

Eigenvalues for some complex infinite tridiagonal matrices

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Abstract

The discrete spectrum for an unbounded operator J defined by a special infinite tridiagonal complex matrix is approximated by the eigenvalues of its orthogonal truncations. Let $\sigma(J)$ means the spectrum of the operator J and

$$\Lambda(J) = \{\lambda \in \text{Lim}_{n \rightarrow \infty} \lambda_n : \lambda_n \text{ is an eigenvalue of } J_n\},$$

where $\text{Lim}_{n \rightarrow \infty} \lambda_n$ is the set of limit points of the sequence (λ_n) , and the $n \times n$ matrix J_n is an orthogonal truncation of J .

We consider classes of tridiagonal complex matrices for which $\sigma(J) = \Lambda(J)$.

Keywords: tridiagonal matrix, complex Jacobi matrix, discrete spectrum, eigenvalue, asymptotic formula, unbounded operator.

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1 Introduction

Examples of infinite non-symmetric tridiagonal matrices and complex Jacobi matrices were investigated by several authors: [Aptekarev et al. (1995)], [Beckermann (2001)], [Beckermann and Kaliaguine (1997)], [Djakov and Mityagin (2005)], [Djakov and Mityagin (2006)], [Egorova and Golinskii (2005)], Ikebe, Asai, Miyazaki and Cai [Ikebe et al. (1996)], and others (see, e.g., [Berezansky et al. (2008)], [Golinskii and Kudryavtsev (2006)], [Malejki (2014)], [Marletta and Naboko (2014)], [Shivakumar and Ji (1996)], [Shivakumar et al. (1987)], [Štampach (2017)] and [Volkmer (2008)]). However, systematic investigations, concerning spectral properties of non-selfadjoint tridiagonal operators, are difficult because the structure of complex sequences could be more complicated than the structure of real sequences. Moreover, standard operator methods,

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which give effective results on real Jacobi matrices, fail in the complex case. The relationships of tridiagonal matrices and the formal orthogonal polynomials on the complex plane can be found, e.g., in [Beckermann (2001)], [Beckermann and Kaliaguine (1997)], [Berezansky et al. (2008)]. Complex tridiagonal infinite matrices are also useful in the theory of special functions like the Bessel functions or the Mathieu functions [Ikebe et al. (1996)], [Volkmer (2008)], and [Ziener et al. (2012)]. Non-symmetric tridiagonal matrices with real entries are essentially associated with the Hill equation [Djakov and Mityagin (2005)], the Ince equation [Volkmer (2008)], the Mathieu equation [Ziener et al. (2012)], and other second order equations [Ikebe et al. (1996)], [Volkmer (2004)].

We consider a complex tridiagonal infinite matrix

$$J((d_n), (a_n), (b_n)) = \begin{pmatrix} d_1 & a_1 & 0 & \cdots & \cdots \\ b_1 & d_2 & a_2 & 0 & \ddots \\ 0 & b_2 & d_3 & a_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad (1.1)$$

where $d_n, a_n, b_n \in \mathbb{C} \setminus \{0\}$.

The matrix $J((d_n), (a_n), (b_n))$ defines a linear operator J in the space $l_2 = l_2(\mathbb{N})$ which acts on a maximal domain

$$Dom(J) = \{(f_n)_{n=1}^\infty \in l_2 : (b_{n-1}f_{n-1} + d_nf_n + a_nf_{n+1})_{n=1}^\infty \in l_2\} \quad (1.2)$$

and

$$(Jf)_n = b_{n-1}f_{n-1} + d_nf_n + a_nf_{n+1}, \quad n \geq 1, \quad (1.3)$$

for $f = (f_n)_{n=1}^\infty \in Dom(J)$ and $b_0 = 0$. The domain $Dom(J)$ of the operator J is dense in l_2 .

