

# There is no standard model of ZFC and ZFC<sub>2</sub>

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**Abstract:** In this paper we view the first order set theory ZFC under the canonical first order semantics and the second order set theory ZFC<sub>2</sub> under the Henkin semantics.

Main results are: (i) Let  $M_{st}^{ZFC}$  be a standard model of ZFC, then  $\neg Con(ZFC + \exists M_{st}^{ZFC})$ .

(ii) Let  $M_{st}^{ZFC_2}$  be a standard model of ZFC<sub>2</sub> with Henkin semantics, then

$\neg Con(ZFC_2 + \exists M_{st}^{ZFC_2})$ . (iii) Let  $\kappa$  be inaccessible cardinal then  $\neg Con(ZFC + \exists \kappa)$ .

In order to obtain the statements (i) and (ii) an examples of the inconsistent countable set

in a set theory  $ZFC + \exists M_{st}^{ZFC}$  and in a set theory  $ZFC_2 + \exists M_{st}^{ZFC_2}$  were derived.

It is widely believed that  $ZFC + \exists M_{st}^{ZFC}$  and  $ZFC_2 + \exists M_{st}^{ZFC_2}$  are inconsistent, i.e. ZFC and ZFC<sub>2</sub> have a standard models. Unfortunately this belief wrong.

**Keywords:** Gödel encoding, Russell's paradox, standard model, Henkin semantics, inaccessible cardinal.

## I. Introduction.

### 1.1. Main results.

Let us remind that accordingly to naive set theory, any definable collection is a set. Let  $R$  be the set of all sets that are not members of themselves. If  $R$  qualifies as a member of itself, it would contradict its own definition as a set containing all sets that are not members of themselves. On the other hand, if such a set is not a member of itself, it would qualify as a member of itself by the same definition. This contradiction is Russell's paradox. In 1908, two ways of avoiding the paradox were proposed, Russell's type theory and Zermelo set theory, the first constructed axiomatic set theory. Zermelo's axioms went well beyond Frege's axioms of extensionality and unlimited set abstraction, and evolved into the now-canonical Zermelo–Fraenkel set theory ZFC. "But how do we know that ZFC is a consistent theory, free of contradictions? The short answer is that we don't; it is a matter of faith (or of skepticism)" — E. Nelson wrote in his paper [1]. However, it is deemed unlikely that even ZFC<sub>2</sub> which is significantly stronger than ZFC harbors an unsuspected contradiction; it is widely believed that if ZFC and ZFC<sub>2</sub> were inconsistent, that fact would have been uncovered by now. This much is certain — ZFC and ZFC<sub>2</sub> is immune to the classic paradoxes of naive set theory: Russell's paradox, the Burali-Forti paradox, and Cantor's paradox.

**Remark 1.1.1.** Note that in this paper we view (i) the first order set theory ZFC under the canonical first order semantics (ii) the second order set theory ZFC<sub>2</sub> under the Henkin semantics [2],[3],[4],[5],[6].

**Remark 1.1.2.** Second-order logic essentially differs from the usual first-order

predicate

calculus in that it has variables and quantifiers not only for individuals but also for subsets

of the universe and variables for  $n$ -ary relations as well [2],[6]. The deductive calculus  $\mathbf{DED}_2$  of second order logic is based on rules and axioms which guarantee that the quantifiers range at least over definable subsets [6]. As to the semantics, there are two types of models: (i) Suppose  $\mathbf{U}$  is an ordinary first-order structure and  $\mathbf{S}$  is a set of subsets of the domain  $A$  of  $\mathbf{U}$ . The main idea is that the set-variables range over  $\mathbf{S}$ , i.e.  $\langle \mathbf{U}, \mathbf{S} \rangle \models \exists X \Phi(X) \Leftrightarrow \exists S (S \in \mathbf{S}) [\langle \mathbf{U}, \mathbf{S} \rangle \models \Phi(S)]$ .

We call  $\langle \mathbf{U}, \mathbf{S} \rangle$  a Henkin model, if  $\langle \mathbf{U}, \mathbf{S} \rangle$  satisfies the axioms of  $\mathbf{DED}_2$  and truth in  $\langle \mathbf{U}, \mathbf{S} \rangle$  is preserved by the rules of  $\mathbf{DED}_2$ . We call this semantics of second-order logic the Henkin semantics and second-order logic with the Henkin semantics the Henkin second-order logic. There is a special class of Henkin models, namely those  $\langle \mathbf{U}, \mathbf{S} \rangle$  where  $\mathbf{S}$  is the set of all subsets of  $A$ . We call these full models. We call this semantics of second-order logic the full semantics and second-order logic with the full semantics the full second-order logic.

**Remark 1.1.3.** We emphasize that the following facts are the main features of second-order logic:

**1. The Completeness Theorem:** A sentence is provable in  $\mathbf{DED}_2$  if and only if it holds in

all Henkin models [2],[6].

**2. The Löwenheim-Skolem Theorem:** A sentence with an infinite Henkin model has a countable Henkin model.

**3. The Compactness Theorem:** A set of sentences, every finite subset of which has a Henkin model, has itself a Henkin model.

**4. The Incompleteness Theorem:** Neither  $\mathbf{DED}_2$  nor any other effectively given deductive calculus is complete for full models, that is, there are always sentences which are true in all full models but which are unprovable.

**5. Failure of the Compactness Theorem for full models.**

**6. Failure of the Löwenheim-Skolem Theorem for full models.**

**7. There is a finite second-order axiom system  $\mathbb{Z}_2$  such that the semiring  $\mathbb{N}$  of natural numbers is the only full model (up to isomorphism) of  $\mathbb{Z}_2$ .**

**8. There is a finite second-order axiom system  $RCF_2$  such that the field  $\mathbb{R}$  of real numbers is the only (up to isomorphism) full model of  $RCF_2$ .**

**Remark 1.1.4.** For let second-order  $ZFC$  be, as usual, the theory that results obtained from  $ZFC$  when the axiom schema of replacement is replaced by its second-order universal closure, i.e.

$$\forall X [Func(X) \Rightarrow \forall u \exists v \forall r [r \in v \Leftrightarrow \exists s (s \in u \wedge (s, r) \in X)]], \quad (1.1.1)$$

where  $X$  is a second-order variable, and where  $Func(X)$  abbreviates " $X$  is a functional relation", see [7].

**Designation 1.1.1.** We will denote (i) by  $ZFC_2^{Hs}$  set theory  $ZFC_2$  with the Henkin semantics, (ii) by  $\overline{ZFC}_2^{Hs}$  set theory  $ZFC_2^{Hs} + \exists M_{st}^{ZFC_2^{Hs}}$  and (iii) by  $ZFC_{st}$  set theory  $ZFC + \exists M_{st}^{ZFC}$ , where  $M_{st}^{Th}$  is a standard model of the theory  $Th$ .

**Axiom  $\exists M^{ZFC}$ .** [8]. There is a set  $M^{ZFC}$  and a binary relation  $\epsilon \subseteq M^{ZFC} \times M^{ZFC}$  which makes  $M^{ZFC}$  a model for  $ZFC$ .

**Remark 1.1.3.**(i) We emphasize that it is well known that axiom  $\exists M^{ZFC}$  a single statement

in  $ZFC$  see [8], Ch.II, section 7. We denote this statement through out all this paper by symbol  $Con(ZFC; M^{ZFC})$ . The completeness theorem says that  $\exists M^{ZFC} \Leftrightarrow Con(ZFC)$ .

(ii) Obviously there exists a single statement in  $ZFC_2^{Hs}$  such that  $\exists M^{ZFC_2^{Hs}} \Leftrightarrow Con(ZFC_2^{Hs})$ .

We denote this statement through out all this paper by symbol  $Con(ZFC_2^{Hs}; M^{ZFC_2^{Hs}})$  and there

exists a single statement  $\exists M^{Z_2^{Hs}}$  in  $Z_2^{Hs}$ . We denote this statement through out all this paper by

symbol  $Con(Z_2^{Hs}; M^{Z_2^{Hs}})$ .

**Axiom**  $\exists M_{st}^{ZFC}$ . [8]. There is a set  $M_{st}^{ZFC}$  such that if  $R$  is  $\{\langle x, y \rangle | x \in y \wedge x \in M_{st}^{ZFC} \wedge y \in M_{st}^{ZFC} \}$

then  $M_{st}^{ZFC}$  is a model for  $ZFC$  under the relation  $R$ .

**Definition 1.1.1.** [8]. The model  $M_{st}^{ZFC}$  and  $M_{st}^{Z_2^{Hs}}$  is called a standard model since the relation  $\in$  used is merely the standard  $\in$ - relation.

**Remark 1.1.4.** [8]. Note that axiom  $\exists M^{ZFC}$  doesn't imply axiom  $\exists M_{st}^{ZFC}$ .

**Remark 1.1.6.** Note that in order to deduce: (i)  $\sim Con(ZFC_2^{Hs})$  from  $Con(ZFC_2^{Hs})$ , and (ii)  $\sim Con(ZFC)$  from  $Con(ZFC)$ , by using Gödel encoding, one needs something more

than the consistency of  $ZFC_2^{Hs}$ , e.g., that  $ZFC_2^{Hs}$  has an omega-model  $M_{\omega}^{ZFC_2^{Hs}}$  or a standard model  $M_{st}^{ZFC_2^{Hs}}$  i.e., a model in which the *integers are the standard integers*. To put it another way, why should we believe a statement just because there's a  $ZFC_2^{Hs}$ -proof of it? It's clear that if  $ZFC_2^{Hs}$  is inconsistent, then we won't believe  $ZFC_2^{Hs}$ -proofs. What's slightly more subtle is that the mere consistency of  $ZFC_2$  isn't quite enough to get us to believe arithmetical theorems of  $ZFC_2^{Hs}$ ; we must also believe that these arithmetical theorems are asserting something about the standard naturals. It is "conceivable" that  $ZFC_2^{Hs}$  might be consistent but that the only nonstandard models  $M_{Nst}^{ZFC_2^{Hs}}$  it has are those in which the integers are nonstandard, in which case we might not "believe" an arithmetical statement such as " $ZFC_2^{Hs}$  is inconsistent" even if there is a  $ZFC_2^{Hs}$ -proof of it.

## 2. Derivation of the inconsistent definable set in set theory $\overline{ZFC}_2^{Hs}$ and in set theory $ZFC_{st}$ .

### 2.1. Derivation of the inconsistent definable set in set theory $\overline{ZFC}_2^{Hs}$ .

We assume now that  $Con(Z_2^{Hs}; M_{st}^{Z_2^{Hs}})$ .

**Designation 2.1.1.** Let  $\Gamma_X^{Hs}$  be the collection of the all 1-place open wff of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 2.1.1.** Let  $\Psi_1(X), \Psi_2(X)$  be 1-place open wff's of the set theory  $\overline{ZFC}_2^{Hs}$ .

