

Modeling Exponential growth and Exponential Decay Real Phenomena of by ψ -Caputo Fractional Derivative

Abstract. The concept of " ψ -Caputo" fractional derivative is discussed in this article. The kernel ψ , is any increasing function with $\psi'(x) \neq 0, \forall x \in [a, b]$. The aim is to show, based on experimental studies and the error rate measurement tool, root-mean-square deviation that fractional approach of differential equations are often better than the classical ordinary approach. Several kernels alongside classical approaches are used in experiments. Two exponential decay models and one exponential growth model are considered.

Key words: Exponential decay, Exponential growth, ψ -Caputo Fractional derivative, Optimization, Initial value problems.

1 Introduction

Fractional calculus was discovered more than three centuries ago. Earlier works on Fractional calculus focused only on theories. Hence, existence and uniqueness of solution to problem were studied, without further materialization. [8-12] are some known references where the theory of existence and the uniqueness of fractional differential and their solutions is found.

The classical approach involving system of integer-order of differentiation, may fail sometime to explain some complicated incidents in real phenomena. Nowadays, fractional order of differential equations have been proven efficient than their integer-order counterparts in the modeling of various world phenomenon. The efficient of the fractional differentiation approach has been proven in various sciences branches such as physics, chemistry, epidemiology, finance and biology sciences [1-7], these are few of them just to mention.

In this work, we added our contribution to what has been done so far by various researchers such as Almeida [21] who introduced fractional derivative of a function with respect to another function, called the ψ -Caputo. Almeida et al [7,21] have proven further that the fractional order of differential equation is better than the classical in modeling the population growth, a country's GDP, the Newtown law of cooling and many others real world phenomenon. We proved that, the ψ -Caputo fractional differential equation is better than their integer-order counterpart for modeling the radioactivity decay and the RC circuit charging and discharging process provided that a suitable kernel ψ is selected [22].

2 Preliminaries

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $g : [0, +\infty) \rightarrow \mathbb{R}$ is defined as

$$({}_{RL}I_{0^+}^\alpha g)(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (t-s)^{\alpha-1} g(s) ds,$$

Provided that the right hand side of the integral is point wise defined on $(0, +\infty)$ and Γ is the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt, \forall \nu > 0, .$

Definition 2.2 The Caputo derivative of order $\alpha > 0$ for a function $g: [0, +\infty] \rightarrow \mathbb{R}$ is defined as

$$({}_CD_{0^+}^\alpha g)(t) = \begin{cases} \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} g^{(n)}(s) ds, & n-1 < \alpha < n \in \mathbb{R}, \\ g^{(n)}(t) & , \alpha \in \mathbb{R}. \end{cases}$$

Where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α .

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Definition 2.3 Let $\alpha > 0, g \in L^1[a, b]$ and $\psi \in C^1[a, b]$ be an increasing function with $\psi'(x) \neq 0, \forall x \in [a, b]$ then $I_{0^+}^{\alpha, \psi} g(t)$ denotes the fractional integral of g w.r.t ψ and it is given by

$$I_{0^+}^{\alpha, \psi} g(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} g(s) ds.$$

Definition 2.4 [21] Let $\alpha > 0, g, \psi \in C^n[a, b]$ with ψ be an increasing function and

$\psi'(x) \neq 0, \forall x \in [a, b]$ then ${}_CD_{0^+}^{\alpha, \psi} g(t)$ denotes the fractional Derivative of g w.r.t ψ and it is given by

$${}_CD_{0^+}^{\alpha, \psi} g(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{n-\alpha-1} \left(\frac{1}{\psi'(s)} \frac{d}{dt} \right)^n g(s) ds.$$

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Lemma 2.1 Let $\alpha > 0, n$ a natural number such that $\alpha \in (n-1, 1)$ if $g, \psi \in C^n[a, b]$ then

$$I_{0^+}^{\alpha, \psi} ({}_CD_{0^+}^{\alpha, \psi} g)(t) = g(t) - \sum_{i=0}^{n-1} \frac{\left(\frac{1}{\psi'(s)} \frac{d}{dt} \right)^i g(0)}{i!} (\psi(t) - \psi(0))^i.$$

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3 Methodology

To investigate the power of the ψ – *Caputo* fractional differential equation over the classical differential equation, we considered some linear model differential equations whose classical solutions are given either in the form of exponential decay or growth curve model. For each model, the true data obtained from experimental study is considered, and then the model based on classical approach of differential equation is defined alongside its counterpart derived from the ψ – *Caputo* fractional differential equation. Both models are used to fit the original data set. The percentage of the root square deviation error is used to compare the performance of each of the method.