Assume that X is a complex Banach space. Denote by $B(X)$ the space of all bounded linear operators on X . Let $A : Dom(A) \rightarrow X$ be a densely defined linear operator in X . The resolvent set of A is the set

$$\rho(A) = \{\lambda \in \mathbb{C} : (A - \lambda I)^{-1} \text{ exists in } B(X)\}. \quad (1.4)$$

The inverse operator $(A - \lambda I)^{-1}$, where $\lambda \in \rho(A)$, is called the resolvent of A and

$$\sigma(A) = \mathbb{C} \setminus \rho(A) \quad (1.5)$$

means the spectrum. We are interested in the classes of unbounded tridiagonal operators, for which the spectrum is discrete. The criteria, which guarantee compactness of the resolvent and a discrete spectrum, for a non-symmetric or complex tridiagonal operator are given in [Ikebe et al. (1996)], [Malejki (2014)], [Shivakumar et al. (1987)] and [Volkmer (2008)].

Different kind of problems in infinite-dimensional spaces are approximated by some problems in finite-dimensional spaces because of the fact that many mathematical software products provide functionality for the solution of matrix equations. To realize this strategy correctly, the projective and iterative methods can be used, see for example [Arqub (2017)], [Cojuhari and Nowak (2009)], [Maayah et al. (2014)], [Marletta and Naboko (2014)] and [Nowak (2009)], and the famed monographs: [Böttcher and Silbermann (1998)], [Krasnosel'skij et al. (1972)], [Lindner (2006)], [Prösdorf and Silbermann (1991)]. The question about approximation of the spectrum of an operator by eigenvalues of properly selected matrices is also unsophisticated.

Denote the orthogonal truncation of $J((d_n), (a_n), (b_n))$ by

$$J_n = \begin{pmatrix} d_1 & a_1 & & & \\ b_1 & d_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & b_{n-1} & d_n & \end{pmatrix} \quad (1.6)$$

for $n \geq 1$.

Define

$$\Lambda(J) = \{\lambda \in \text{Lim}_{n \rightarrow \infty} \lambda_n : \lambda_n \text{ is an eigenvalue of } J_n\}, \quad (1.7)$$

where $\text{Lim}_{n \rightarrow \infty} \lambda_n$ is the set of limit points of the sequence (λ_n) .

The aim of the research is finding classes of tridiagonal operators for which $\sigma(J) \subset \Lambda(J)$ or $\sigma(J) \supset \Lambda(J)$. This can be done if we assume that J is an unbounded operator with compact resolvent.

2 $\sigma(J) \subset \Lambda(J)$

Remark 2.1. In the case of self-adjoint operators, among other, the following results are known.

1. If A is a bounded self-adjoint operator in l_2 then $\sigma(A) \subset \Lambda(A)$ [Arveson (1994)].
2. If J is a self-adjoint operator given by a real Jacobi matrix, then $\sigma(J) \subset \Lambda(J)$ (classical result for real Jacobi matrices).

The problem of approximation of eigenvalues for unbounded self-adjoint Jacobi matrices acting in l_2 by the eigenvalues of finite matrices was considered also in [Boutet de Monvel and Zielinski (2014)], [Ifantis and Panagopoulos (2001)], [Ifantis et al. (2007)], [Malejki (2007)], [Malejki (2010)], [Volkmer (2004)] and in others.

The following results are known for complex or non-selfadjoint tridiagonal operators.

Remark 2.2. If J is represented by a complex tridiagonal matrix $J((d_n), (a_n), (b_n))$ and it is a compact operator, then $\sigma(J) \subset \Lambda(J)$ (classical result, Theorem 18.1, [Krasnosel'skij et al. (1972)]).

Remark 2.3. Real tridiagonal infinite matrices were investigated by H. Volkmer. The enough conditions on real sequences $(a_n), (b_n), (d_n)$ to obtain the inclusion $\sigma(J) \subset \Lambda(J)$ are presented in [Volkmer (2008)].

Remark 2.4. Y. Ikebe, N. Asai, Y. Miyazaki and D. Cai have proved that if $\lim_{n \rightarrow \infty} |d_n| = \infty$ and $(a_n) = (b_n)$ are bounded complex sequences then $\sigma(J) \subset \Lambda(J)$ [Ikebe et al. (1996)].

We generalize the result given in [Ikebe et al. (1996)]. The new result for complex tridiagonal operators is as follow.

