\partial x \partial y^2 \partial y'} + 3(y')^2 \frac{\partial^5 f}{\partial x \partial y^2 (\partial y')^2} \right] \end{aligned} \right] \right\}$$

125

$$e = \left\{ f \left[\begin{aligned} &y' \frac{\partial^4 f}{\partial x^2 \partial y^2} + 2y' \frac{\partial^5 f}{\partial x^3 \partial y \partial y'} + 4y' \frac{\partial^4 f}{\partial x \partial y^2 \partial y'} + \\ &2 \frac{\partial^4 f}{\partial x^3 (\partial y')^2} + y' \frac{\partial^3 f}{\partial y^2 \partial y} \end{aligned} \right] + f' \left[\begin{aligned} &3y' \frac{\partial^4 f}{\partial x^2 \partial y \partial y'} + \frac{\partial^4 f}{\partial x^2 \partial y'} + f \frac{\partial^4 f}{\partial x^2 (\partial y')^2} + \\ &y' \frac{\partial^3 f}{\partial y^2 \partial y'} + 2y' \frac{\partial^3 f}{\partial y^2 \partial y} + (y')^2 \frac{\partial^4 f}{\partial x \partial y^2 \partial y'} \end{aligned} \right] \right\}$$

126

$$g = \left\{ 6f^2 \left[\begin{aligned} &\frac{\partial^5 f}{\partial x^3 (\partial y')^4} + 4y' \frac{\partial^4 f}{\partial x^2 \partial y (\partial y')^2} + \frac{\partial^5 f}{\partial x \partial y (\partial y')^2} + \\ &y' \frac{\partial^4 f}{\partial x \partial y^2 \partial y'} + 4 \frac{\partial^5 f}{\partial x^2 \partial y \partial y'} + \frac{\partial^3 f}{\partial x \partial y^2} + 3 \frac{\partial^4 f}{\partial y^2 (\partial y')^2} \end{aligned} \right] + (f')^2 \left[\begin{aligned} &4 \frac{\partial^5 f}{\partial x \partial y'} + y' \frac{\partial^4 f}{\partial y (\partial y')^2} + f \frac{\partial^5 f}{(\partial y')^3} + \\ &2 \frac{\partial^4 f}{\partial x^2 \partial y} + 4y' \frac{\partial^5 f}{\partial x \partial y} + 2 \frac{\partial^4 f}{\partial x^2 \partial y} + 2f \frac{\partial^4 f}{\partial x \partial y \partial y'} \end{aligned} \right] + \right. \\ \left. f \left[\begin{aligned} &y' \frac{\partial^4 f}{\partial x \partial y^2 \partial y'} + 16(y')^2 \frac{\partial^5 f}{\partial x^2 \partial y^2 \partial y'} + 2 \frac{\partial^3 f}{\partial x \partial y} + \right. \right. \\ \left. \left. 8 \frac{\partial^4 f}{\partial x \partial y^2} + 8y' \frac{\partial^5 f}{\partial x \partial y^2 \partial y'} + 16y' \frac{\partial^4 f}{\partial y^2 \partial y'} \right] + \left[\begin{aligned} &5f^2(y')^2 \frac{\partial^4 f}{\partial y^3 \partial y'} + 8f^2 \frac{\partial^5 f}{\partial x^2 \partial y'} + \\ &2(f')^2 \frac{\partial^4 f}{\partial x (\partial y')^2} + 8f^2 y' \frac{\partial^4 f}{\partial y^2 (\partial y')^2} \end{aligned} \right] + 6f^3 y' \left[\frac{\partial^4 f}{\partial y^2 (\partial y')^2} + 2 \frac{\partial^4 f}{\partial x \partial y (\partial y')^3} \right] \right\}$$