Definition 3.1 Consider an experimental data vector of size n , obtained from a time dependent process such that the i^{th} value is denoted $y_i = y(t_i)$. If the original data is predicted by a function $\hat{y}$, in a way that the i^{th} fitted data value is denoted $\hat{y} = \hat{y}(t_i)$, the root-mean-squared deviation is denoted (see [19]) by

$$RMSD = \sqrt{\frac{\sum_{t=0}^n (y(t) - \hat{y}(t))^2}{n}} \quad (1)$$

The quantity given by (1) can be used to evaluate the percentage of the error that occurs while using predicted data instead of original data. The ratio below is used for the purpose

$$R = \sqrt{\frac{\sum_{t=0}^n (y(t) - \hat{y}(t))^2}{n}} \bigg/ \sqrt{\frac{\sum_{t=0}^n (y(t))^2}{n}} \quad (2)$$

Equation (2) gives the magnitude of the error did while using estimated value.

4 Examples

In this section, we proved by the mean of examples that the ψ – *Caputo* fractional differential equation can fit the data better than the classical differential equation, provided that a suitable kernel ψ is selected.

4.1 Radioactivity Experiment

Atoms present in the matter are characterized by their nuclear state. The nuclear state of an element can change due to some chemical reaction or even when it is heated to some extent. Radioactivity studies focus on nuclear state change. In general, the nuclear state change of an atom produces either electromagnetic radiation or some particles. In the process of nuclear state change, the number of atoms involve in the process may change as well. Hence one can observe for instance a fusion of two neutrons with two protons or fission of nucleus. The known forms of radiation are Alpha, Beta and Gamma. An experiment found in [18], studies the radioactivity of Silver and Aluminum (^{110}Ag , ^{108}Ag and ^{28}Al), which is a decay as particles are emitted over the time. The experimental data of ^{28}Al is found in [1,page 6]. A

107 scatter plot of the data shows that an exponential decay model is suitable to fit the data set.
108 The analytic form of the model is given by

$$109 \quad \frac{dn(t)}{dt} = -\lambda n(t), \quad t \geq 0, \quad (3)$$

110 A classical solution to the differential equation (3) is defined as

$$111 \quad n(t) = n_0 \exp\left(-\frac{t}{\tau}\right) \quad (4)$$

112 Where $\lambda = \frac{1}{\tau}$, t is the time elapsed, n is the number of particle yet to disintegrate at the time
113 t . The quantity n_0 is the initial number of particles at $t = 0$, and τ is the mean life-time. The
114 purpose of the experiment is to estimate the value of τ that would lead to a best fit of the
115 experimental data using formula (4). Experimental data and *Matlab* optimization routine
116 *lsqcurvefit* [20], returned $\lambda = 0.0121 \leftrightarrow \tau = 82.6446$ as the best value that would help in
117 minimizing the error while fitting the original data using (4).

118 Let us now admit that the Radioactivity model defined above is ruled by the ψ – *Caputo*
119 fractional differential equation. This equation is derived from the classical equation (3), with
120 the assumption that the order of derivative is not integer. The fractional counterpart of (3)
121 using ψ – *Caputo* derivative is defined as

$$122 \quad {}_C D_{0^+}^{\alpha, \psi} n(t) = -\lambda n(t) \quad (5)$$

123 Applying the fractional integration $I_{0^+}^{\alpha, \psi}$ to both sides of (5) and considering theorem (7.2) in
124 [13], the exact solution of (5) can be written as

$$125 \quad n(t) = n_0 E_{\alpha} \left(-\lambda (\psi(t) - \psi(0))^{\alpha} \right) \quad (6)$$

126 Where $E_{\alpha}(\cdot)$ is the Mittag-Leffler function given by $E_{\alpha}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\alpha i + 1)}$, $t \in \mathbb{R}$. α is the
127 fractional order of derivative and λ is the inverse of the mean life-time τ .

