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Lomax-Modified Weibull Distribution: A New Generalization

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Abstract

In this paper, the Lomax-Modified Weibull distribution with five parameters is introduced as a new generalization of Modified Weibull distribution introduced by Sarhan and Zaindin (2009). Some properties of the new distribution are obtained. Estimation of the five unknown parameters will be discussed in the view of the likelihood and Bayesian principals. Also, a numerical example is provided.

Keywords: Bayesian estimation; Lomax-Modified Weibull(L-MW) distribution; maximum likelihood estimates.

2017 Statistics Subject Classification: ;;

1 Introduction

In statistical inference, parametric modeling of data is the main source of obtaining new families of distributions in the hope of adding flexibility for existing one. So many authors suggested diffrent new families which are extension, modification or combinations of old ones. Among these families we find: the beta-G family [Eugene et al. (2002) and Jones (2004)]; the Kumaraswamy-G (Kw-G)family [Cordeiro and de Castro (2011)]; the McDonald-G(Mc-G)family [Alexander et al. (2012)]; the transformed-transformer T-X family [Alzaatreh et al. (2013)]; the Lomax-Fréchet and he Lomax-Gumbel distributions [Gupta et al. (2015)] and [Gupta et al. (2016)],and more. This paper will be concerned with the last one. Gupta *et al., 2015 and Gupta* et al. (2016) suggested a new obtaining family of distributions from two other families F and G with the following cdf:

$$F_G(x) = \frac{F(G(x))}{F(1)} \tag{1.1}$$

and the corresponding pdf is

$$f_G(x) = \frac{F'(G(x))g(x)}{F(1)} \tag{1.2}$$

where F(x) and f(x) be the cdf and pdf of the first distribution with non-negative interval $[0, a]$,where a is in the range $1 \leq a < \infty$ and G(x) and g(x) be the cdf and pdf of the second distribution with interval $[a, b]$, where $b, c \in \mathbb{R}$ or $[0, \infty)\, or\, (-\infty, \infty)$. One can construct a large number of distributions by choosing different forms of first and second distributions. for example: Gupta *et al., 2015 introduced the Lomax-Fréchet distribution; Gupta* et al., 2016 introduced the Lomax-Gumbel distribution and others.

2 Lomax-Modified Weibull Distribution

Modified Weibull distribution is considered a generalization of some important lifetime distributions such as exponential, Rayleigh, Weibull and Linear failure rate distributions. Many authors showed interest in obtaining a generalization of this distribution in hope of adding flexibility in parametric modeling of data. Among these authors Zaindin and Sarhan (2011) who introduced the exponentiation of Modified Weibull distribution, Mandouh (2018) who discussed Logistic-Modified Weibull distribution, Mahmoud *et al.* (2018) who introduced the P-A-L modified Weibull distribution among other authors. Here we will introduce a new generalization of modified Weibull which called Lomax-Modified Weibull distribution. The new distribution is obtained by using the idea of Gupta *et al.*, 2015 and Gupta *et al.*, (2016). Let the first distribution is Lomax distribution with the cdf $F(x) = 1 - (1 + \frac{x}{\lambda})^{-\alpha}$ and the second distribution is Modified Weibull distribution introduced by Sarhan and Zaindin (2009) with the cdf $G(x) = 1 - e^{-\beta x^{\gamma}}$. Putting F(x) and G(x) in formulas (1.1) and (1.2), we get the new distribution in terms of the Lomax-Modified Weibull (Lomax-MW) distribution.
The cdf of the Lomax-MW distribution is

$$F_{LMW}(x) = K[1 - (1 + \frac{1}{\lambda}(1 - e^{-cx-\beta x^{\gamma}}))^{-\alpha}], \tag{2.1}$$

and

$$f_{LMW}(x) = \frac{K\alpha}{\lambda}(c + \beta\gamma x^{\gamma-1})e^{-cx-\beta x^{\gamma}}[(1 + \frac{1}{\lambda}(1 - e^{-cx-\beta x^{\gamma}}))^{-(\alpha+1)}] \tag{2.2}$$