127

$$h = \left\{ f^3 y' \left[\begin{aligned} &2 \frac{\partial^3 f}{\partial x \partial y^2} + 5f \frac{\partial^3 f}{\partial y (\partial y')^2} + 10 \frac{\partial^3 f}{\partial x \partial y \partial y'} \\ &+ 4 \frac{\partial^3 f}{\partial x \partial y \partial y'} + 4f' \frac{\partial^3 f}{\partial y^2 \partial y'} + 2f \frac{\partial^3 f}{\partial x \partial y^2} \end{aligned} \right] + 2fy' \left[\begin{aligned} &\frac{\partial^3 f}{\partial x \partial y} + f \frac{\partial^3 f}{\partial y^2 \partial y'} + 5 \frac{\partial^3 f}{\partial y (\partial y')^2} + \\ &2f \frac{\partial^3 f}{\partial y^2 \partial y'} + 2 \frac{\partial^3 f}{\partial x \partial y^2} + 9 \frac{\partial^4 f}{\partial x^2 \partial y \partial y'} + \\ &13f \frac{\partial^4 f}{\partial x \partial y (\partial y')^2} + 5f^2 \frac{\partial^4 f}{\partial y (\partial y')^3} + f \frac{\partial^4 f}{\partial x \partial y (\partial y')^3} \\ &+ 2f \frac{\partial^4 f}{\partial x^2 \partial y^2} + f^2 \frac{\partial^4 f}{\partial x \partial y^2 \partial y'} \end{aligned} \right] + f^3 y'' \left[\begin{aligned} &4f \frac{\partial^3 f}{\partial y (\partial y')^2} \end{aligned} \right] + 20y' \frac{\partial^4 f}{\partial x^2 \partial y^2} + 7f'' \frac{\partial^3 f}{\partial x^2 \partial y'} \right\}$$

128

$$i = f'' \left\{ \left[\frac{\partial^3 f}{\partial x (\partial y')^2} + 3 \frac{\partial^2 f}{\partial y \partial y'} + \frac{\partial^3 f}{\partial y (\partial y')^2} + \right. \right. \\ \left. \left. 4f'' \frac{\partial^3 f}{\partial x (\partial y')^3} + 2f^2 \frac{\partial^3 f}{\partial x (\partial y')^2} + 2 \frac{\partial^2 f}{\partial x \partial y} \right] + f \left[\frac{\partial^4 f}{\partial x^2 (\partial y')^2} + 19 \frac{\partial^3 f}{\partial x \partial y \partial y'} + 4 \frac{\partial^2 f}{\partial y \partial y'} + \right. \right. \\ \left. \left. 2 \frac{\partial^4 f}{\partial x^3 \partial y} + 8 \frac{\partial^3 f}{\partial x (\partial y')^2} + 4f^2 \frac{\partial^4 f}{\partial x^2 \partial y \partial y'} \right] \right\}$$

129

$$j = ff'' \left\{ \left[\left[6 \frac{\partial^3 f}{\partial x (\partial y')^2} + 2 \frac{\partial^4 f}{\partial x (\partial y')^3} + \right. \right. \right. \\ \left. \left. \left. f' \frac{\partial^3 f}{\partial y (\partial y')^2} + 2 \frac{\partial^2 f}{\partial x \partial y} \right] \right] + ff'' \left[3 \frac{\partial^2 f}{\partial x \partial y} + f \frac{\partial^2 f}{\partial y \partial y'} + \right. \right. \\ \left. \left. 4 \frac{\partial^3 f}{\partial x^2 \partial y} + f \frac{\partial^3 f}{\partial x \partial y \partial y'} \right] \right\}$$

130

$$k = \left\{ 2f''' \left[\frac{\partial^3 f}{\partial x \partial y} + 2f' \frac{\partial^2 f}{\partial y \partial y'} + f \frac{\partial^2 f}{\partial x \partial y} \right] + 2f(y')^2 \left[f' \frac{\partial^4 f}{\partial x \partial y^3} + \frac{\partial^4 f}{\partial x \partial y} \right] + \right. \\ \left. 4f^2 \left[8 \frac{\partial^3 f}{\partial y (\partial y')^2} + \frac{\partial^3 f}{\partial x \partial y^2} \right] + fy' \left[\frac{\partial^4 f}{\partial x \partial y (\partial y')^2} \right] + \left[28f^3 \frac{\partial^4 f}{\partial x \partial y (\partial y')^2} + 5f^3 \frac{\partial^3 f}{\partial y^2 \partial y'} + \right. \right. \\ \left. \left. 2f^2 f' \frac{\partial^3 f}{\partial x \partial y^2} + 26f \frac{\partial^3 f}{\partial x \partial y \partial y'} + 4(f'')^2 \frac{\partial^2 f}{\partial y \partial y'} \right] \right\}$$

131

4. Analysis of the Basic Properties of Method

132

In verifying the accuracy and applicability of our method, we examine the basic properties

133

which include order, error constant, consistency and zero stability.