128 Two parameters are to be optimized in equation (6). These are the fractional order of
129 derivative α and the inverse of the mean life-time λ . Moreover, the kernel function is
130 selected with properties defined in definition 2.4. Hence for each selected kernel function,
131 corresponding optimal parameters are obtained. Table 1 is a summary of sample kernel their
132 corresponding optimal parameters and their root mean squared deviation.

133 Table 1: Radioactivity error comparison.

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	Classical Model	Kernel $\psi(x) = x$	Kernel $\psi(x) = \sqrt{x+1}$	Kernel $\psi(x) = \ln(x+1)$
λ	0.0121	0.0314	0.1177	0.1046
α	1	0.8252	1.0571	1.6420
SSE	1.17×10^5	2.9×10^3	8.87×10^3	2.65×10^4
RMSD	67.2560	10.5557	18.4694	31.9074
R	18.58%	2.92%	5.10%	8.81%

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140 The optimal parameters minimize the total error term when the original data is fitted by the
 141 fractional model defined by equation (6)

142 Figure 1a, 1b and 1c below show the original experimental data, the fitted model using
 143 classical and ψ – Caputo fractional order of derivative approaches with different kernels as
 144 defined in table 1. It is observable that in general fitting the data with the fractional approach
 145 provides less sum of squared error than the classical approach. The performance is evaluated
 146 in table 1.

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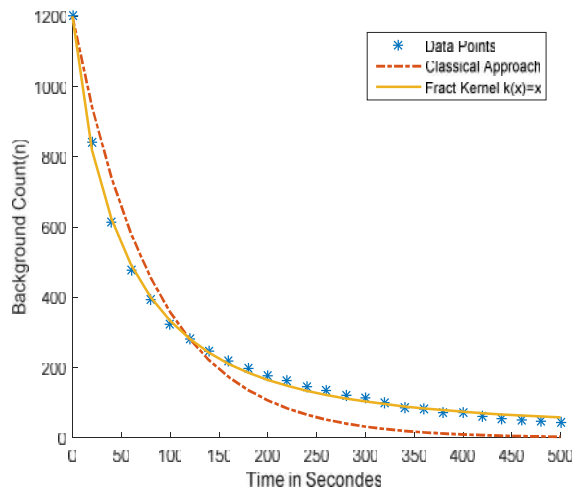


Figure 1 a

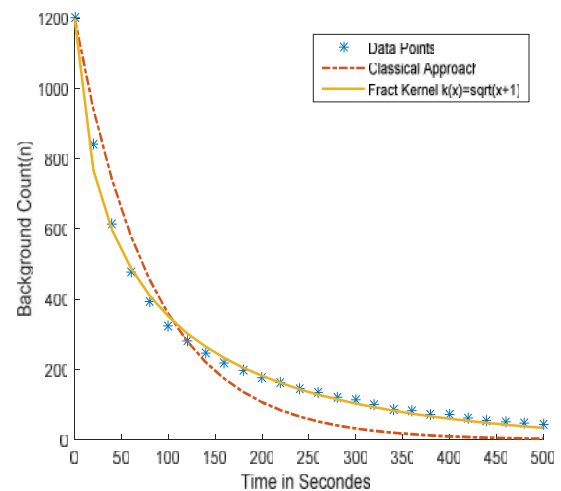


Figure 1 b

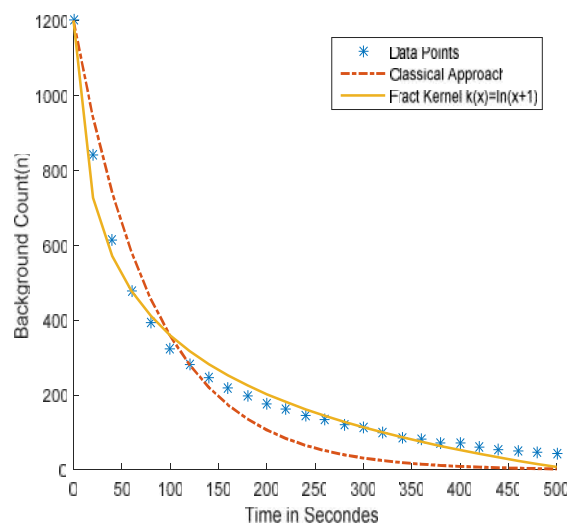


Figure 1 c

148 From both table 1 and figure 1. It is observable that the ψ –Caputo fractional differential
149 equation has an error rate smaller than the classical method of differential equations.