where the constant $K = [F(1)]^{-1} = [1 - (1 + \frac{1}{\lambda})^{-\alpha}]^{-1}$, and $\alpha, \lambda, \beta, \gamma > 0, c > 0, x \geq 0$.
Using Binomial Expansion, one can put formulas (2.1) and (2.2) in the following forms:

$$F_{LMW}(x) = \sum_{i=1}^{\infty} b_i (G(x))^i$$

where $b_i = -K \sum_{i=1}^{\infty} \binom{-\alpha}{i} \lambda^{-i}$ and G(x) is the cdf of the modified Weibull distribution or $G(x)^i$ is the cdf of the exponentiated Weibull distribution.
and

$$f_{LMW}(x) = \sum_{i=1}^{\infty} b_i i (G(x))^{i-1} g(x)$$

where $i(G(x))^{i-1}g(x)$is the cdf of the exponentiated Weibull distribution.

The hazard function can can be obtained using (2.1) and (2.2)as follows:

$$h_{LMW}(x) = \frac{\frac{K\alpha}{\lambda}(c + \beta\gamma x^{\gamma-1})e^{-cx-\beta x^{\gamma}}[(1 + \frac{1}{\lambda}(1 - e^{-cx-\beta x^{\gamma}}))^{-(\alpha+1)}]}{1 - K[1 - (1 + \frac{1}{\lambda}(1 - e^{-cx-\beta x^{\gamma}}))^{-\alpha}]} \tag{2.3}$$

Figures 1 and 2 show the plots of the pdf and hazard function of the Lomax-MW distribution for various parameter values.

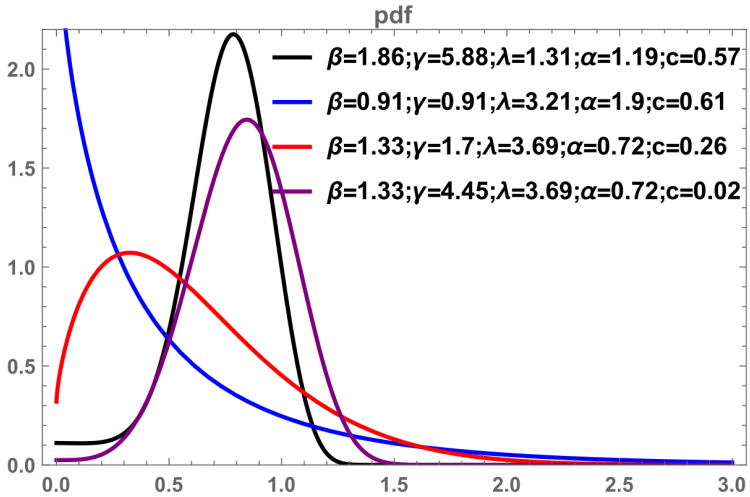

Figure 1: pdf of Lomax-MW distribution for various values of the parameters

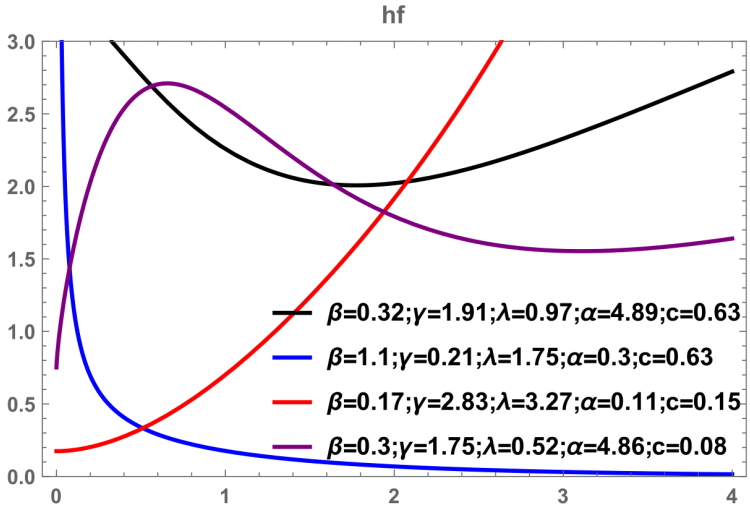

Figure 2: hf of Lomax-MW distribution for various values of the parameters

The quantile x_q of the Lomax-MW distribution has no closed form so one can solve the following equation numerically to obtain x_q,

$$\beta x_q^{\gamma} + c x_q + log[1 - \lambda((1 - \frac{q}{K})^{-\frac{1}{\alpha}} - 1)] = 0. \tag{2.4}$$