134

4.1 Order and Error Constant:

135

Definition 1: According to (Lambert 1973), Linear Multistep Method (9) is said to be of

136

order p , if p is the largest positive integer for which $c_0 = c_1 = c_2 = \dots = c_p = c_{p+1} = 0$

137

but $c_{p+2} \neq 0$.

138

Expanding (9) by Taylor series and comparing coefficients of the expansion equating it to

139

zero, we get the c_i values for the method:

140

$$c_0 = c_1 = c_2 = \dots = c_6$$

141

Hence, the method is of order $p = 5$ with principal truncation error

142

$$c_{p+2} = - \frac{108507}{1666666667} \approx 6.51 \times 10^{-5}$$

143

4.2 Consistency

144

For (9) to be consistent, the following criteria must be met.

145

Condition 1: $p \geq 1$

146

Condition 2: $\sum_{j=0}^k \alpha_j = 0$ where $j = (0 \dots 2)$

147 Condition 3: $\rho'(r) = 0$ when $r = 1$

148 Condition 4: $\rho''(r) = 2!\sigma(r)$ when $r = 1$

149 where ρ and σ are the first and second characteristic polynomials of (9), applying

150 these conditions to (9), the method was found to be consistent.

151 4.3 Zero Stability

152 **Definition 2** (Lambert1973): A linear multistep method is said to be zero-stable if no root
153 $\rho(r)$ has modulus greater than one (that is, if all roots of $\rho(r)$ lie in or on the unit circle). A
154 numerical solution to class of system (1) is stable if the difference between the numerical and
155 the theoretical solutions can be made as small as possible. Hence, (9) is found to be zero-
156 stable since none of the roots has modulus greater than one.

157 4.4 Convergence

158 **Definition 3:** A linear multistep method of the form (9) is convergent if it is consistent and
159 zero stable. Hence the necessary and sufficient conditions for the method (9) to be convergent
160 is that it must both be consistent and zero-stable. Since, these conditions are satisfied, then
161 the method (9) is said to be convergent.

162 5. Numerical Experiments

163 The accuracy of the method (9) for the direct solution of (1) is tested on some linear and non-
164 linear problems.

165 Problem 1:

166
$$y'' = \frac{(y')^2}{2y}, y\left(\frac{\pi}{6}\right) = \frac{1}{4}, y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, h = \frac{1}{320}$$

167 Analytical Solution

168
$$y(x) = \sin^2 x$$

169 Problem 2

170
$$y'' = x(y')^2, y(0) = 1, y'(0) = \frac{1}{2}, h = \frac{1}{320}$$

171 Analytical Solution

172
$$y(x) = 1 + \frac{1}{2} \log\left(\frac{2+x}{2-x}\right)$$

Problem 3

The temperature y degrees of a body, t minutes after being placed in a certain room, satisfies the differential equation $3 \frac{d^2 y}{dt^2} + \frac{dy}{dt} = 0$. By using the substitution $z = \frac{dy}{dt}$, or the otherwise, find y in terms of t given that $y = 60$ when $t = 0$ and $y = 35$ when $t = 6 \ln 4$. Find after how many minutes the rate of cooling of the body will have fallen below one degree per minute, giving your answer correct to the nearest minute. How cool does the body get?

Formulating the Problem, we have;

$$y'' = \frac{(-y')}{3}, y(0) = 60, y'(0) = -\frac{80}{9}, h = \frac{1}{320}$$

Analytical Solution

$$y(x) = \frac{80}{3} e^{-\left(\frac{1}{3}\right)x} + \frac{100}{3}$$

5.1 Results

TABLE-1 OF RESULT FOR PROBLEM 1

x	Exact Solution	Computed Solution	Error
0.1	0.2554320488157864	0.2554320488157865	1.11E-16
0.2	0.2581624564719335	0.2581624564719332	3.88E-16
0.3	0.2609023108763734	0.2609023108763727	7.21E-16
0.4	0.2636515050038920	0.2636515050038909	1.05E-15
0.5	0.2664099314644430	0.2664099314644412	1.83E-15
0.6	0.2691774825073438	0.2691774825073411	2.72E-15
0.7	0.2719540500254835	0.2719540500254801	3.44E-15
0.8	0.2747395255595467	0.2747395255595421	4.55E-15
0.9	0.2775338003022494	0.2775338003022436	5.77E-15
1.0	0.2803367651025898	0.2803367651025824	7.32E-15