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151 4.2 RC circuit.

152 The RC circuit is a well known experimental setting used to measure the charging and
153 discharging time of a capacitor, C. Figure 2 shows the experimental setting of an RC circuit.

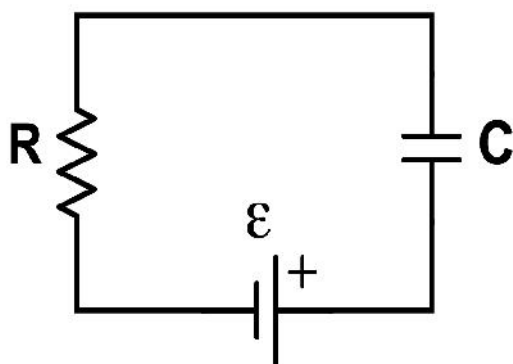


Figure 2a : RC in charging mode

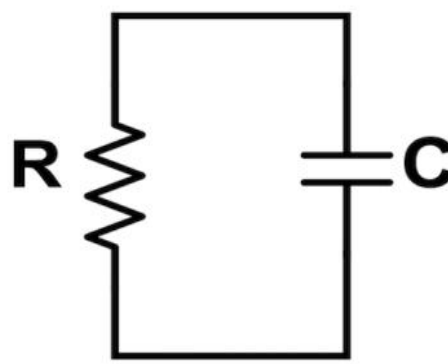


Figure 2b : RC in discharging mode

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155 In the charging mode , see figure 2a, The capacitor C , the resistor R , and the DC/voltage
156 source ε are connected . This leads to an accumulation of electric charges in the capacitor C
157 over the time. In the discharging mode, the potential source is removed from the circuit. Only
158 a capacitor initially full of electric charges Q is directly connected to a resistor R . Such
159 setting leads to a loss of electric charges over the time. An experimental data set of both
160 charging and discharging of a RC circuit is found at [14] .

161 Any electric component that resists to electric charges flow in an electric circuit is called
162 resistor. Such component usually produces heat as a results of it electric resistance. That is

the principle used in electric heater, iron for instance. A resistor of resistance R in a circuit obeys to the Ohm's law

$$V = I \times R \quad (7)$$

Where V is the voltage applied to the resistor, I is the rate at which charge flows through the resistor, and R is a constant called the resistance.

A capacitor is any electric circuit component with the ability to store from a circuit connected to a voltage source and to supply electric charge Q , in a circuit without voltage source. When it is attached to a resistor, the capacitor will push the electric charges through the resistor, creating electric current. Capacitors obey the equation

$$V = \frac{Q}{C} \quad (8)$$

where Q is the charge stored in the capacitor, C is a constant called the capacitance of the capacitor, and V is the resulting voltage.

Setting the formulas (7) and (8) for V equal to each other gives

$$\frac{Q}{C} = IR \quad (9)$$

Moreover, the rate I at which charges flow through the resistor is the same as the rate at which charges flow out of the capacitor, so

$$-\frac{dQ}{dt} = I \quad (10)$$

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A combination of (9) and (10) introduces the RC circuit model which can be written as

$$\frac{dQ}{dt} = -\frac{1}{RC}Q \quad (11)$$

It is familiar that the solution for the differential equation (11) is given by

$$Q = Q_0 e^{\frac{-1}{RC}t} \quad (12)$$

where Q_0 is the amount held in the capacitor at $t=0$, using the fact that $Q=CV$, (12) becomes

$$V = V_0 e^{\frac{-1}{RC}t} \quad (13)$$

It is common in RC circuit to estimate the capacitance C . Experimental data retrieval from [14], has the following initial parameters. The resistance $R=2200\Omega$, the potential of the voltage source $V_0=4.897$. The purpose of the experiment is to estimate the C value given in (13), using the experimental data and optimization routine, the value $C=0.0057$ appears to

be the best that would help minimizing the total error if equation(13) was used to fit the original data.