(i) We define now the equivalence relation  $(\cdot \sim_X \cdot) \subset \Gamma_X^{Hs} \times \Gamma_X^{Hs}$  by

$$\Psi_1(X) \sim \Psi_2(X) \Leftrightarrow \forall X[\Psi_1(X) \Leftrightarrow \Psi_2(X)] \quad (2.1.1)$$

(ii) A subset  $\Lambda_X^{Hs}$  of  $\Gamma_X^{Hs}$  such that  $\Psi_1(X) \sim \Psi_2(X)$  holds for all  $\Psi_1(X)$  and  $\Psi_2(X)$  in  $\Lambda_X^{Hs}$ , and never for  $\Psi_1(X)$  in  $\Lambda_X^{Hs}$  and  $\Psi_2(X)$  outside  $\Lambda_X^{Hs}$ , is called an equivalence class of  $\Gamma_X^{Hs}$ .

(iii) The collection of all possible equivalence classes of  $\Gamma_X^{Hs}$  by  $\sim_X$ , denoted  $\Gamma_X^{Hs}/\sim_X$

$$\Gamma_X^{Hs}/\sim_X \triangleq \{[\Psi(X)]_{Hs} | \Psi(X) \in \Gamma_X^{Hs}\}. \quad (2.1.2)$$

(iv) For any  $\Psi(X) \in \Gamma_X^{Hs}$  let  $[\Psi(X)]_{Hs} \triangleq \{\Phi(X) \in \Gamma_X^{Hs} | \Psi(X) \sim \Phi(X)\}$  denote the equivalence class to which  $\Psi(X)$  belongs. All elements of  $\Gamma_X^{Hs}$  equivalent to each other are also elements of the same equivalence class.

**Definition 2.1.2.**[9]. Let  $Th$  be any theory in the recursive language  $\mathcal{L}_{Th} \supset \mathcal{L}_{PA}$ , where  $\mathcal{L}_{PA}$  is a language of Peano arithmetic. We say that a number-theoretic relation  $R(x_1, \dots, x_n)$  of

$n$  arguments is expressible in  $Th$  if and only if there is a wff  $\hat{R}(x_1, \dots, x_n)$  of  $Th$  with the free

variables  $x_1, \dots, x_n$  such that, for any natural numbers  $k_1, \dots, k_n$ , the following hold:

(i) If  $R(k_1, \dots, k_n)$  is true, then  $\vdash_{Th} \hat{R}(\bar{k}_1, \dots, \bar{k}_n)$ .

(ii) If  $R(k_1, \dots, k_n)$  is false, then  $\vdash_{Th} \neg \hat{R}(\bar{k}_1, \dots, \bar{k}_n)$ .

**Designation 2.1.2.**(i) Let  $g_{ZFC_2^{Hs}}(u)$  be a Gödel number of given an expression  $u$  of the set theory  $\overline{ZFC}_2^{Hs} \triangleq ZFC_2^{Hs} + \exists M_{st}^{ZFC_2^{Hs}}$ .

(ii) Let  $\mathbf{Fr}_2^{Hs}(y, v)$  be the relation :  $y$  is the Gödel number of a wff of the set theory  $\overline{ZFC}_2^{Hs}$  that contains free occurrences of the variable  $X$  with Gödel number  $v$  [9].

(iii) Note that the relation  $\mathbf{Fr}_2^{Hs}(y, v)$  is expressible in  $\overline{ZFC}_2^{Hs}$  by a wff  $\widehat{\mathbf{Fr}}_2^{Hs}(y, v)$

(iv) Note that for any  $y, v \in \mathbb{N}$  by definition of the relation  $\mathbf{Fr}_2^{Hs}(y, v)$  follows that

$$\widehat{\mathbf{Fr}}_2^{Hs}(y, v) \Leftrightarrow \exists ! \Psi(X) \left[ \left( g_{\overline{ZFC}_2^{Hs}}(\Psi(X)) = y \right) \wedge \left( g_{\overline{ZFC}_2^{Hs}}(X) = v \right) \right], \quad (2.1.3)$$

where  $\Psi(X)$  is a unique wff of  $\overline{ZFC}_2^{Hs}$  which contains free occurrences of the variable  $X$  with Gödel number  $v$ . We denote a unique wff  $\Psi(X)$  defined by using equivalence (1.2.3)

by symbol  $\Psi_{y,v}(X)$ , i.e.

$$\widehat{\mathbf{Fr}}_2^{Hs}(y, v) \Leftrightarrow \exists ! \Psi_{y,v}(X) \left[ \left( g_{\overline{ZFC}_2^{Hs}}(\Psi_{y,v}(X)) = y \right) \wedge \left( g_{\overline{ZFC}_2^{Hs}}(X) = v \right) \right], \quad (2.1.4)$$

(v) Let  $\wp_2^{Hs}(y, v, v_1)$  be a Gödel number of the following wff:  $\exists ! X[\Psi(X) \wedge Y = X]$ , where  $g_{\overline{ZFC}_2^{Hs}}(\Psi(X)) = y, g_{\overline{ZFC}_2^{Hs}}(X) = v, g_{\overline{ZFC}_2^{Hs}}(Y) = v_1$ .

**Definition 2.1.3.** Let  $\Gamma_X^{Hs}$  be the countable collection of the all 1-place open wff's of the set theory  $\overline{ZFC}_2^{Hs}$  that contains free occurrences of the variable  $X$ .

**Definition 2.1.4.** Let  $g_{\overline{ZFC}_2^{Hs}}(X) = v$ . Let  $\Gamma_v^{Hs}$  be a set of the all Gödel numbers of the 1-place open wff's of the set theory  $\overline{ZFC}_2^{Hs}$  that contains free occurrences of the variable  $X$  with Gödel number  $v$ , i.e.

$$\Gamma_v^{Hs} = \{y \in \mathbb{N} | \langle y, v \rangle \in \mathbf{Fr}_2^{Hs}(y, v)\}, \quad (2.1.5)$$

or in the following equivalent form:

$$\forall y(y \in \mathbb{N}) \left[ y \in \Gamma_v \Leftrightarrow (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_2^{Hs}(y, v) \right]. \quad (2.1.6)$$

**Remark 2.1.1.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{Hs}$  is a set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 2.1.5.**(i) We define now the equivalence relation

$$(\cdot \sim_v \cdot) \subset \Gamma_v^{Hs} \times \Gamma_v^{Hs} \quad (2.1.7)$$

in the sense of the set theory  $\overline{ZFC}_2^{Hs}$  by

$$y_1 \sim_v y_2 \Leftrightarrow (\forall X[\Psi_{y_1, v}(X) \Leftrightarrow \Psi_{y_2, v}(X)]) \quad (2.1.8)$$

Note that from the axiom of separation it follows directly that the equivalence relation  $(\cdot \sim_v \cdot)$  is a relation in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

(ii) A subset  $\Lambda_v^{Hs}$  of  $\Gamma_v^{Hs}$  such that  $y_1 \sim_v y_2$  holds for all  $y_1$  and  $y_2$  in  $\Lambda_v^{Hs}$ , and never for  $y_1$  in

$\Lambda_v^{Hs}$  and  $y_2$  outside  $\Lambda_v^{Hs}$ , is an equivalence class of  $\Gamma_v^{Hs}$ .

(iii) For any  $y \in \Gamma_v^{Hs}$  let  $[y]_{Hs} \triangleq \{z \in \Gamma_v^{Hs} | y \sim_v z\}$  denote the equivalence class to which  $y$  belongs. All elements of  $\Gamma_v^{Hs}$  equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of  $\Gamma_v^{Hs}$  by  $\sim_v$ , denoted  $\Gamma_v^{Hs} / \sim_v$

$$\Gamma_v^{Hs} / \sim_v \triangleq \{[y]_{Hs} | y \in \Gamma_v^{Hs}\}. \quad (2.1.9)$$

**Remark 2.1.2.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{Hs} / \sim_v$  is a

set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 2.1.6.** Let  $\mathfrak{I}_2^{Hs}$  be the countable collection of the all sets definable by 1-place open wff of the set theory  $\overline{ZFC}_2^{Hs}$ , i.e.

$$\forall Y \{Y \in \mathfrak{I}_2^{Hs} \Leftrightarrow \exists \Psi(X) [([\Psi(X)]_{Hs} \in \Gamma_X^{Hs} / \sim_X) \wedge [\exists! X[\Psi(X) \wedge Y = X]]]\}. \quad (2.1.10)$$

**Definition 2.1.7.** We rewrite now (2.1.10) in the following equivalent form

$$\forall Y \{Y \in \mathfrak{I}_2^{Hs} \Leftrightarrow \exists \Psi(X) [([\Psi(X)]_{Hs} \in \Gamma_X^{*Hs} / \sim_X) \wedge (Y = X)]\}, \quad (2.1.11)$$

where the countable collection  $\Gamma_X^{*Hs} / \sim_X$  is defined by

$$\forall \Psi(X) \{[\Psi(X)] \in \Gamma_X^{*Hs} / \sim_X \Leftrightarrow [([\Psi(X)] \in \Gamma_X^{Hs} / \sim_X) \wedge \exists! X \Psi(X)]\} \quad (2.1.12)$$

**Definition 2.1.8.** Let  $\mathfrak{R}_2^{Hs}$  be the countable collection of the all sets such that

$$\forall X (X \in \mathfrak{I}_2^{Hs}) [X \in \mathfrak{R}_2^{Hs} \Leftrightarrow X \notin X]. \quad (2.1.13)$$

**Remark 2.1.3.** Note that  $\mathfrak{R}_2^{Hs} \in \mathfrak{I}_2^{Hs}$  since  $\mathfrak{R}_2^{Hs}$  is a collection definable by 1-place open wff

is definable by formula

$$\Psi(Z, \mathfrak{I}_2^{Hs}) \triangleq \forall X (X \in \mathfrak{I}_2^{Hs}) [X \in Z \Leftrightarrow X \notin X].$$

From (2.1.13) one obtains

$$\mathfrak{R}_2^{Hs} \in \mathfrak{R}_2^{Hs} \Leftrightarrow \mathfrak{R}_2^{Hs} \notin \mathfrak{R}_2^{Hs}. \quad (2.1.14)$$

But (2.1.14) gives a contradiction

$$(\mathfrak{R}_2^{Hs} \in \mathfrak{R}_2^{Hs}) \wedge (\mathfrak{R}_2^{Hs} \notin \mathfrak{R}_2^{Hs}). \quad (2.1.15)$$