From formula (2.4) one can get some special cases such as: putting $\gamma = 2$, we obtain the quantile of the Lomax-linear failure rate distribution with the following form

$$x_q = \frac{-c + \sqrt{c^2 - 4\beta log[1 - \lambda((1 - \frac{q}{K})^{-\frac{1}{\alpha}} - 1)]}}{2\beta}. \tag{2.5}$$

Also, putting $c = 0$, we obtain the quantile of the Lomax-Weibull distribution with the form

$$x_q = [-\frac{1}{\beta} log[1 - \lambda((1 - \frac{q}{K})^{-\frac{1}{\alpha}} - 1)]]^{\frac{1}{\gamma}}. \tag{2.6}$$

and others.

The rth raw moments for the Lomax-MW distribution is

$$E(X^r) = \frac{K\alpha}{\lambda} \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{\infty} (-1)^j \frac{(-1)^k (\beta(j+1))^k}{\Gamma(k+1)} \lambda^{-i} C_i^{-(\alpha+1)} C_j^i [c(c(j+1))^{-r-k\gamma-1} \Gamma(r + k\gamma + 1)$$
$$+\gamma\beta(c(j+1))^{-r-(k+1)\gamma} \Gamma(r + (k+1)\gamma)]. \tag{2.7}$$

where $C_j^i = \frac{\Gamma(i+1)}{\Gamma(i-j+1)\Gamma(j+1)}$.

The Rényi entropy for the Lomax-MW distribution is

$$I_R(\delta) = \frac{1}{1-\delta} log[\sum_{i,j,k,l=0}^{\infty} (\frac{K\alpha}{\lambda})^{\delta} (\frac{\beta\gamma}{c})^k \frac{(\beta(\delta+j))^l}{\Gamma(l+1)} (-1)^{j+l} \lambda^{-i} C_j^i C_k^{\delta} C_i^{-\delta(\alpha+1)} \frac{\Gamma(\gamma(k+l) - k + 1))}{(c(j+\delta))^{\gamma(k+l)-k+1}}]. \tag{2.8}$$

One can use formula (2.8) as a measure of variation or uncertainty of a Lomax-MW random variable.

3 Parameter Estimation

Now, Estimation of the five unknown parameters will be discussed in the view of the likelihood and Bayesian principals.

3.1 Maximum Likelihood Estimation

Suppose that $x_1, x_2, ..., x_n$ be a random sample of size n follows the Lomax-MW distribution. To estimate the five unknown parameters $(\alpha, \lambda, \beta, \gamma, c)$ of the Lomax-MW distribution we first obtain the likelihood function as follows:

$$L = (\frac{K\alpha}{\lambda})^n \prod_{i=1}^{n} (c + \gamma\beta x_i^{\gamma-1}) e^{-c\sum_{i=1}^{n} x_i - \beta \sum_{i=1}^{n} x_i^{\gamma}} \prod_{i=1}^{n} (1 + \frac{1}{\lambda}(1 - e^{-c\sum_{i=1}^{n} x_i - \beta x_i^{\gamma-1}}))^{-(\alpha+1)}, \tag{3.1}$$

Taking the logarithm of likelihood function, we obtain

$$lnL = nlnK + nln\alpha - nln\lambda + \sum_{i=1}^{n} ln(c + \gamma\beta x_i^{\gamma-1}) - c\sum_{i=1}^{n} x_i - \beta\sum_{i=1}^{n} x_i^{\gamma} - (\alpha+1)\sum_{i=1}^{n} ln(1 + \frac{1}{\lambda}(1 - e^{-cx_i - \beta x_i^{\gamma}})). \tag{3.2}$$