Theorem 2.1. If J has a compact resolvent,

$$\lim_{n \rightarrow \infty} |d_n| = \infty$$

and

$$\lim_{n \rightarrow \infty} \frac{|a_{n-1}| + |a_n| + |b_{n-1}| + |b_n|}{|d_n|} = 0,$$

then $\sigma(J)$ is discrete and $\sigma(J) \subset \Lambda(J)$.

Proof. The idea of the proof from [Ikebe et al. (1996)] works in this case. Let C be a diagonal operator in l_2 given by the matrix

$$J \left(\left(\frac{1}{\sqrt{d_n}} \right), (0), (0) \right). \quad (2.1)$$

Obviously, C is compact.

Next, suppose that L is an operator in l_2 defined by

$$J((0), (a_n), (b_n)). \quad (2.2)$$

Observe that the matrix

$$J \left((0), \left(\frac{a_n}{\sqrt{d_n} \sqrt{d_{n+1}}} \right), \left(\frac{b_n}{\sqrt{d_n} \sqrt{d_{n+1}}} \right) \right) \quad (2.3)$$

represents a compact operator in l_2 if and only if

$$\lim_{n \rightarrow \infty} \frac{|a_n| + |b_n|}{\sqrt{|d_n|} \sqrt{|d_{n+1}|}} = 0, \quad (2.4)$$

but the last equation is true because of the assumption. Moreover, CLC has the matrix representation (2.3). Then we can write

$$J \supseteq L + C^{-2} = C^{-1}(CLC + I)C^{-1}. \quad (2.5)$$

Without loss of generality we can assume that J is invertible in $B(l_2)$. Thus $CLC + I$ is also invertible in $B(l_2)$, so

$$A := J^{-1} = C(CL C + I)^{-1}C \quad (2.6)$$

is a compact operator. Then $Jx = \lambda x$ if and only if $Ax = \frac{1}{\lambda}x$ for $\lambda \in \mathbb{C}$ and $x \in l_2 \setminus \{0\}$.

Let $\{e_n : n = 1, 2, \dots\}$ be a canonical basis in l_2 and denote by P_n the orthogonal projection on $H_n = \text{span}\{e_1, \dots, e_n\}$. Then P_n admits the block matrix representation

$$P_n = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.7)$$

where I_n is the identity on H_n .

Denote

$$A_n = P_n C (P_n CL C P_n + I)^{-1} C P_n \quad (2.8)$$

and observe that

$$P_n CL C P_n + I = \begin{pmatrix} C_n J_n C_n & 0 \\ 0 & I'_n \end{pmatrix}, \quad (2.9)$$

where

$$C_n = \begin{pmatrix} \frac{1}{\sqrt{d_1}} & 0 & & \\ 0 & \frac{1}{\sqrt{d_2}} & \ddots & \\ & \ddots & \ddots & 0 \\ & & 0 & \frac{1}{\sqrt{d_n}} \end{pmatrix}$$

and I'_n is the identity on $l_2 \ominus H_n$.

The operator $K = CLC$ is compact; therefore,

$$\|P_n K P_n - K\| \rightarrow 0, \quad n \rightarrow \infty, \quad (2.10)$$

and also

$$\|(P_n K P_n + I) - (K + I)\| \rightarrow 0, \quad n \rightarrow \infty. \quad (2.11)$$

Notice that $K + I$ is invertible in $B(l_2)$, so there exists n_0 such that $P_n K P_n + I$ is invertible for $n \geq n_0$ and

$$\|(P_n K P_n + I)^{-1} - (K + I)^{-1}\| \rightarrow 0, \quad n \rightarrow \infty. \quad (2.12)$$

Thus A_n exists for $n \geq n_0$; moreover,

$$\begin{aligned} \|A_n - A\| &\leq \\ &\leq \|P_n A P_n - A\| + \|C\|^2 \|(P_n K P_n + I)^{-1} - (K + I)^{-1}\| \end{aligned} \quad (2.13)$$

and

$$\|A_n - A\| \rightarrow 0, \quad n \rightarrow \infty. \quad (2.14)$$

The equation (2.8) implies that

$$A_n = \begin{pmatrix} J_n^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \quad (2.15)$$