Comparison of Errors in our New Method and Kayode (2011) for Problem 1

X	Error in New Method	Error in Kayode(2011)
0.1	1.110223E-16	0.64811445E-07
0.2	3.885781E-16	0.80343529E-07
0.3	7.216450E-16	0.93317005E-07

0.4	1.054712E-15	0.10334724E-06
0.5	1.831868E-15	0.11012633E-06
0.6	2.720044E-15	0.11342972E-06
0.7	3.441691E-15	0.11312237E-06
0.8	4.551914E-15	0.10916432E-06
0.9	5.773160E-15	0.10161543E-06
1.0	7.327472E-15	0.90639024E-07

189

190 TABLE-2 OF RESULT FOR PROBLEM 2

X	Exact Solution	Computed Solution	Error
0.1	1.05004172927849	1.05004172927849	0.000000E+000
0.2	1.10033534773107	1.10033534773106	6.217249E-15
0.3	1.15114043593646	1.15114043593644	2.153833E-14
0.4	1.20273255405408	1.20273255405402	5.573320E-14
0.5	1.25541281188299	1.25541281188287	1.161293E-13
0.6	1.30951960420311	1.30951960420288	2.220446E-13
0.7	1.36544375427139	1.36544375427099	3.994582E-13
0.8	1.42364893019360	1.42364893019290	6.943335E-13
0.9	1.48470027859405	1.48470027859286	1.187273E-12
1.0	1.54930614433405	1.54930614433203	2.026601E-12

191

192 Comparison of Errors in our New Method with Kayode (2011) and Adesanya (2011) for
193 Problem 2.

X	Error in New Method	Error in Kayode (2011)	Adesanya (2011)
0.1	0.000000E+000	0.61853700E-08	6.4420468E-11
0.2	6.217249E-15	0.31695117E-07	5.4567017E-10
0.3	2.153833E-14	0.75714456E-07	1.921674E-09
0.4	5.573320E-14	0.14304432E-06	4.797029E-09
0.5	1.161293E-13	0.24120724E-06	9.998000E-09
0.6	2.220446E-13	0.38177170E-06	1.871478E-08

0.7	3.994582E-13	0.58268768E-06	3.272868E-08
0.8	6.943335E-13	0.87233773E-07	5.4792477E-08
0.9	1.187273E-12	0.12968951E-07	8.929446E-08
1.0	2.026601E-12	0.19343897E-06	1.4347036E-07

194 TABLE-3 OF RESULTS FOR COOLING PROBLEM

X	Exact Solution	Computed Solution	Error
0.1	59.125762679520	59.125770155947	7.476427E-06
0.2	58.280186267509	58.280215661700	2.939419E-05
0.3	57.462331147625	57.462395949280	6.480165E-05
0.4	56.671288507811	56.671401298345	1.127905E-05
0.5	55.906179330416	55.906351828020	1.724976E-04
0.6	55.166153415412	55.166396518144	2.431027E-04
0.7	54.450388435647	54.450712262611	3.238270E-04
0.8	53.758089023057	53.758502953777	4.139307E-04
0.9	53.088485884845	53.088998596888	5.127120E-04
1.0	52.440834948634	52.441454453559	6.195049E-04

195

196 6. Discussion of Results

197 Tables 1-3 presented the numerical solutions in terms of the maximum errors obtained for
 198 each of the problems considered respectively. The error of the new method is compared with
 199 those of predictor-corrector and block method of Kayode (2011) and Adesanya (2011)
 200 respectively.

201 In tables 1, the new method converges faster than Kayode (2011) when solving problem 1
 202 with the same order but different approach. This makes the new method to be more efficient
 203 than previous method as displayed in global maximum errors obtained for the method in (9).
 204 Also, we compared the new method with Kayode (2011) and Adesanya (2011) in table 2.
 205 These authors respectively solved problem 2 with predictor-corrector and Block mode. With
 206 our results as displayed in table 2, the error in our new method shows to be more efficient and
 207 converges faster. Also the new method was used to solve an engineering (cooling) problem
 208 which shows that the body become more cooler and fallen below one degree ($1^{\circ}C$) as the

step length (h) of our method is been reduced which makes the temperature of the body in the room to satisfy our differential equations and the new method developed as been demonstrated in the computed result and the error displayed in table 3.