However contradiction (2.1.15) it is not a contradiction inside  $\overline{ZFC}_2^{Hs}$  for the reason that

the countable collection  $\mathfrak{S}_2^{Hs}$  is not a set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**In order to obtain a contradiction inside  $\overline{ZFC}_2^{Hs}$  we introduce the following definitions.**

**Definition 2.1.9.** We define now the countable set  $\Gamma_v^{*Hs} / \sim_v$  by

$$\forall y \left\{ [y]_{Hs} \in \Gamma_v^{*Hs} / \sim_v \Leftrightarrow ([y]_{Hs} \in \Gamma_v^{Hs} / \sim_v) \wedge \widehat{\mathbf{Fr}}_2^{Hs}(y, v) \wedge [\exists! X \Psi_{y,v}(X)] \right\}. \quad (2.1.16)$$

**Remark 2.1.4.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{*Hs}$  is a set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 2.1.10.** We define now the countable set  $\mathfrak{S}_2^{*Hs}$  by formula

$$\forall Y \left\{ Y \in \mathfrak{S}_2^{*Hs} \Leftrightarrow \exists y \left[ ([y]_{Hs} \in \Gamma_v^{*Hs} / \sim_v) \wedge (g_{\overline{ZFC}_2^{Hs}}(X) = v) \wedge Y = X \right] \right\}. \quad (2.1.17)$$

Note that from the axiom schema of replacement (1.1.1) it follows directly that  $\mathfrak{S}_2^{*Hs}$  is a

set in the sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Definition 2.1.11.** We define now the countable set  $\mathfrak{R}_2^{*Hs}$  by formula

$$\forall X (X \in \mathfrak{S}_2^{*Hs}) [X \in \mathfrak{R}_2^{*Hs} \Leftrightarrow X \notin X]. \quad (2.1.18)$$

Note that from the axiom schema of separation it follows directly that  $\mathfrak{R}_2^{*Hs}$  is a set in the

sense of the set theory  $\overline{ZFC}_2^{Hs}$ .

**Remark 2.1.5.** Note that  $\mathfrak{R}_2^{*Hs} \in \mathfrak{S}_2^{*Hs}$  since  $\mathfrak{R}_2^{*Hs}$  is definable by the following formula

$$\Psi^*(Z) \triangleq \forall X (X \in \mathfrak{S}_2^{*Hs}) [X \in Z \Leftrightarrow X \notin X]. \quad (2.1.19)$$

**Theorem 2.1.1.** Set theory  $\overline{ZFC}_2^{Hs}$  is inconsistent.

Proof. From (2.1.18) and Remark 2.1.5 we obtain  $\mathfrak{R}_2^{*Hs} \in \mathfrak{R}_2^{*Hs} \Leftrightarrow \mathfrak{R}_2^{*Hs} \notin \mathfrak{R}_2^{*Hs}$  from which immediately one obtains a contradiction

$$(\mathfrak{R}_2^{*Hs} \in \mathfrak{R}_2^{*Hs}) \wedge (\mathfrak{R}_2^{*Hs} \notin \mathfrak{R}_2^{*Hs}). \quad (2.1.20)$$

## 2.2. Derivation of the inconsistent definable set in set theory $ZFC_{st}$ .

**Designation 2.2.1.** (i) Let  $g_{ZFC_{st}}(u)$  be a Gödel number of given an expression  $u$  of the set theory  $ZFC_{st} \triangleq ZFC + \exists M_{st}^{ZFC}$ .

(ii) Let  $\mathbf{Fr}_{st}(y, v)$  be the relation :  $y$  is the Gödel number of a wff of the set theory  $ZFC_{st}$  that contains free occurrences of the variable  $X$  with Gödel number  $v$  [9].

(iii) Note that the relation  $\mathbf{Fr}_{st}(y, v)$  is expressible in  $ZFC_{st}$  by a wff  $\widehat{\mathbf{Fr}}_{st}(y, v)$

(iv) Note that for any  $y, v \in \mathbb{N}$  by definition of the relation  $\mathbf{Fr}_{st}(y, v)$  follows that

$$\widehat{\mathbf{Fr}}_{st}(y, v) \Leftrightarrow \exists! \Psi(X) [(g_{ZFC_{st}}(\Psi(X)) = y) \wedge (g_{ZFC_{st}}(X) = v)], \quad (2.2.1)$$

where  $\Psi(X)$  is a unique wff of  $ZFC_{st}$  which contains free occurrences of the variable  $X$  with Gödel number  $v$ . We denote a unique wff  $\Psi(X)$  defined by using equivalence (2.2.1)

by symbol  $\Psi_{y,v}(X)$ , i.e.

$$\widehat{\mathbf{Fr}}_{st}(y, v) \Leftrightarrow \exists! \Psi_{y,v}(X) [(g_{ZFC_{st}}(\Psi_{y,v}(X)) = y) \wedge (g_{ZFC_{st}}(X) = v)], \quad (2.2.2)$$

(v) Let  $\wp_{st}(y, v, v_1)$  be a Gödel number of the following wff:  $\exists! X[\Psi(X) \wedge Y = X]$ , where  $g_{ZFC_{st}}(\Psi(X)) = y, g_{ZFC_{st}}(X) = v, g_{ZFC_{st}}(Y) = v_1$ .

(2.6) in section 2, see Remark 2.2 and Designation 2.3, (see also [8]-[9]).

**Definition 2.2.1.** Let  $\Gamma_X^{st}$  be the countable collection of the all 1-place open wff's of the set theory  $ZFC_{st}$  that contains free occurrences of the variable  $X$ .

**Definition 2.2.2.** Let  $g_{ZFC_{st}}(X) = v$ . Let  $\Gamma_v^{st}$  be a set of the all Gödel numbers of the 1-place open wff's of the set theory  $ZFC_{st}$  that contains free occurrences of the variable  $X$  with Gödel number  $v$ , i.e.

$$\Gamma_v^{st} = \{y \in \mathbb{N} | \langle y, v \rangle \in \mathbf{Fr}_{st}(y, v)\}, \quad (2.2.3)$$

or in the following equivalent form:

$$\forall y(y \in \mathbb{N}) \left[ y \in \Gamma_v^{st} \Leftrightarrow (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_{st}(y, v) \right].$$

**Remark 2.2.1.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{st}$  is a set in the sense of the set theory  $ZFC_{st}$ .

**Definition 2.2.3.** (i) We define now the equivalence relation  $(\cdot \sim_X \cdot) \subset \Gamma_X^{st} \times \Gamma_X^{st}$  by

$$\Psi_1(X) \sim_X \Psi_2(X) \Leftrightarrow (\forall X[\Psi_1(X) \Leftrightarrow \Psi_2(X)]) \quad (2.2.4)$$

(ii) A subcollection  $\Lambda_X^{st}$  of  $\Gamma_X^{st}$  such that  $\Psi_1(X) \sim_X \Psi_2(X)$  holds for all  $\Psi_1(X)$  and  $\Psi_2(X)$  in

$\Lambda_X^{st}$ , and never for  $\Psi_1(X)$  in  $\Lambda_X^{st}$  and  $\Psi_2(X)$  outside  $\Lambda_X^{st}$ , is an equivalence class of  $\Gamma_X^{st}$ .

(iii) For any  $\Psi(X) \in \Gamma_X^{st}$  let  $[\Psi(X)]_{st} \triangleq \{\Phi(X) \in \Gamma_X^{st} | \Psi(X) \sim_X \Phi(X)\}$  denote the equivalence

class to which  $\Psi(X)$  belongs. All elements of  $\Gamma_X^{st}$  equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of  $\Gamma_X^{st}$  by  $\sim_X$ , denoted  $\Gamma_X^{st}/\sim_X$

$$\Gamma_X^{st}/\sim_X \triangleq \{[\Psi(X)]_{st} | \Psi(X) \in \Gamma_X^{st}\}. \quad (2.2.5)$$

**Definition 2.2.4.** (i) We define now the equivalence relation  $(\cdot \sim_v \cdot) \subset \Gamma_v^{st} \times \Gamma_v^{st}$  in the sense of the set theory  $ZFC_{st}$  by

$$y_1 \sim_v y_2 \Leftrightarrow (\forall X[\Psi_{y_1, v}(X) \Leftrightarrow \Psi_{y_2, v}(X)]) \quad (2.2.6)$$

Note that from the axiom of separation it follows directly that the equivalence relation  $(\cdot \sim_v \cdot)$  is a relation in the sense of the set theory  $ZFC_{st}$ .

(ii) A subset  $\Lambda_v^{st}$  of  $\Gamma_v^{st}$  such that  $y_1 \sim_v y_2$  holds for all  $y_1$  and  $y_2$  in  $\Lambda_v^{st}$ , and never for  $y_1$  in  $\Lambda_v^{st}$  and  $y_2$  outside  $\Lambda_v^{st}$ , is an equivalence class of  $\Gamma_v^{st}$ .

(iii) For any  $y \in \Gamma_v^{st}$  let  $[y]_{st} \triangleq \{z \in \Gamma_v^{st} | y \sim_v z\}$  denote the equivalence class to which  $y$  belongs. All elements of  $\Gamma_v^{st}$  equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of  $\Gamma_v^{st}$  by  $\sim_v$ , denoted  $\Gamma_v^{st}/\sim_v$

$$\Gamma_v^{st}/\sim_v \triangleq \{[y]_{st} | y \in \Gamma_v^{st}\}. \quad (2.2.7)$$

**Remark 2.2.2.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{st}/\sim_v$  is a

set in the sense of the set theory  $ZFC_{st}$ .