So if $\lambda \neq 0$ is an eigenvalue of J then $\mu = \frac{1}{\lambda}$ is an eigenvalue of A . According to the projective method approach ([Krasnosel'skij et al. (1972)], Th. 18.1) we deduce that there exists a sequence $(\mu_n)_{n \geq n_0}$ such that μ_n is an eigenvalue of A_n and $\mu = \lim_{n \rightarrow \infty} \mu_n$. Notice that if $\mu_n \neq 0$ is an eigenvalue of A_n then $\lambda_n = \frac{1}{\mu_n}$ is an eigenvalue of J_n . Then $\lambda = \lim_{n \rightarrow \infty} \lambda_n$ and finally $\sigma(J) \subset \Lambda(J)$. $\square$

3 $\Lambda(J) \subset \sigma(J)$

The investigations related to the inclusion $\Lambda(J) \subset \sigma(J)$ in the class of tridiagonal infinite matrices are complicated. If $J((d_n), (a_n), (b_n))$ is real and symmetric, then the inclusion $\Lambda(J) \subset \sigma(J)$ holds under some conditions (see [Arveson (1994)], [Ifantis and Panagopoulos (2001)], [Ifantis et al. (2007)], [Petropoulou and Velázquez (2014)] and others). The complex case looks interesting too. We successfully use the approach presented in [Petropoulou and Velázquez (2014)], applied to real symmetric Jacobi matrices, for complex tridiagonal matrices.

Theorem 3.1. *Let $\{e_n : n = 1, 2, 3, \dots\}$ be the canonical basis for l_2 and P_n be the orthogonal projection on $H_n = \text{span}\{e_1, \dots, e_n\}$. Assume that the matrix $J((d_n), (a_n), (b_n))$ defines a linear operator J in l_2 and J_n is given by (1.6). If for all bounded complex sequences $(\lambda_n)_{n=1}^\infty$ such that $\lambda_n \in \sigma(J_n)$, $n \geq 1$, and for all sequences $(x_n)_{n=1}^\infty$ of eigenvectors such that $P_n J x_n = \lambda_n x_n$, $\|x_n\| = 1$ and $x_n \in H_n$ for $n \geq 1$,*

$$\lim_{n \rightarrow \infty} |b_n(x_n, e_n)| = 0, \quad (3.1)$$

then $\Lambda(J) \subset \sigma(J)$.

Proof. We follow [Petropoulou and Velázquez (2014)] and [Ifantis et al. (2007)]. Let $\lambda \in \Lambda(J)$. Without loss of generality we can assume $\lambda = \lim_{n \rightarrow \infty} \lambda_n$, where λ_n is an eigenvalue of J_n . Let $\xi_n = (\xi_{n,1}, \xi_{n,2}, \dots, \xi_{n,n})^\top$ be an eigenvector of J_n such that $J_n \xi_n = \lambda_n \xi_n$ and $\|\xi_n\| = 1$ for $n \geq 1$. Then

$$P_n J x_n = P_n J P_n x_n = \lambda_n x_n,$$

where $x_n = \sum_{k=1}^n \xi_{n,k} e_k$.

Next notice that

$$J x_n = \lambda_n x_n + b_n(x_n, e_n) e_{n+1}, \quad (3.2)$$

and

$$\|J x_n\|^2 = \|\lambda_n x_n\|^2 + |b_n(x_n, e_n)|^2. \quad (3.3)$$

Then from

$$\|(J - \lambda)x_n\|^2 = \|J x_n\|^2 + |\lambda|^2 - 2\text{Re}\bar{\lambda}(J x_n, x_n), \quad (3.4)$$

(3.2) and (3.3) we derive

$$\|(J - \lambda)x_n\|^2 = |\lambda_n|^2 + |\lambda|^2 - 2\text{Re}\bar{\lambda}\lambda_n + |b_n(x_n, e_n)|^2 = \quad (3.5)$$

$$= |\lambda_n - \lambda|^2 + |b_n(x_n, e_n)|^2. \quad (3.6)$$

So, there exists $(x_n)_{n=1}^\infty$ such that $\|x_n\| = 1$ for $n \geq 1$ and $\|(J - \lambda)x_n\| \rightarrow 0$, as $n \rightarrow \infty$, if (3.1) holds. This ensures that λ belongs to the spectrum of J . $\square$