7. Conclusion

In this paper, a new method with the use of Taylor's series for the approximation of y variables has enabled us to compute the derivatives of the method to any possible order which allows direct solution of Initial Value Problems (IVPs) of ordinary differential equations. Using this new method with all computation with the aid of MATLAB generated codes; this enables us to compute directly the solution of second order ordinary differentials equations with initial value problems (IVPs) directly without reducing to system of first order. Based on this new approach, it is evident that the new method is considerably more efficient than other numerical methods with the same properties of consistency, zero-stability and convergence.

References

- [1]. Adesanya A.O., Anake, T.A., and Udoh, M.O. (2008): Improved continuous method for direct solution of general second order ordinary differential equations. *Journal of Nigerian Association of Mathematical Physics* 13, 59-62.
- [2]. Adesanya A.O. (2011): Block method for direct solution of general higher order initial value problems of ordinary differential equations. Ph.D Thesis, Federal University of Technology Akure. (Unpublished).
- [3]. Awoyemi D.O. (2001): A new sixth order algorithm for general second order ordinary differential equations. *International Journal of Computer and Mathematics* 77, 117-124.
- [4]. Awoyemi, D. O. (2003): A p-stable linear multistep method for solving third order ordinary differential equations. *International Journal of Computer and Mathematics* 80(8), 85 – 991.
- [5]. Awoyemi, D. O. (1999): A class of continuous methods for general second order initial value problem in ordinary differential equation. *International Journal of Computer and Mathematics* 72, 29-37.
- [6]. Awoyemi, D. O., and Kayode S. J. (2005): A maximal order collocation method for direct solution of initial value problems of general second ordinary differential equations. *Proceedings of the conference organized by the National Mathematical Centre, Abuja.*

- 240 [7]. Badmus A. M. and Yahaya, Y.A. (2009): An Accurate Uniform Order 6 Block Method
241 for Direct Solution of General Second Order Ordinary Differential Equations. *The Pacific*
242 *Journal of Science and Technology*, 10 (2), 248-254.
- 243 [8]. Bolarinwa B, Akinduko O. B., Duromola M. K. (2013): A Fourth Order One-Step Hybrid
244 Method for the Numerical Solution of Initial Value Problems of Second Order Ordinary
245 Differential Equations. *Journal of Natural Sciences* 1(2): 79-85.
- 246 [9]. Brown, R. L. (1977): Some characteristics of implicit multistep Multi derivative
247 Integration formulas. *Society for Industrial and Applied Mathematics Journal on Numerical*
248 *Analysis*. 14 (6), 982 – 993.
- 249 [10]. Bruguano L. and Trigiante D (1998): Solving Differential Problems by Multistep Initial
250 and Boundary Value Methods. *Gordon and Breach Science Publishers, Amsterdam*, 280-299.
- 251 [11]. Fatunla, S. O. (1988): Numerical methods for ivps in odes, Academic Press Inc. Harcourt
252 Brace Jovanovich Publishers, New York.
- 253 [12]. Kayode, S. J. (2011): A class of one-point zero-stable continuous hybrid methods for
254 direct solution of second-order differential equations. *African Journal of Mathematics and*
255 *Computer Science Research* 4(3), 93-99.
- 256 [13]. Lambert, J. D. (1973): Computational methods in Ordinary Differential Equations, John
257 Wiley & Sons Inc. New York.
- 258 [14]. Majid, Z.A., Suleiman, M.B. and Omar Z. (2006): “3 Points Implicit Block Method for
259 Solving Ordinary Differential Equations. *Bulletin of the Malaysian Mathematical Sciences*
260 *Society* (2) 29(1) 23–31.
- 261 [15] Ogunware, B. G., Omole, E. O., and Olanegan, O.O. (2015): Hybrid and Non-hybrid
262 Implicit Schemes for Solving Third Order ODEs Using Block Method as Predictors,
263 *Journal of Mathematical Theory and Modelling*, 5(3), 10-25.
- 264 [16]. Olanegan, O. O., Awoyemi, D. O., Ogunware, B. G. and Obarhua, F. O. (2015):
265 Continuous Double-Hybrid Point Method for the Solution of Second Order Ordinary
266 Differential Equations. *International Journal of Advanced Scientific and Technical Research*
267 (5)2, 549 – 562.
- 268 [17]. Onumanyi, P. Awoyemi, D. O., Jator, S. N. and Serisena U. W. (1994): New Linear
269 Multistep Method with Constant Coefficient for First Order Initial Value Problems. *Journal*
270 *of Nigerian Mathematical Society*, 13(7), 37 – 51.