**Definition 2.2.5.** Let  $\mathfrak{S}_{st}$  be the countable collection of all sets definable by 1-place

open wff of the set theory  $ZFC_{st}$ , i.e.

$$\forall Y \{ Y \in \mathfrak{S}_{st} \Leftrightarrow \exists \Psi(X) [ ([\Psi(X)]_{st} \in \Gamma_X^{st} / \sim_X ) \wedge [\exists ! X [\Psi(X) \wedge Y = X]] ] \}. \quad (2.2.8)$$

**Definition 2.2.6.** We rewrite now (2.2.8) in the following equivalent form

$$\forall Y \{ Y \in \mathfrak{S}_{st} \Leftrightarrow \exists \Psi(X) [ ([\Psi(X)]_{st} \in \Gamma_X^{*st} / \sim_X ) \wedge (Y = X) ] \}, \quad (2.2.9)$$

where the countable collection  $\Gamma_X^{*st} / \sim_X$  is defined by

$$\forall \Psi(X) \{ [\Psi(X)]_{st} \in \Gamma_X^{*st} / \sim_X \Leftrightarrow [ ([\Psi(X)]_{st} \in \Gamma_X^{st} / \sim_X ) \wedge \exists ! X \Psi(X) ] \} \quad (2.2.10)$$

**Definition 2.2.7.** Let  $\mathfrak{R}_{st}$  be the countable collection of the all sets such that

$$\forall X (X \in \mathfrak{S}_{st}) [ X \in \mathfrak{R}_{st} \Leftrightarrow X \notin X ]. \quad (2.2.11)$$

**Remark 2.2.3.** Note that  $\mathfrak{R}_{st} \in \mathfrak{S}_{st}$  since  $\mathfrak{R}_{st}$  is a collection definable by 1-place open wff

is definable by formula

$$\Psi(Z, \mathfrak{S}_{st}) \triangleq \forall X (X \in \mathfrak{S}_{st}) [ X \in Z \Leftrightarrow X \notin X ].$$

From (2.2.11) and Remark 2.2.3 one obtains directly

$$\mathfrak{R}_{st} \in \mathfrak{R}_{st} \Leftrightarrow \mathfrak{R}_{st} \notin \mathfrak{R}_{st}. \quad (2.2.12)$$

But (2.2.12) immediately gives a contradiction

$$(\mathfrak{R}_{st} \in \mathfrak{R}_{st}) \wedge (\mathfrak{R}_{st} \notin \mathfrak{R}_{st}). \quad (2.2.13)$$

However contradiction (2.2.13) it is not a true contradiction inside  $ZFC_{st}$  for the reason that the countable collection  $\mathfrak{S}_{st}$  is not a set in the sense of the set theory  $ZFC_{st}$ .

**In order to obtain a true contradiction inside  $ZFC_{st}$  we introduce the following definitions.**

**Definition 2.2.8.** We define now the countable set  $\Gamma_v^{*st} / \sim_v$  by formula

$$\forall y \left\{ [y]_{st} \in \Gamma_v^{*st} / \sim_v \Leftrightarrow ([y]_{st} \in \Gamma_v^{st} / \sim_v ) \wedge \widehat{\mathbf{Fr}}_{st}(y, v) \wedge [\exists ! X \Psi_{y,v}(X)] \right\}. \quad (2.2.14)$$

**Remark 2.2.4.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{*st} / \sim_v$  is a set in the sense of the set theory  $ZFC_{st}$ .

**Definition 2.2.9.** We define now the countable set  $\mathfrak{S}_{st}^*$  by formula

$$\forall Y \{ Y \in \mathfrak{S}_{st}^* \Leftrightarrow \exists y [ ([y]_{st} \in \Gamma_v^{*st} / \sim_v ) \wedge (g_{ZFC_{st}}(X) = v) \wedge Y = X ] \}. \quad (2.2.15)$$

Note that from the axiom schema of replacement it follows directly that  $\mathfrak{S}_{st}^*$  is a set in the sense of the set theory  $ZFC_{st}$ .

**Definition 2.2.10.** We define now the countable set  $\mathfrak{R}_{st}^*$  by formula

$$\forall X (X \in \mathfrak{S}_{st}^*) [ X \in \mathfrak{R}_{st}^* \Leftrightarrow X \notin X ]. \quad (2.2.16)$$

Note that from the axiom schema of separation it follows directly that  $\mathfrak{R}_{st}^*$  is a set in the sense of the set theory  $ZFC_{st}$ .

**Remark 2.2.5.** Note that  $\mathfrak{R}_{st}^* \in \mathfrak{S}_{st}^*$  since  $\mathfrak{R}_{st}^*$  is definable by the following formula

$$\Psi^*(Z) \triangleq \forall X (X \in \mathfrak{S}_{st}^*) [ X \in Z \Leftrightarrow X \notin X ]. \quad (2.2.17)$$

**Theorem 2.2.1.**[11]. Set theory  $ZFC_{st}$  is inconsistent.

Proof. From (2.2.17) and Remark 2.2.5 we obtain  $\mathfrak{R}_{st}^* \in \mathfrak{R}_{st}^* \Leftrightarrow \mathfrak{R}_{st}^* \notin \mathfrak{R}_{st}^*$  from which immediately one obtains a contradiction

$$(\mathfrak{R}_{st}^* \in \mathfrak{R}_{st}^*) \wedge (\mathfrak{R}_{st}^* \notin \mathfrak{R}_{st}^*). \quad (2.2.18)$$

**Remark 2.2.6.** Theorem 2.2.1 originally was proved in papers [11]-[13] by using another essentially complicated approach.

## 2.3. Derivation of the inconsistent definable set in $ZFC_{Nst}$ .

**Definition 2.3.1.** Let  $\overline{PA}$  be a first order theory which contain usual postulates of Peano arithmetic [9] and recursive defining equations for every primitive recursive function as desired. So for any  $(n + 1)$ -place function  $f$  defined by primitive recursion over any  $n$ -place

base function  $g$  and  $(n + 2)$ -place iteration function  $h$  there would be the defining equations:

$$(i) f(0, y_1, \dots, y_n) = g(y_1, \dots, y_n), (ii) f(x + 1, y_1, \dots, y_n) = h(x, f(x, y_1, \dots, y_n), y_1, \dots, y_n).$$

**Designation 2.3.1.** (i) Let  $M_{Nst}^{ZFC}$  be a nonstandard model of  $ZFC$  and let  $M_{st}^{\overline{PA}}$  be a standard

model of  $\overline{PA}$ . We assume now that  $M_{st}^{\overline{PA}} \subset M_{Nst}^{ZFC}$  and denote such nonstandard model of the set theory  $ZFC$  by  $M_{Nst}^{ZFC}[\overline{PA}]$ . (ii) Let  $ZFC_{Nst}$  be the theory

$$ZFC_{Nst} = ZFC + M_{Nst}^{ZFC}[\overline{PA}].$$

**Designation 2.3.2.** (i) Let  $g_{ZFC_{Nst}}(u)$  be a Gödel number of given an expression  $u$  of the set theory  $ZFC_{Nst} \triangleq ZFC + \exists M_{Nst}^{ZFC}[\overline{PA}]$ .

(ii) Let  $\mathbf{Fr}_{Nst}(y, v)$  be the relation :  $y$  is the Gödel number of a wff of the set theory  $ZFC_{Nst}$

that contains free occurrences of the variable  $X$  with Gödel number  $v$  [9].

(iii) Note that the relation  $\mathbf{Fr}_{Nst}(y, v)$  is expressible in  $ZFC_{Nst}$  by a wff  $\widehat{\mathbf{Fr}}_{Nst}(y, v)$

(iv) Note that for any  $y, v \in \mathbb{N}$  by definition of the relation  $\mathbf{Fr}_{Nst}(y, v)$  follows that

$$\widehat{\mathbf{Fr}}_{Nst}(y, v) \Leftrightarrow \exists! \Psi(X) [(g_{ZFC_{Nst}}(\Psi(X)) = y) \wedge (g_{ZFC_{Nst}}(X) = v)], \quad (2.3.1)$$

where  $\Psi(X)$  is a unique wff of  $ZFC_{st}$  which contains free occurrences of the variable  $X$  with Gödel number  $v$ . We denote a unique wff  $\Psi(X)$  defined by using equivalence (2.3.1)

by symbol  $\Psi_{y,v}(X)$ , i.e.

$$\widehat{\mathbf{Fr}}_{Nst}(y, v) \Leftrightarrow \exists! \Psi_{y,v}(X) [(g_{ZFC_{Nst}}(\Psi_{y,v}(X)) = y) \wedge (g_{ZFC_{Nst}}(X) = v)], \quad (2.3.2)$$

(v) Let  $\wp_{Nst}(y, v, v_1)$  be a Gödel number of the following wff:  $\exists! X [\Psi(X) \wedge Y = X]$ , where

$$g_{ZFC_{Nst}}(\Psi(X)) = y, g_{ZFC_{Nst}}(X) = v, g_{ZFC_{Nst}}(Y) = v_1.$$

**Definition 2.3.2.** Let  $\Gamma_X^{Nst}$  be the countable collection of the all 1-place open wff's of the set theory  $ZFC_{Nst}$  that contains free occurrences of the variable  $X$ .

**Definition 2.3.3.** Let  $g_{ZFC_{Nst}}(X) = v$ . Let  $\Gamma_v^{Nst}$  be a set of the all Gödel numbers of the 1-place open wff's of the set theory  $ZFC_{Nst}$  that contains free occurrences of the variable  $X$

with Gödel number  $v$ , i.e.

$$\Gamma_v^{Nst} = \{y \in \mathbb{N} | \langle y, v \rangle \in \mathbf{Fr}_{Nst}(y, v)\}, \quad (2.3.3)$$

or in the following equivalent form:

$$\forall y (y \in \mathbb{N}) \left[ y \in \Gamma_v^{Nst} \Leftrightarrow (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_{Nst}(y, v) \right].$$

**Remark 2.3.1.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{st}$  is a set in the sense of the set theory  $ZFC_{Nst}$ .

**Definition 2.3.3.**(i) We define now the equivalence relation  $(\cdot \sim_X \cdot) \subset \Gamma_X^{Nst} \times \Gamma_X^{Nst}$  by

$$\Psi_1(X) \sim_X \Psi_2(X) \Leftrightarrow (\forall X[\Psi_1(X) \Leftrightarrow \Psi_2(X)]) \quad (2.3.4)$$

(ii) A subcollection  $\Lambda_X^{st}$  of  $\Gamma_X^{st}$  such that  $\Psi_1(X) \sim_X \Psi_2(X)$  holds for all  $\Psi_1(X)$  and  $\Psi_2(X)$  in  $\Lambda_X^{st}$ , and never for  $\Psi_1(X)$  in  $\Lambda_X^{st}$  and  $\Psi_2(X)$  outside  $\Lambda_X^{st}$ , is an equivalence class of  $\Gamma_X^{Nst}$ .

(iii) For any  $\Psi(X) \in \Gamma_X^{Nst}$  let  $[\Psi(X)]_{Nst} \triangleq \{\Phi(X) \in \Gamma_X^{Nst} | \Psi(X) \sim_X \Phi(X)\}$  denote the equivalence class to which  $\Psi(X)$  belongs. All elements of  $\Gamma_X^{st}$  equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of  $\Gamma_X^{Nst}$  by  $\sim_X$ , denoted  $\Gamma_X^{Nst} / \sim_X$

$$\Gamma_X^{Nst} / \sim_X \triangleq \{[\Psi(X)]_{Nst} | \Psi(X) \in \Gamma_X^{Nst}\}. \quad (2.3.5)$$

**Definition 2.3.4.**(i) We define now the equivalence relation  $(\cdot \sim_v \cdot) \subset \Gamma_v^{Nst} \times \Gamma_v^{Nst}$  in the sense of the set theory  $ZFC_{Nst}$  by

$$y_1 \sim_v y_2 \Leftrightarrow (\forall X[\Psi_{y_1,v}(X) \Leftrightarrow \Psi_{y_2,v}(X)]) \quad (2.3.6)$$

Note that from the axiom of separation it follows directly that the equivalence relation  $(\cdot \sim_v \cdot)$  is a relation in the sense of the set theory  $ZFC_{Nst}$ .