Now we are going to analyze the condition (3.1). Let (x_n) be a sequence considered in Theorem 3.1, so $x_n = \sum_{k=1}^n \xi_{n,k} e_k$ for $n \geq 1$ and $\xi_n = (\xi_{n,1}, \xi_{n,2}, \dots, \xi_{n,n})^\top \in \mathbb{C}^n$ is an eigenvector for J_n such that $J_n \xi_n = \lambda_n \xi_n$ and $\|\xi_n\| = \|x_n\| = 1$. Then (3.1) could be written as

$$\lim_{n \rightarrow \infty} |b_n \xi_{n,n}| = 0. \quad (3.7)$$

Next we follow [Petropoulou and Velázquez (2014)] to estimate $|b_n \xi_{n,n}|$ for large n . The equality $J_n \xi_n = \lambda_n \xi_n$ is equivalent to the system

$$\begin{cases} b_{n-1} \xi_{n,n-1} + (d_n - \lambda_n) \xi_{n,n} = 0, \\ b_{n-k-1} \xi_{n,n-k-1} + (d_{n-k} - \lambda_n) \xi_{n,n-k} + a_{n-k} \xi_{n,n-k+1} = 0, \\ \quad k = 1, \dots, n-2; \\ (d_1 - \lambda_n) \xi_{n,1} + a_1 \xi_{n,2} = 0. \end{cases} \quad (3.8)$$

We have assumed that the sequence (λ_n) is bounded, so $|\lambda_n| \leq M$ for $n \geq 1$. Let $K \geq 1$ be an integer. There exists n_0 such that $|d_{n-K}| > M + 1$ for $n \geq n_0$. Then we use (3.8) to obtain the following estimates

$$|\xi_{n,n-k}| \leq (M + 1) \left(\frac{|b_{n-k-1}|}{|d_{n-k}|} |\xi_{n,n-k-1}| + \frac{|a_{n-k}|}{|d_{n-k}|} |\xi_{n,n-k+1}| \right) \quad (3.9)$$

for $k = 1, \dots, K$, and

$$|\xi_{n,n}| \leq (M + 1) \frac{|b_{n-1}|}{|d_n|} |\xi_{n,n-1}|, \quad (3.10)$$

where $n > n_0$.

Denote

$$B_n = \max\{|a_{n-1}|, \dots, |a_{n-K}|, |b_{n-1}|, \dots, |b_{n-K}|\} \quad (3.11)$$

and

$$D_n = \min\{|d_n|, \dots, |d_{n-K}|\}. \quad (3.12)$$

Follow the idea that was used in [Petropoulou and Velázquez (2014)], from (3.9), (3.10), (3.11) and (3.12) we derive

$$|\xi_{n,n}| \leq c(M, K) \left(\frac{B_n}{D_n} \right)^K \quad (3.13)$$

for large n , where a constant $c(M, K) > 0$ is independent on n .

So if

$$\lim_{n \rightarrow \infty} |b_n| \left(\frac{B_n}{D_n} \right)^K = 0, \quad (3.14)$$

then (3.1) holds.

Corollary 3.2. *If J is an operator given by a tridiagonal matrix (1.1), where $\lim_{n \rightarrow \infty} |d_n| = +\infty$, and there exists an integer $K \geq 1$ such that*

$$\lim_{n \rightarrow \infty} |b_n| \left(\frac{B_n}{D_n} \right)^K = 0,$$

where B_n, D_n are given by (3.11) and (3.12), then $\Lambda(J) \subset \sigma(J)$.

Corollary 3.3. *Let J be an operator given by a tridiagonal matrix (1.1) and $|a_n|, |b_n| = O(n^\beta)$ and $\frac{1}{|d_n|} = O(n^{-\alpha})$, as $n \rightarrow \infty$. If $\alpha > \beta$ and $\alpha \geq 0$ then $\Lambda(J) \subset \sigma(J)$.*

Proof. If $\alpha > \beta$ then there exists an integer $K \geq 1$ such that $K\alpha > (K+1)\beta$. So

$$|b_n| \left(\frac{B_n}{D_n} \right)^K = O \left(n^{(K+1)\beta - K\alpha} \right), \quad n \rightarrow \infty$$

and (3.14) holds. □

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