Let us consider now that the RC circuit model (11) is built based on a fractional differential equation. Such equation can be written using the ψ -Caputo derivative as

$$\begin{cases} {}^c D_{0^+}^{\alpha, \psi} Q(t) = -\frac{1}{RC} Q(t) \\ Q(0) = Q_0 \end{cases} \quad (14)$$

Apply $I_{0^+}^{\alpha, \psi}$ to both sides of (14) and using theorem (7.2) in [13], the exact solution of (14) can be written as

$$Q(t) = Q_0 E_{\alpha} \left(-\frac{1}{RC} (\psi(t) - \psi(0))^{\alpha} \right) \quad (15)$$

Using the relation $Q = CV$, equation (15) becomes,

$$V(t) = V_0 E_{\alpha} \left(-\frac{1}{RC} (\psi(t) - \psi(0))^{\alpha} \right) \quad (16)$$

Different kernel functions ψ are considered for experiment. Table 2 summarizes the experiment results providing optimal values of the parameters involved alongside the error rate obtained when fitting the original data by a model. The estimated parameters in this experiment are the capacitance value C and the fractional order of derivative α .

Table 2: RC discharging performances.

	Classical Model	Kernel $\psi(x) = x$	Kernel $\psi(x) = \sqrt{x+1}$	Kernel $\psi(x) = \ln(x+1)$
C	0.0057	0.0051	0.0014	0.0018
α	1	0.9871	1.1798	1.5855
SSE	0.8503	0.0098	0.7431	1.5906
RMSD	0.0922	0.0099	0.0862	0.1261
R	5.43%	0.58%	5.08%	7.43%

It is observable from table 2 that the kernel function $\psi(x) = x$ performs better than others. Indeed it requires an optimal capacitance which is almost equal to the capacitance value of the classical approach, a fractional order of derivative close to 1, and a 0% error rate. The kernel $\psi(x) = \ln(x+1)$, produces an error rate slightly greater than the rate of the classical approach. The other kernel namely $\psi(x) = \sqrt{x+1}$ has an error rate close to the 5% observable in the classical case.

Figure 3 is a set of graphs displaying the behavior of the original data and the experimental results.