(ii) A subset  $\Lambda_v^{Nst}$  of  $\Gamma_v^{Nst}$  such that  $y_1 \sim_v y_2$  holds for all  $y_1$  and  $y_2$  in  $\Lambda_v^{Nst}$ , and never for  $y_1$  in  $\Lambda_v^{Nst}$  and  $y_2$  outside  $\Lambda_v^{Nst}$ , is an equivalence class of  $\Gamma_v^{Nst}$ .

(iii) For any  $y \in \Gamma_v^{Nst}$  let  $[y]_{Nst} \triangleq \{z \in \Gamma_v^{Nst} | y \sim_v z\}$  denote the equivalence class to which  $y$  belongs. All elements of  $\Gamma_v^{Nst}$  equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of  $\Gamma_v^{Nst}$  by  $\sim_v$ , denoted  $\Gamma_v^{Nst} / \sim_v$

$$\Gamma_v^{Nst} / \sim_v \triangleq \{[y]_{Nst} | y \in \Gamma_v^{Nst}\}. \quad (2.3.7)$$

**Remark 2.3.2.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{Nst} / \sim_v$  is a

set in the sense of the set theory  $ZFC_{Nst}$ .

**Definition 2.3.5.** Let  $\mathfrak{S}_{Nst}$  be the countable collection of the all sets definable by 1-place open wff of the set theory  $ZFC_{Nst}$ , i.e.

$$\forall Y \{Y \in \mathfrak{S}_{Nst} \Leftrightarrow \exists \Psi(X)[([\Psi(X)]_{Nst} \in \Gamma_X^{Nst} / \sim_X) \wedge [\exists! X[\Psi(X) \wedge Y = X]]]\}. \quad (2.3.8)$$

**Definition 2.3.6.** We rewrite now (2.3.8) in the following equivalent form

$$\forall Y \{Y \in \mathfrak{S}_{Nst} \Leftrightarrow \exists \Psi(X)[([\Psi(X)]_{Nst} \in \Gamma_X^{*Nst} / \sim_X) \wedge (Y = X)]\}, \quad (2.3.9)$$

where the countable collection  $\Gamma_X^{*Nst} / \sim_X$  is defined by formula

$$\forall \Psi(X) \{[\Psi(X)]_{Nst} \in \Gamma_X^{*Nst} / \sim_X \Leftrightarrow ([\Psi(X)]_{Nst} \in \Gamma_X^{Nst} / \sim_X) \wedge \exists! X \Psi(X)\} \quad (2.3.10)$$

**Definition 2.3.7.** Let  $\mathfrak{R}_{Nst}$  be the countable collection of the all sets such that

$$\forall X (X \in \mathfrak{S}_{Nst}) [X \in \mathfrak{R}_{Nst} \Leftrightarrow X \notin X]. \quad (2.3.11)$$

**Remark 2.3.3.** Note that  $\mathfrak{R}_{Nst} \in \mathfrak{S}_{Nst}$  since  $\mathfrak{R}_{Nst}$  is a collection definable by 1-place open

wff is definable by formula

$$\Psi(Z, \mathfrak{I}_{Nst}) \triangleq \forall X(X \in \mathfrak{I}_{Nst})[X \in Z \Leftrightarrow X \notin X].$$

From (2.3.11) one obtains

$$\mathfrak{R}_{Nst} \in \mathfrak{R}_{Nst} \Leftrightarrow \mathfrak{R}_{Nst} \notin \mathfrak{R}_{Nst}. \quad (2.3.12)$$

But (2.3.12) gives a contradiction

$$(\mathfrak{R}_{Nst} \in \mathfrak{R}_{Nst}) \wedge (\mathfrak{R}_{Nst} \notin \mathfrak{R}_{Nst}). \quad (2.3.13)$$

However a contradiction (2.3.13) it is not a true contradiction inside  $ZFC_{Nst}$  for the reason

that the countable collection  $\mathfrak{I}_{Nst}$  is not a set in the sense of the set theory  $ZFC_{Nst}$ .

**In order to obtain a true contradiction inside  $ZFC_{Nst}$  we introduce the following definitions.**

**Definition 2.3.8.** We define now the countable set  $\Gamma_v^{*Nst} / \sim_v$  by formula

$$\forall y \left\{ [y]_{Nst} \in \Gamma_v^{*Nst} / \sim_v \Leftrightarrow ([y]_{Nst} \in \Gamma_v^{Nst} / \sim_v) \wedge \widehat{\mathbf{Fr}}_{Nst}(y, v) \wedge [\exists! X \Psi_{y,v}(X)] \right\}. \quad (2.3.14)$$

**Remark 2.3.4.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{*Nst} / \sim_v$  is

a set in the sense of the set theory  $ZFC_{st}$ .

**Definition 2.3.9.** We define now the countable set  $\mathfrak{I}_{Nst}^*$  by formula

$$\forall Y \{ Y \in \mathfrak{I}_{Nst}^* \Leftrightarrow \exists y ([y]_{Nst} \in \Gamma_v^{*Nst} / \sim_v) \wedge (g_{ZFC_{Nst}}(X) = v) \wedge Y = X \}. \quad (2.3.15)$$

Note that from the axiom schema of replacement it follows directly that  $\mathfrak{I}_{st}^*$  is a set in the

sense of the set theory  $ZFC_{Nst}$ .

**Definition 2.3.10.** We define now the countable set  $\mathfrak{R}_{Nst}^*$  by formula

$$\forall X(X \in \mathfrak{I}_{Nst}^*)[X \in \mathfrak{R}_{Nst}^* \Leftrightarrow X \notin X]. \quad (2.3.16)$$

Note that from the axiom schema of separation it follows directly that  $\mathfrak{R}_{Nst}^*$  is a set in the

sense of the set theory  $ZFC_{Nst}$ .

**Remark 2.3.5.** Note that  $\mathfrak{R}_{Nst}^* \in \mathfrak{I}_{Nst}^*$  since  $\mathfrak{R}_{Nst}^*$  is definable by the following formula

$$\Psi^*(Z) \triangleq \forall X(X \in \mathfrak{I}_{Nst}^*)[X \in Z \Leftrightarrow X \notin X]. \quad (2.3.17)$$

**Theorem 2.3.1.** Set theory  $ZFC_{Nst}$  is inconsistent.

Proof. From (2.3.16) and Remark 2.3.5 we obtain  $\mathfrak{R}_{Nst}^* \in \mathfrak{R}_{Nst}^* \Leftrightarrow \mathfrak{R}_{Nst}^* \notin \mathfrak{R}_{Nst}^*$  from which one obtains a contradiction

$$(\mathfrak{R}_{Nst}^* \in \mathfrak{R}_{Nst}^*) \wedge (\mathfrak{R}_{Nst}^* \notin \mathfrak{R}_{Nst}^*). \quad (2.3.18)$$

### 3. Avoiding the contradictions from set theory $\overline{ZFC}_2^{Hs}$ and set theory $ZFC_{st}$ using Quinean approach.

In order to avoid difficulties mentioned above we use well known Quinean approach [14].

#### 3.1. Quinean set theory NF.

Remind that the primitive predicates of Russellian unramified typed set theory (TST),

a streamlined version of the theory of types, are equality = and membership  $\in$ . TST has a linear hierarchy of types: type 0 consists of individuals otherwise undescribed. For each (meta-) natural number  $n$ , type  $n + 1$  objects are sets of type  $n$  objects; sets of type  $n$  have members of type  $n - 1$ . Objects connected by identity must have the same type. The following two atomic formulas succinctly describe the typing rules:  $x^n = y^n$  and  $x^n \in y^{n+1}$ .

The axioms of TST are:

**Extensionality:** sets of the same (positive) type with the same members are equal;

**Axiom schema of comprehension:**

If  $\Phi(x^n)$  is a formula, then the set  $\{x^n \mid \Phi(x^n)\}^{n+1}$  exists i.e., given any formula  $\Phi(x^n)$ , the formula

$$\exists A^{n+1} \forall x^n [x^n \in A^{n+1} \leftrightarrow \Phi(x^n)] \quad (3.1.1)$$

is an axiom where  $A^{n+1}$  represents the set  $\{x^n \mid \Phi(x^n)\}^{n+1}$  and is not free in  $\Phi(x^n)$ .

Quinean set theory [14] (New Foundations) seeks to eliminate the need for such superscripts.

New Foundations has a universal set, so it is a non-well founded set theory. That is to say, it is a logical theory that allows infinite descending chains of membership such as

...

$x_n \in x_{n-1} \in \dots x_3 \in x_2 \in x_1$ . It avoids Russell's paradox by only allowing stratifiable formulae in the axiom of comprehension. For instance  $x \in y$  is a stratifiable formula, but  $x \in x$  is not (for details of how this works see below).

**Definition 3.1.1.** In New Foundations (*NF*) and related set theories, a formula  $\Phi$  in the language of first-order logic with equality and membership is said to be stratified if and only if there is a function  $\sigma$  which sends each variable appearing in  $\Phi$  [considered as an item of syntax] to a natural number (this works equally well if all integers are used) in such a way that any atomic formula  $x \in y$  appearing in  $\Phi$  satisfies  $\sigma(x) + 1 = \sigma(y)$  and any atomic formula  $x = y$  appearing in  $\Phi$  satisfies  $\sigma(x) = \sigma(y)$ .

Quinean set theory.

**Axioms and stratification are:**

The well-formed formulas of New Foundations (*NF*) are the same as the well-formed formulas of TST, but with the type annotations erased. The axioms of *NF* are:

**Extensionality:** Two objects with the same elements are the same object;

A comprehension schema: All instances of TST Comprehension but with type indices dropped (and without introducing new identifications between variables).

By convention, *NF*'s Comprehension schema is stated using the concept of stratified formula and making no direct reference to types. Comprehension then becomes.

**Axiom schema of comprehension:**

$\{x \mid \Phi^s\}$  exists for each stratified formula  $\Phi^s$ .

Even the indirect reference to types implicit in the notion of stratification can be eliminated. Theodore Hailperin showed in 1944 that Comprehension is equivalent to a finite conjunction of its instances,[15] so that *NF* can be finitely axiomatized without any reference to the notion of type. Comprehension may seem to run afoul of problems similar to those in naive set theory, but this is not the case. For example, the existence of the impossible Russell class  $\{x \mid x \notin x\}$  is not an axiom of *NF*, because  $x \notin x$  cannot be stratified.

### 3.2. Set theory $\overline{ZFC}_2^{Hs}$ , $ZFC_{st}$ and set theory $ZFC_{Nst}$ with stratified axiom schema of replacement.