The first order derivative of the log-likelihood function with respect to $\alpha, \lambda, \beta, \gamma$ and c are

$$\begin{aligned}
\frac{\partial lnL}{\partial \alpha} &= \frac{n}{\alpha} - \frac{n(1+\lambda^{-1})^{-\alpha} ln(1+\lambda^{-1})}{(1-(1+\frac{1}{\lambda})^{-\alpha})} - \sum_{i=1}^{n} ln(1 + \frac{1}{\lambda}(1 - e^{-cx_i - \beta x_i^{\gamma}})), \\
\frac{\partial lnL}{\partial \lambda} &= -\frac{n}{\lambda} + \frac{n\alpha(1+\lambda^{-1})^{-(\alpha+1)}}{\lambda^2(1-(1+\frac{1}{\lambda})^{-\alpha})} + \frac{(\alpha+1)(1 - e^{-cx_i - \beta x_i^{\gamma}})}{\lambda^2(1 + \lambda^{-1}(1 - e^{-cx_i - \beta x_i^{\gamma}}))}, \\
\frac{\partial lnL}{\partial \beta} &= -\sum_{i=1}^{n} x_i^{\gamma} + \sum_{i=1}^{n} \frac{\gamma x_i^{\gamma-1}}{c + \beta\gamma x_i^{\gamma}} - \frac{(\alpha+1)}{\lambda} \sum_{i=1}^{n} \frac{x_i^{\gamma} e^{-cx_i - \beta x_i^{\gamma}}}{(1 + \lambda^{-1}(1 - e^{-cx_i - \beta x_i^{\gamma}})}, \\
\frac{\partial lnL}{\partial \gamma} &= -\beta\sum_{i=1}^{n} x_i^{\gamma} lnx_i - \frac{\beta(\alpha+1)}{\lambda} \sum_{i=1}^{n} \frac{e^{-cx_i - \beta x_i^{\gamma}} x_i^{\gamma} lnx_i}{(1 + \lambda^{-1}(1 - e^{-cx_i - \beta x_i^{\gamma}}))} + \sum_{i=1}^{n} \frac{\beta x_i^{\gamma-1} + \beta\gamma x_i^{\gamma-1} ln(x_i)}{c + \beta\gamma x_i^{\gamma}}, \\
\frac{\partial lnL}{\partial c} &= -\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{1}{cx_i + \beta x_i^{\gamma-1}} - \sum_{i=1}^{n} \frac{(\alpha+1)(1 - e^{-cx_i - \beta x_i^{\gamma}}) x_i}{\lambda(1 + \lambda^{-1}(1 - e^{-cx_i - \beta x_i^{\gamma}}))}.
\end{aligned} \tag{3.3}$$

To obtain the estimates of the five unknown parameters, one can equate the first derivatives in (3.3) to zero and solve them numerically. Also, to obtain the asymptotic variance-covariance matrix of $(\hat{\alpha}, \hat{\lambda}, \hat{\beta}, \hat{\gamma}, \hat{c})$, we get the information matrix using the second order derivative of log-likelihood function. Inverting the information matrix and replacing the unknown parameters by their mles, we have the asymptotic variance-covariance matrix. So $100(1-\nu)\%$ approximate confidence intervals(CIs) for the parameter θ(say)is:
$(\hat{\theta} - z_{\frac{\nu}{2}} \sqrt{var(\hat{\theta})}, \hat{\theta} + z_{\frac{\nu}{2}} \sqrt{var(\hat{\theta})})$.

3.2 Bayesian Estimation

Approximate Bayes estimates are computed using the Gibbs sampling procedure by generating samples from posterior distributions. With non-informative prior distribution $\pi_0(\alpha, \lambda, \beta, \gamma, c) \propto \frac{1}{\alpha\lambda\beta\gamma c}$ where α, λ, β,γ and $c > 0.$, the joint posterior distribution take the form:
$\pi(\alpha, \lambda, \beta, \gamma, c|x) \propto \pi_0(\alpha, \lambda, \beta, \gamma, c) \exp l(x, \alpha, \lambda, \beta, \gamma, c)$,
where $l(x, \alpha, \lambda, \beta, \gamma, c)$ is the logarithm of the likelihood function which given in (3.2). When we set $\rho_1 = \log(\beta), \rho_2 = \log(\gamma), \rho_3 = \log(\alpha)$, $\rho_4 = \log(\lambda)$ and $\rho_5 = \log(c)$, the joint prior distribution will be $\pi(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) \propto$ constant,$-\infty < \rho_1, \rho_2, \rho_3, \rho_4$and $\rho_5 < \infty$ and the joint posterior distribution becomes