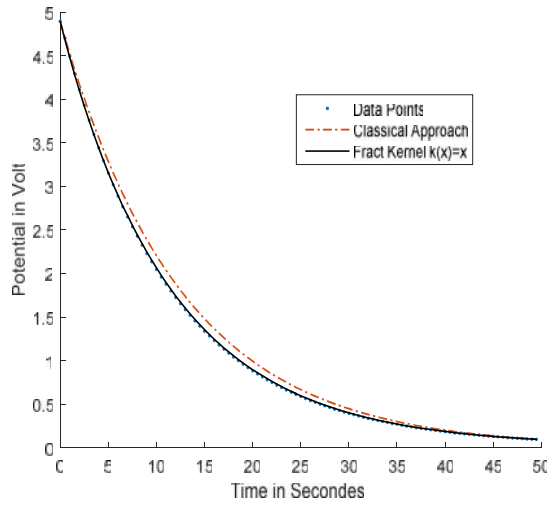


Figure 3a

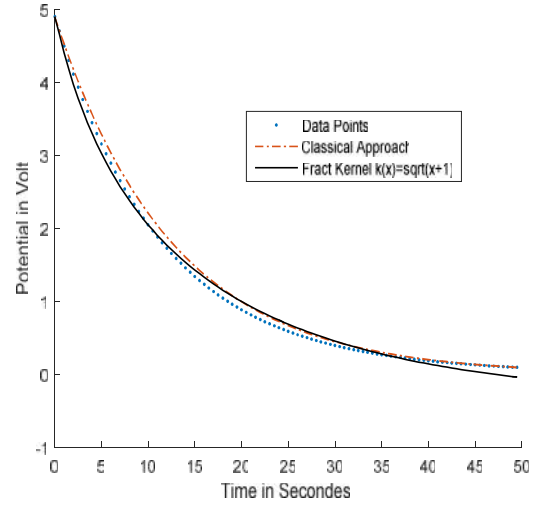


Figure 3b

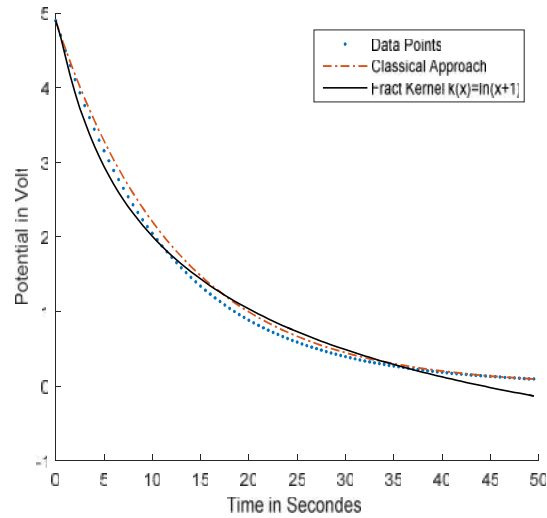


Figure 3c

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218 For charging mode the RC circuit has the following elements, a capacitance C , the
 219 resistor R and a potential source ε , that maintains the potential leading to the accumulation
 220 of electrical charges in the circuit (see fig 2a). The Kirchhof's current law of the circuit is
 221 written as

$$222 \quad V_0 + IR - \frac{Q}{C} = 0, \quad (17)$$

223 Since $Q'(t) = I(t)$ the charging differential equation model is derived from (17) as

$$224 \quad V_0 + \frac{dQ}{dt} R - \frac{Q}{C} = 0 \quad (18)$$

225 The exact solution of the ordinary differential equation (18) is given by

$$Q(t) = CV_0 \left(1 - e^{\frac{-1}{RC}t} \right), \quad (19)$$

Which is equivalent to

$$V(t) = V_0 \left(1 - e^{\frac{-1}{RC}t} \right) \quad (20)$$

The use of experimental data retrieved from [14], gives a same value of $C = 0.0057$ as obtained in the discharging process.

To solve the fractional counterpart model equation derived from (18), let us defined (18) using the ψ -Caputo derivative as

$$\begin{cases} {}^c D_{0^+}^{\alpha, \psi} Q(t) = \frac{1}{RC} Q(t) - \frac{V_0}{R}, \\ Q(0) = Q_0 \end{cases} \quad (21)$$

Applying $I_{0^+}^{\alpha, \psi}$ to both sides of (21) and using theorem (7.2) in [13] the exact solution of (21) can be written as

$$Q(t) = CV_0 \left(1 - E_\alpha \left(\frac{-1}{RC} (\psi(t) - \psi(0))^\alpha \right) \right) \quad (22)$$

Which is equivalent to

$$V(t) = V_0 \left(1 - E_\alpha \left(\frac{-1}{RC} (\psi(t) - \psi(0))^\alpha \right) \right) \quad (23)$$

Different kernel functions ψ are considered for experiment. Table 3 summarizes the experimental results providing optimal values of the parameters involved alongside the error rate obtained when fitting the original data by a model. The estimated parameters in this experiment are the capacitance value C and the fractional order of derivative α .

Table 3: RC charging performances.

	Classical Model	Kernel $\psi(x) = x$	Kernel $\psi(x) = \sqrt{x+1}$	Kernel $\psi(x) = \ln(x+1)$
C	0.0057	0.0051	0.0014	0.0018
α	1	0.9962	1.1864	1.5904
SSE	1.2646	0.0127	0.7277	1.5925
RMSD	0.1125	0.0113	0.0853	0.1262
R	2.83%	0.28%	2.15%	3.18%

It is observable from table 3 that the kernel function $\psi(x) = x$ performs better than others. Indeed it requires an optimal capacitance which is almost equal to the capacitance value of the classical approach, a fractional order of derivative close to 1, and a 0% error rate. The kernel $\psi(x) = \sqrt{x+1}$ also perform better than the classical approach. However the kernel $\psi(x) = \ln(x+1)$ is a bit weaker than the classical approach.

Figure 4 is a set of graphs displaying the behavior of the original data and the experimental results.