The stratified axiom schema of replacement asserts that the image of a set under any function definable by stratified formula of the theory  $ZFC_{st}$  will also fall inside a set.

**Stratified Axiom schema of replacement:**

Let  $\Phi^s(x, y, w_1, w_2, \dots, w_n)$  be any stratified formula in the language of  $ZFC_{st}$  whose free variables are among  $x, y, A, w_1, w_2, \dots, w_n$ , so that in particular  $B$  is not free in  $\Phi^s$ . Then

$$\begin{aligned} \forall A \forall w_1 \forall w_2 \dots \forall w_n [\forall x (x \in A \Rightarrow \exists! y \Phi^s(x, y, w_1, w_2, \dots, w_n)) \Rightarrow \\ \Rightarrow \exists B \forall x (x \in A \Rightarrow \exists y (y \in B \wedge \Phi^s(x, y, w_1, w_2, \dots, w_n)))] \end{aligned} \quad (3.2.1)$$

i.e., if the relation  $\Phi^s(x, y, \dots)$  represents a definable function  $f$ ,  $A$  represents its domain, and  $f(x)$  is a set for every  $x \in A$ , then the range of  $f$  is a subset of some set  $B$ .

**Stratified Axiom schema of separation:**

Let  $\Phi^s(x, w_1, w_2, \dots, w_n)$  be any stratified formula in the language of  $ZFC_{st}$  whose free variables are among  $x, A, w_1, w_2, \dots, w_n$ , so that in particular  $B$  is not free in  $\Phi^s$ . Then

$$\forall w_1 \forall w_2 \dots \forall w_n \forall A \exists B \forall x [x \in B \Leftrightarrow (x \in A \wedge \Phi^s(x, w_1, w_2, \dots, w_n))], \quad (3.2.2)$$

**Remark 3.2.1.** Notice that the stratified axiom schema of separation follows from the stratified axiom schema of replacement together with the axiom of empty set.

**Remark 3.2.2.** Notice that the stratified axiom schema of replacement (separation) obviously violated any contradictions (2.1.20), (2.2.18) and (2.3.18) mentioned above. The existence of the countable Russell sets  $\mathfrak{R}_2^{*Hs}$ ,  $\mathfrak{R}_{st}^*$  and  $\mathfrak{R}_{Nst}^*$  impossible, because  $x \notin x$  cannot be stratified.

## 4. Second-order set theory $ZFC_2$ with the full second-order semantics.

### 4.1. Second order set theory $ZFC_2$ with urlogic.

Remind that urlogic has the following characteristics [6]:

1. Sentences of urlogic are finite strings of symbols. That a string of symbols is a sentence of urlogic, is a non-mathematical judgement.
2. Some sentences are accepted as axioms. That a sentence is an axiom, is a non-mathematical judgement.
3. Derivations are made from axioms. The derivations obey certain rules of proof. That a derivation obeys the rules of proof, is a non-mathematical judgement.
4. Derived sentences can be asserted as facts.

**Remark 4.1.1.** Let  $ZFC_2^{Ul}$  be second order set theory  $ZFC_2$  with urlogic. Note that in  $ZFC_2^{Ul}$

by using the rules of **DED**<sub>2</sub> we dealing without any reference to semantics, i.e. satisfiability in some standard model, validity etc.

**Definition 4.1.1.** Let  $\Gamma_X^{Ul}$  be the countable collection of the all 1-place open wff's of the set theory  $ZFC_2^{Ul}$  that contains free occurrences of the variable  $X$ .

Let  $\Psi_1(X), \Psi_2(X)$  be 1-place open wff's of the set theory  $ZFC_2^{Ul}$ . We define now the equivalence relation  $(\cdot \sim_X \cdot) \subset \Gamma_X^{Ul} \times \Gamma_X^{Ul}$  by

$$\Psi_1(X) \sim_X \Psi_2(X) \Leftrightarrow \forall X[\Psi_1(X) \Leftrightarrow \Psi_2(X)] \quad (4.1.1)$$

For any  $\Psi(X) \in \Gamma_X^{Ul}$  let  $[\Psi(X)]_{Ul} \triangleq \{\Phi(X) \in \Gamma_X^{Ul} | \Psi(X) \sim \Phi(X)\}$  denote the equivalence class to which  $\Psi(X)$  belongs. All elements of  $\Gamma_X^{Ul}$  equivalent to each other are also elements of the same equivalence class. The collection of all possible equivalence classes of  $\Gamma_X^{Ul}$  by  $\sim_X$ , denoted  $\Gamma_X^{Ul} / \sim_X$

$$\Gamma_X^{Ul} / \sim_X \triangleq \{[\Psi(X)]_{Ul} | \Psi(X) \in \Gamma_X^{Ul}\}. \quad (4.1.2)$$

Let  $\mathbf{Fr}_2^{Ul}(y, v)$  be the relation :  $y$  is the Gödel number of a wff of the set theory  $ZFC_2^{Ul}$  that

contains free occurrences of the variable  $X$  with Gödel number  $v$  [9].

Note that the relation  $\mathbf{Fr}_2^{Ul}(y, v)$  is expressible in  $ZFC_2^{Ul}$  by a wff  $\widehat{\mathbf{Fr}}_2^{Ul}(y, v)$

Note that for any  $y, v \in \mathbb{N}$  by definition of the relation  $\mathbf{Fr}_2^{Ul}(y, v)$  follows that

$$\widehat{\mathbf{Fr}}_2^{Ul}(y, v) \Leftrightarrow \exists ! \Psi(X) \left[ \left( g_{ZFC_2^{Ul}}(\Psi(X)) = y \right) \wedge \left( g_{ZFC_2^{Ul}}(X) = v \right) \right], \quad (4.1.3)$$

where  $\Psi(X)$  is a unique wff of  $ZFC_2^{Ul}$  which contains free occurrences of the variable  $X$  with Gödel number  $v$ . We denote a unique wff  $\Psi(X)$  defined by using equivalence

(4.1.3)

by symbol  $\Psi_{y,v}^{Ul}(X)$ , i.e.

$$\widehat{\mathbf{Fr}}_2^{Ul}(y, v) \Leftrightarrow \exists ! \Psi_{y,v}^{Ul}(X) \left[ \left( g_{ZFC_2^{Ul}}(\Psi_{y,v}^{Ul}(X)) = y \right) \wedge \left( g_{ZFC_2^{Ul}}(X) = v \right) \right]. \quad (4.1.4)$$

**Definition 4.1.2.** Let  $g_{ZFC_2^{Ul}}(X) = v$ . Let  $\Gamma_v^{Ul}$  be a set of the all Gödel numbers of the 1-place open wff's of the set theory  $ZFC_2^{Ul}$  that contains free occurrences of the variable  $X$  with Gödel number  $v$ , i.e.

$$\Gamma_v^{Ul} = \{y \in \mathbb{N} | \langle y, v \rangle \in \mathbf{Fr}_2^{Ul}(y, v)\}, \quad (4.1.5)$$

or in the following equivalent form:

$$\forall y(y \in \mathbb{N}) \left[ y \in \Gamma_v^{Ul} \Leftrightarrow (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_2^{Ul}(y, v) \right]. \quad (4.1.6)$$

**Remark 4.1.2.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{Ul}$  is a set in the sense of the set theory  $ZFC_2^{Ul}$ .

**Definition 4.1.3.**(i) We define now the equivalence relation

$$(\cdot \sim_v \cdot) \subset \Gamma_v^{Ul} \times \Gamma_v^{Ul} \quad (4.1.7)$$

in the sense of the set theory  $ZFC_2^{Ul}$  by

$$y_1 \sim_v y_2 \Leftrightarrow (\forall X[\Psi_{y_1,v}^{Ul}(X) \Leftrightarrow \Psi_{y_2,v}^{Ul}(X)]) \quad (4.1.8)$$

For any  $y_1 \in \Gamma_v^{Ul}$  let  $[y_1]_{Ul} \triangleq \{y \in \Gamma_v^{Ul} | y_1 \sim_v y\}$  denote the equivalence class to which  $y_1$

belongs. The collection of all possible equivalence classes of  $\Gamma_v^{Ul}$  by  $\sim_v$ , denoted  $\Gamma_v^{Ul} / \sim_v$

$$\Gamma_v^{Ul} / \sim_v \triangleq \{[y]_{Ul} | y \in \Gamma_v^{Ul}\}. \quad (4.1.9)$$

**Remark 4.1.3.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{Hs} / \sim_v$  is a

set in the sense of the set theory  $ZFC_2^{Ul}$ .

**Definition 4.1.4.** Let  $\mathfrak{I}_2^{Ul}$  be the countable collection of all sets definable by 1-place

open

wff of the set theory  $ZFC_2^{Ul}$ , i.e.

$$\forall Y \{ Y \in \mathfrak{I}_2^{Ul} \Leftrightarrow \exists \Psi(X) [ ([\Psi(X)]_{Ul} \in \Gamma_X^{Ul} / \sim_X ) \wedge [\exists ! X [\Psi(X) \wedge Y = X]] ] \}. \quad (4.1.10)$$

**Definition 4.1.5.** We rewrite now (4.1.10) in the following equivalent form

$$\forall Y \{ Y \in \mathfrak{I}_2^{Ul} \Leftrightarrow \exists \Psi(X) [ ([\Psi(X)]_{Ul} \in \Gamma_X^{*Ul} / \sim_X ) \wedge (Y = X) ] \}, \quad (4.1.11)$$

where the countable collection  $\Gamma_X^{*Ul} / \sim_X$  is defined by

$$\forall \Psi(X) \{ [\Psi(X)]_{Ul} \in \Gamma_X^{*Ul} / \sim_X \Leftrightarrow [ ([\Psi(X)]_{Ul} \in \Gamma_X^{Ul} / \sim_X ) \wedge \exists ! X \Psi(X) ] \} \quad (4.1.12)$$

**Definition 4.1.6.** Let  $\mathfrak{R}_2^{Ul}$  be the countable collection of all sets such that

$$\forall X (X \in \mathfrak{I}_2^{Ul}) [X \in \mathfrak{R}_2^{Ul} \Leftrightarrow X \notin X]. \quad (4.1.13)$$

**Remark 4.1.4.** Note that  $\mathfrak{R}_2^{Ul} \in \mathfrak{I}_2^{Ul}$  since  $\mathfrak{R}_2^{Ul}$  is a collection definable by 1-place open wff

$$\Psi(Z, \mathfrak{I}_2^{Ul}) \triangleq \forall X (X \in \mathfrak{I}_2^{Ul}) [X \in Z \Leftrightarrow X \notin X]. \quad (4.1.14)$$

From (4.1.13) one obtains

$$\mathfrak{R}_2^{Ul} \in \mathfrak{R}_2^{Ul} \Leftrightarrow \mathfrak{R}_2^{Ul} \notin \mathfrak{R}_2^{Ul}. \quad (4.1.15)$$

But (4.1.15) gives a contradiction

$$(\mathfrak{R}_2^{Ul} \in \mathfrak{R}_2^{Ul}) \wedge (\mathfrak{R}_2^{Ul} \notin \mathfrak{R}_2^{Ul}). \quad (4.1.16)$$

However contradiction (2.1.16) it is not a contradiction inside  $ZFC_2^{Ul}$  for the reason that the countable collection  $\mathfrak{I}_2^{Ul}$  is not a set in the sense of the set theory  $ZFC_2^{Ul}$ .