$$\begin{aligned}\pi(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5|x) \quad \propto \quad & \pi(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5) \, . - n \log[1 - (1 + exp(-\rho_4))^{-exp(\rho 3)}] + \rho_3 \\ & -\rho_4 + \sum_{i=1}^{n} log[exp(\rho_5) + exp(\rho_1)exp(\rho_2)x_i^{(exp(\rho_2)-1)}] - \sum_{i=1}^{n} exp(\rho_5)x_i \\ & - \sum_{i=1}^{n} exp(\rho_1)x_i^{exp(\rho_2)} - (exp(\rho_3) + 1) \log[1 + exp(-\rho_4)(1 - exp^{-exp(\rho_1 x_i^{exp(\rho_2)}}],\end{aligned}$$

Using the WinBUGS software, posterior summaries such as: the mean; the standard deviation; credible interval and others can be computed.

4 A Numerical Example

Here, a sample of size $n = 100$ from Lomax-MW distribution is generated to estimate the five unknown parameters α, λ, β, γ and c. we use the MLE and Bayesian procedures for this purpose with assuming $\alpha 0 = 0.5, \lambda_0 = 0.5, \beta_0 = 1.5$, $\gamma_0 = 1.5$, and $c_0 = 1.5$. First, the MLEs and 95 % confidence intervals for the five parameters α,λ, β, γ and c are listed in table(1):

Table 1: the MLEs and CIs for the four parameters α,λ, β, γ and c of the Lomax-MW model

Parameter	Estimate	Standard Deviation	95% Confidence Interval
α	0.9010	2.3392	(0.0000, 5.4857)
λ	1.5896	0.1623	(1.2715, 1.9077)
β	0.0020	0.0153	(0.0000, 0.0319)
γ	0.5790	0.1315	(0.1998, 0.8367)
c	3.3690	1.2265	(0.9650, 5.7730)

Second, Approximate Bayes estimates are obtained using Gibbs Sampling procedure. Under the assumption of the following non-informative uniform priors U(0.00, 3.00), U(0.001, 0.01), U(0.001, 0.10), U(-4.00, 3.00), U(-4.00, 3.00) and U(-4.00, 3.00) for$\rho_1, \rho_2, \rho_3, \rho_4$ and ρ_5 respectively, a set of 10000 Gibbs samples are generated after burn in sample of size 1000. As table (2) shows the MC error is less than 5 % of the sample standard deviation which indicating convergence of the algorithm.

Table 2: Summary results for the posterior parameters in the case of the Lomax-MW model

Parameter	Estimate	Standard Deviation	MC error	95 % Credible Interval
α	1.005	0.00260	2.788E-5	(1.001, 1.010)
λ	17.73	1.93500	0.02635	(12.98, 20.01)
β	1.044	0.04415	6.63E-4	(1.001, 1.163)
γ	1.006	0.00264	2.922E-5	(1.001, 1.010)
c	20.08	0.00486	8.058E-5	(20.07, 20.09)

From Tables 1-2, one can note that in most cases the CIs based on maximum likelihood inference is larger than the credible intervals based on the posterior summaries.

5 CONCLUSION

Modified Weibull distribution which is considered a generalization of some important lifetime distributions such as exponential, Rayleigh, Weibull and Linear failure rate distributions. Many authors interested in obtaining a generalization of this distribution in hope of adding flexibility in parametric modeling of data. Here a new generalization of modified Weibull was introduced which called Lomax-Modified Weibull distribution.Some properties of the new distribution are investigated and maximum likelihood and Bayesian estimation of the five unknown parameters are discussed.

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