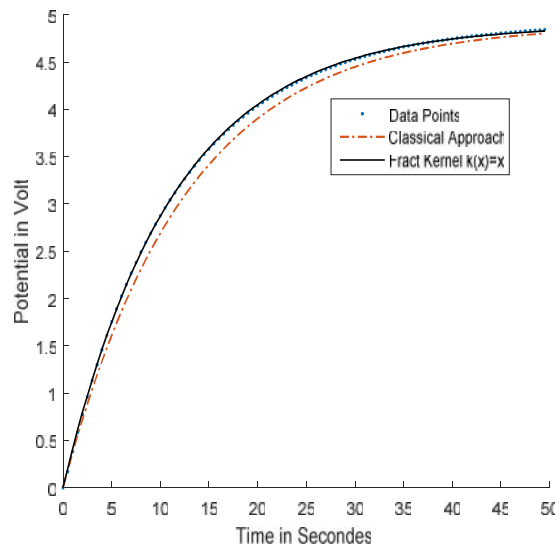


Figure 4a

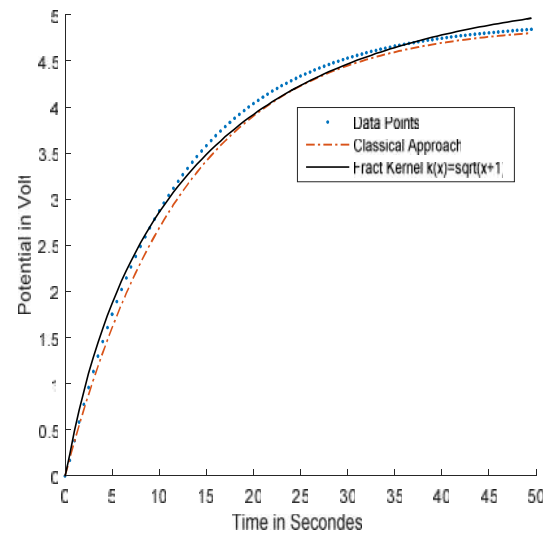


Figure 4b

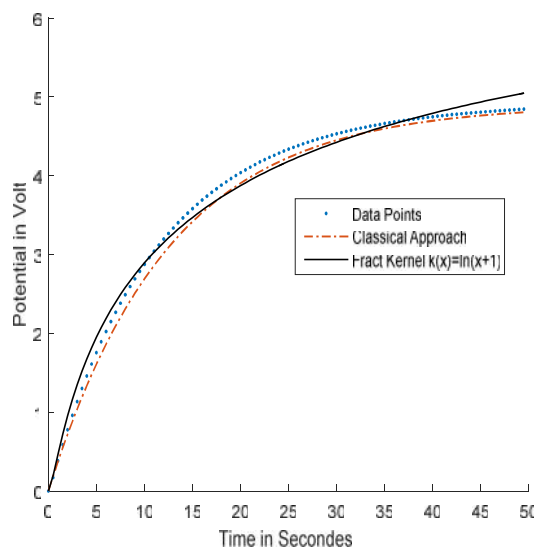


Figure 4c

5. Discussion

1-) Interpreting the meaning of a fractional derivative is often difficult. In the physical sense for instance, the first derivative found in the RC or radioactivity differential equations represent the speed with which charges flow in the circuit respectively the number of radioactive element decrease. It is therefore important to give a meaning to the fractional derivative order α , found in the counterpart fractional differential equation of a given problem. It is not easy to provide a meaning to α , without considering the classical model of the same problem. It is notable the fact that the fractional order derivative of the problem studied in this paper is a value approaching one. Moreover, if $\alpha = 1$, then the Mittag Leffler function equals to the classical exponential function. That is

$$E_{\alpha}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\alpha i + 1)} = e^t, t \in \mathbb{R}. \text{ It follows that the solution to the classical approach}$$

coincides exactly with the solution of to the fractional approach. The statement above shows that fractional derivative is effective and meaningful.

2-) In this work, we considered models whose fractional model are easily obtained from their classical counterpart. However, this should not be used as a property. The Carleman Embedding technique [15], which is applicable to classical linear systems, is not always applicable on fractional system. For instance West.B [16] proposed a function as the exact solution to the fractional logistic equation, his allegation was later proven incorrect by Area et al. [17] because the function proposed by West was derived using the Carleman Embedding technique.

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