**In order to obtain a contradiction inside  $ZFC_2^{Ul}$  we introduce the following definitions.**

**Definition 4.1.7.** We define now the countable set  $\Gamma_v^{*Ul} / \sim_v$  by

$$\forall y \left\{ [y]_{Ul} \in \Gamma_v^{*Ul} / \sim_v \Leftrightarrow ([y]_{Ul} \in \Gamma_v^{Ul} / \sim_v ) \wedge \widehat{\mathbf{Fr}}_2^{Ul}(y, v) \wedge [\exists ! X \Psi_{y,v}^{Ul}(X)] \right\}. \quad (4.1.17)$$

**Remark 4.1.5.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{*Ul} / \sim_v$  is a

set in the sense of the set theory  $ZFC_2^{Ul}$ .

**Definition 4.1.8.** We define now the countable set  $\mathfrak{I}_2^{*Ul}$  by formula

$$\forall Y \left\{ Y \in \mathfrak{I}_2^{*Ul} \Leftrightarrow \exists y [ ([y]_{Ul} \in \Gamma_v^{*Ul} / \sim_v ) \wedge (g_{ZFC_2^{Ul}}(X) = v) \wedge Y = X ] \right\}. \quad (4.1.18)$$

Note that from the axiom schema of replacement (1.1.1) it follows directly that  $\mathfrak{I}_2^{*Ul}$  is a

set in the sense of the set theory  $ZFC_2^{Ul}$ .

**Definition 4.1.9.** We define now the countable set  $\mathfrak{R}_2^{*Ul}$  by formula

$$\forall X (X \in \mathfrak{I}_2^{*Ul}) [X \in \mathfrak{R}_2^{*Ul} \Leftrightarrow X \notin X]. \quad (4.1.19)$$

Note that from the axiom schema of separation it follows directly that  $\mathfrak{R}_2^{*Ul}$  is a set in the

sense of the set theory  $ZFC_2^{Ul}$ .

**Remark 4.1.6.** Note that  $\mathfrak{R}_2^{*Ul} \in \mathfrak{I}_2^{*Ul}$  since  $\mathfrak{R}_2^{*Ul}$  is definable by the following formula

$$\Psi^*(Z) \triangleq \forall X (X \in \mathfrak{I}_2^{*Ul}) [X \in Z \Leftrightarrow X \notin X]. \quad (4.1.20)$$

**Theorem 4.1.1.** Set theory  $ZFC_2^{Ul}$  is inconsistent.

Proof. From (4.1.19) and Remark 4.1.6 we obtain  $\mathfrak{R}_2^{*Ul} \in \mathfrak{R}_2^{*Ul} \Leftrightarrow \mathfrak{R}_2^{*Ul} \notin \mathfrak{R}_2^{*Ul}$  from which immediately one obtains a contradiction

$$(\mathfrak{R}_2^{*Ul} \in \mathfrak{R}_2^{*Ul}) \wedge (\mathfrak{R}_2^{*Ul} \notin \mathfrak{R}_2^{*Ul}). \quad (4.1.21)$$

## 4.2. Second-order set theory $ZFC_2$ with the full second-order semantics.

Remind that the canonical approach of second order logic with full second-order semantics to the foundations of mathematics is that mathematical propositions have the form

$$\mathbf{U} \models \Phi \quad (4.2.1)$$

where  $\mathbf{U}$  is a mathematical structure, such as integers, reals etc., and  $\Phi$  is a mathematical statement written in second order logic. If  $\mathbf{A}$  is one of the structures, such as  $(\mathbb{N}, +, \times, <)$  or  $(\mathbb{R}, +, \times, <)$ , for which there is a second order sentence  $\Xi_{\mathbf{U}}$  such that

$$\forall \mathbf{W} (\mathbf{W} \models \Xi_{\mathbf{U}} \Leftrightarrow \mathbf{W} \cong \mathbf{U}) \quad (4.2.2)$$

then (4.2.2) can be expressed as a second order semantic logical truth

$$\models \Xi_{\mathbf{U}} \Rightarrow \Phi \quad (4.2.3)$$

**Remark 4.2.1.** Let  $ZFC_2^{fss}$  be second order set theory  $ZFC_2$  with the full second-order semantics.

(1) There is no completeness theorem for second-order logic.

(2) Nor do the axioms of second-order  $ZFC_2^{fss}$  imply a reflection principle which ensures

that if a sentence of second-order set theory is true, then it is true in some standard model.

**Remark 4.2.2.** Thus there may be sentences of the language of second-order set theory

$ZFC_2^{fss}$  :

(ii) that are true but unsatisfiable, or

(i) sentences that are valid, but false.

**Remark 4.2.3.** For example let  $Z$  be the conjunction of all the axioms of second-order  $ZFC_2^{fss}$ .  $Z$  is surely true. But the existence of a model for  $Z$  requires the existence of strongly inaccessible cardinals. The axioms of  $ZFC_2^{fss}$  don't entail the existence of strongly

inaccessible cardinals, and hence the satisfiability of  $Z$  is independent of  $ZFC_2^{fss}$ . Thus,  $Z$  is

true but its unsatisfiability is consistent with  $ZFC_2^{fss}$ .

**Definition 4.2.1.** Well formed formula  $\Psi$  of  $ZFC_2^{fss}$  is a well formed formula of the first order (wff<sub>1</sub>) if  $\Psi$  contain only first-order variables and first-order quantifiers.

Let  $\Gamma_X^{fss}$  be the countable collection of the all 1-place open wff<sub>1</sub>'s of the set theory  $ZFC_2^{fss}$

that contains free occurrences of the first-order variable  $X$ .

Let  $\Psi_1(X), \Psi_2(X)$  be 1-place open wff<sub>1</sub>'s of the set theory  $ZFC_2^{fss}$ . We define

now the equivalence relation  $(\cdot \sim_X \cdot) \subset \Gamma_X^{fss} \times \Gamma_X^{fss}$  by

$$\Psi_1(X) \sim_X \Psi_2(X) \Leftrightarrow \forall X[\Psi_1(X) \Leftrightarrow \Psi_2(X)] \quad (4.2.4)$$

For any  $\Psi(X) \in \Gamma_X^{\#fss}$  let  $[\Psi(X)]_{\#fss} \triangleq \{\Phi(X) \in \Gamma_X^{\#fss} | \Psi(X) \sim \Phi(X)\}$  denote the equivalence

class to which  $\Psi(X)$  belongs. All elements of  $\Gamma_X^{\#fss}$  equivalent to each other are also elements of the same equivalence class. The collection of all possible equivalence classes of  $\Gamma_X^{\#fss}$  by  $\sim_X$ , denoted  $\Gamma_X^{\#fss} / \sim_X$

$$\Gamma_X^{\#fss} / \sim_X \triangleq \{[\Psi(X)]_{\#fss} | \Psi(X) \in \Gamma_X^{\#fss}\}. \quad (4.2.5)$$

Let  $\mathbf{Fr}_2^{\#fss}(y, v)$  be the relation :  $y$  is the Gödel number of a wff of the set theory  $ZFC_2^{\#fss}$  that

contains free occurrences of the first-order variable  $X$  with Gödel number  $v$  [9].

Note that the relation  $\mathbf{Fr}_2^{\#fss}(y, v)$  is expressible in  $ZFC_2^{\#fss}$  by a wff  $\widehat{\mathbf{Fr}}_2^{\#fss}(y, v)$ .

Note that for any  $y, v \in \mathbb{N}$  by definition of the relation  $\mathbf{Fr}_2^{\#fss}(y, v)$  follows that

$$\widehat{\mathbf{Fr}}_2^{\#fss}(y, v) \Leftrightarrow \exists ! \Psi(X) \left[ \left( g_{ZFC_2^{\#fss}}(\Psi(X)) = y \right) \wedge \left( g_{ZFC_2^{\#fss}}(X) = v \right) \right], \quad (4.2.6)$$

where  $\Psi(X)$  is a unique wff<sub>1</sub> of  $ZFC_2^{\#fss}$  which contains free occurrences of the variable  $X$

with Gödel number  $v$ . We denote a unique wff  $\Psi(X)$  defined by using equivalence (4.2.6)

by symbol  $\Psi_{y,v}^{\#fss}(X)$ , i.e.

$$\widehat{\mathbf{Fr}}_2^{\#fss}(y, v) \Leftrightarrow \exists ! \Psi_{y,v}^{\#fss}(X) \left[ \left( g_{ZFC_2^{\#fss}}(\Psi_{y,v}^{\#fss}(X)) = y \right) \wedge \left( g_{ZFC_2^{\#fss}}(X) = v \right) \right]. \quad (4.2.7)$$

**Remark 4.2.4.** In order to avoid difficulties mentioned above, see Remark 4.2.1-Remark

4.2.3 we dealing with the countable collection  $\Gamma_X^{\#fss}$  of the all 1-place open wff<sub>1</sub>'s of the set theory  $ZFC_2^{\#fss}$ .

**Definition 4.2.2.** Let  $g_{ZFC_2^{\#fss}}(X) = v$ . Let  $\Gamma_v^{\#fss}$  be a set of all Gödel numbers of 1-place open

wff<sub>1</sub>'s of the set theory  $ZFC_2^{\#fss}$  that contains free occurrences of the first-order variable  $X$  with Gödel number  $v$ , i.e.

$$\Gamma_v^{\#fss} = \{y \in \mathbb{N} | \langle y, v \rangle \in \mathbf{Fr}_2^{\#fss}(y, v)\}, \quad (4.2.8)$$

or in the following equivalent form:

$$\forall y(y \in \mathbb{N}) \left[ y \in \Gamma_v^{\#fss} \Leftrightarrow (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_2^{\#fss}(y, v) \right]. \quad (4.2.9)$$

**Remark 4.2.5.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{\#fss}$  is a set

in the sense of the set theory  $ZFC_2^{\#fss}$ .

**Definition 4.2.3.(i)** We define now the equivalence relation

$$(\cdot \sim_v \cdot) \subset \Gamma_v^{\#fss} \times \Gamma_v^{\#fss} \quad (4.2.10)$$

in the sense of the set theory  $ZFC_2^{\#fss}$  by

$$y_1 \sim_v y_2 \Leftrightarrow \left( \forall X \left[ \Psi_{y_1, v}^{\#fss}(X) \Leftrightarrow \Psi_{y_2, v}^{\#fss}(X) \right] \right) \quad (4.2.11)$$

The collection of all possible equivalence classes of  $\Gamma_v^{\#fss}$  by  $\sim_v$ , denoted  $\Gamma_v^{\#fss} / \sim_v$

$$\Gamma_v^{\#fss} / \sim_v \triangleq \{[y]_{\#fss} | y \in \Gamma_v^{\#fss}\}. \quad (4.2.12)$$

**Remark 4.2.6.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{\#fss} / \sim_v$  is a

set in the sense of the set theory  $ZFC_2^{fss}$ .

**Definition 4.2.4.** Let  $\mathfrak{S}_2^{\#fss}$  be the countable collection of the all sets definable by 1-place

open first order wff of the set theory  $ZFC_2^{fss}$ , i.e.

$$\forall Y \left\{ Y \in \mathfrak{S}_2^{\#fss} \Leftrightarrow \exists \Psi(X) \left[ \left( [\Psi(X)]_{\#fss} \in \Gamma_X^{\#fss} / \sim_X \right) \wedge [\exists! X [\Psi(X) \wedge Y = X]] \right] \right\}. \quad (4.2.13)$$

**Definition 4.2.5.** We rewrite now (4.2.13) in the following equivalent form

$$\forall Y \left\{ Y \in \mathfrak{S}_2^{\#fss} \Leftrightarrow \exists \Psi(X) \left[ \left( [\Psi(X)]_{\#fss} \in \Gamma_X^{\#fss} / \sim_X \right) \wedge (Y = X) \right] \right\}, \quad (4.2.14)$$

where the countable collection  $\Gamma_X^{\#fss} / \sim_X$  is defined by

$$\forall \Psi(X) \left\{ [\Psi(X)]_{\#fss} \in \Gamma_X^{\#fss} / \sim_X \Leftrightarrow \left[ \left( [\Psi(X)]_{\#fss} \in \Gamma_X^{\#fss} / \sim_X \right) \wedge \exists! X \Psi(X) \right] \right\} \quad (4.2.15)$$

**Definition 4.2.6.** Let  $\mathfrak{R}_2^{\#fss}$  be the countable collection of all sets such that

$$\forall X \left( X \in \mathfrak{S}_2^{\#fss} \right) \left[ X \in \mathfrak{R}_2^{\#fss} \Leftrightarrow X \notin X \right]. \quad (4.2.16)$$

**Remark 4.2.7.** Note that  $\mathfrak{R}_2^{\#fss} \in \mathfrak{S}_2^{\#fss}$  since  $\mathfrak{R}_2^{\#fss}$  is a collection definable by 1-place open wff<sub>1</sub>

$$\Psi \left( Z, \mathfrak{S}_2^{\#fss} \right) \triangleq \forall X \left( X \in \mathfrak{S}_2^{\#fss} \right) [X \in Z \Leftrightarrow X \notin X]. \quad (4.2.17)$$

From (4.2.16) one obtains

$$\mathfrak{R}_2^{\#fss} \in \mathfrak{R}_2^{\#fss} \Leftrightarrow \mathfrak{R}_2^{\#fss} \notin \mathfrak{R}_2^{\#fss}. \quad (4.2.18)$$

But (4.2.18) gives a contradiction

$$\left( \mathfrak{R}_2^{\#fss} \in \mathfrak{R}_2^{\#fss} \right) \wedge \left( \mathfrak{R}_2^{\#fss} \notin \mathfrak{R}_2^{\#fss} \right). \quad (4.2.19)$$

However contradiction (2.2.19) it is not a contradiction inside  $ZFC_2^{fss}$  for the reason that the countable collection  $\mathfrak{S}_2^{\#fss}$  is not a set in the sense of the set theory  $ZFC_2^{fss}$ .

**In order to obtain a contradiction inside  $ZFC_2^{fss}$  we introduce the following definitions.**

**Definition 4.2.7.** We define now the countable set  $\Gamma_v^{\#fss} / \sim_v$  by

$$\forall y \left\{ [y]_{UI} \in \Gamma_v^{\#fss} / \sim_v \Leftrightarrow \left( [y]_{\#fss} \in \Gamma_v^{\#fss} / \sim_v \right) \wedge \widehat{\mathbf{Fr}}_2^{\#fss}(y, v) \wedge [\exists! X \Psi_{y, v}^{\#fss}(X)] \right\}. \quad (4.2.20)$$

**Remark 4.2.8.** Note that from the axiom of separation it follows directly that  $\Gamma_v^{\#fss} / \sim_v$  is a

set in the sense of the set theory  $ZFC_2^{fss}$ .

**Definition 4.2.8.** We define now the countable set  $\mathfrak{S}_2^{\#fss}$  by formula

$$\forall Y \left\{ Y \in \mathfrak{S}_2^{\#fss} \Leftrightarrow \exists y \left[ \left( [y]_{\#fss} \in \Gamma_v^{\#fss} / \sim_v \right) \wedge \left( g_{ZFC_2^{fss}}(X) = v \right) \wedge Y = X \right] \right\}. \quad (4.2.21)$$

Note that from the axiom schema of replacement (1.1.1) it follows directly that  $\mathfrak{S}_2^{*\#fss}$  is a set in the sense of the set theory  $ZFC_2^{fss}$ .

**Definition 4.2.9.** We define now the countable set  $\mathfrak{R}_2^{*\#fss}$  by formula

$$\forall X (X \in \mathfrak{S}_2^{*\#fss}) [X \in \mathfrak{R}_2^{*\#fss} \Leftrightarrow X \notin X]. \quad (4.2.22)$$

Note that from the axiom schema of separation it follows directly that  $\mathfrak{R}_2^{*\#fss}$  is a set in the sense of the set theory  $ZFC_2^{fss}$ .

**Remark 4.1.6.** Note that  $\mathfrak{R}_2^{*\#fss} \in \mathfrak{S}_2^{*Ul}$  since  $\mathfrak{R}_2^{*Ul}$  is definable by the following formula

$$\Psi^*(Z) \triangleq \forall X (X \in \mathfrak{S}_2^{*\#fss}) [X \in Z \Leftrightarrow X \notin X]. \quad (4.2.23)$$

**Theorem 4.1.1.** Set theory  $ZFC_2^{fss}$  is inconsistent.

Proof. From (4.2.22) and Remark 4.1.6 we obtain  $\mathfrak{R}_2^{*\#fss} \in \mathfrak{R}_2^{*\#fss} \Leftrightarrow \mathfrak{R}_2^{*\#fss} \notin \mathfrak{R}_2^{*Ul}$  from which immediately one obtains a contradiction

$$(\mathfrak{R}_2^{*\#fss} \in \mathfrak{R}_2^{*\#fss}) \wedge (\mathfrak{R}_2^{*\#fss} \notin \mathfrak{R}_2^{*\#fss}). \quad (4.2.24)$$

## 5. Conclusion

In this paper we have proved that set theory  $ZFC + \exists M_{st}^{ZFC}$  is inconsistent. This result originally was obtained in [11] by using essentially complicated approach.

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## References.

- [1] E. Nelson. Warning Signs of a Possible Collapse of Contemporary Mathematics. <https://web.math.princeton.edu/~nelson/papers/warn.pdf>  
In Infinity: New Research Frontiers, by Michael Heller (Editor), W. Hugh Woodin  
Pages 75–85, 2011. Published February 14th 2013 by Cambridge University  
Press Hardcover, 311 pages. ISBN: 1107003873 (ISBN13: 9781107003873)
- [2] L. Henkin, "Completeness in the theory of types". Journal of  
Symbolic Logic 15 (2): 81–91. doi:10.2307/2266967. JSTOR 2266967
- [3] M. Rossberg, "First-Order Logic, Second-Order Logic, and Completeness".  
In V. Hendricks et al., eds. First-order logic revisited. Berlin: Logos-Verlag.
- [4] S. Shapiro, Foundations without Foundationalism: A Case for Second-order  
Logic. Oxford University Press. ISBN 0-19-825029-0
- [5] A. Rayo and G. Uzquiano, Toward a Theory of Second-Order Consequence,  
Notre Dame Journal of Formal Logic Volume 40, Number 3, Summer 1999.
- [6] J. Vaananen, Second-Order Logic and Foundations of Mathematics,  
The Bulletin of Symbolic Logic, Vol.7, No. 4 (Dec., 2001), pp. 504-520.
- [7] G. Uzquiano, Quantification without a domain. New Waves in Philosophy of  
Mathematics. Springer, 29 Sep 2009 - Philosophy - 327 pp.  
ISBN 0230245196, 9780230245198
- [8] P. Cohen, Set Theory and the continuum hypothesis. Reprint of the  
W. A. Benjamin, Inc., New York, 1966 edition. ISBN-13: 978-0486469218

- [9] E. Mendelson, Introduction to mathematical logic. 1997. ISBN-10: 0412808307. ISBN-13: 978-0412808302
- [10] R. Lemhoff, 2016 'European summer meeting of the association for symbolic logic, LOGIC COLLOQUIUM'16, Leeds, UK, July 31-August 6, 2016, The Bulletin of Symbolic Logic Vol. 23, No. 2 (June 2017), pp. 213-266  
J. Foukzon, Inconsistent countable set in second order ZFC and unexistence of the strongly inaccessible cardinals, pp. 240.
- [11] J. Foukzon, E. R. Men'kova, Generalized Löb's Theorem. Strong Reflection Principles and Large Cardinal Axioms, Advances in Pure Mathematics, Vol. 3 No. 3, 2013.  
<http://dx.doi.org/10.4236/apm.2013.33053>
- [12] J. Foukzon, Inconsistent Countable Set in Second Order ZFC and Nonexistence of the Strongly Inaccessible Cardinals, British Journal of Mathematics & Computer Science, ISSN: 2231-0851, Vol.: 9, Issue.: 5  
<http://www.sciencedomain.org/abstract/9622>
- [13] J. Foukzon, Generalized Lob's Theorem. Strong Reflection Principles and Large Cardinal Axioms. Consistency Results in Topology.  
<http://arxiv.org/abs/1301.5340v10>
- [14] W. V. Quine, (1937), "New Foundations for Mathematical Logic", The American Mathematical Monthly, Mathematical Association of America, 44 (2): 70–80, doi:10.2307/2300564, JSTOR 2300564
- [15] T. Hailperin, "A set of axioms for logic," Journal of Symbolic Logic 9, pp. 1-19.