

Cubic B-Spline Collocation and Adomian Decomposition Methods on 4th Order Multi-point Boundary value Problems

**Original Research
Article**

Abstract

In this paper, Cubic B-Spline collocation method (CBSCM) and Adomian Decomposition Method (ADM) are applied to obtain numerical solutions to fourth-order linear and nonlinear differential equations. The CBSCM was based on finite element method involving collocation method with cubic B-spline as a basis function. While ADM was based on multistage decomposition method. We discovered in the illustrative examples considered, that result by ADM were compatible with the closed form solutions well over twenty, in some cases over thirty, decimal places and with extremely minimal absolute errors. Results by CBSCM gave correct solutions to atmost six decimal places with sizeable absolute errors. These has further revealed the importance and superiority of ADM over CBSCM in providing semi-analytic solution to this class of differential equations.

Keywords: Cubic B-Spline; Adomian Decomposition Method; Boundary Value Problems; 4th Order Differential Equations

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1 Introduction

4th order differential equations occur in both pure and applied Mathematics. In applied Mathematics they result from mathematical models that describe deformable systems. And because of their

presence in the development and application of latest technologies, researchers have developed interest in them. For nonlinear boundary value problems their analytic solution are obtained only in rare cases. Recently, [Agom et al (2017)] applied the multistage decomposition method to obtain numerical solutions to this class of equations, while the linear case is given in [Agom et al (2016)]. The existence and uniqueness solution for the linear case can be found everywhere in literature especially in [Agarwal (1986)]. Some other method like sinc-Galekin method has been used as well. [Viswanadham and Raju (2012)] used Cubic B-Spline collocation method to obtain numerical solution. [Jincai et al (2011)] compared cubic B-spline and finite difference methods using two point boundary value problems. Their emphasis was placed on second order linear differential equations. The work revealed that the accuracy of the results obtained by cubic B-spline were better than those obtained through finite difference method. [Taiwo and Ogunlaran (2008)] applied cubic spline collocation Tau method on linear 4th order differential equation. The method could not handle 4th order nonlinear ordinary differential equation directly. But, through Newton's linearisation scheme this was made possible and numerical result were obtained as well.

2 Description of Cubic B-Spline Collocation Method and Adomian Decomposition Metthod

2.1 Concept of CBSCM

In operator form the generalised 4th order boundary value differential equation in an expanded form is given as

$$\beta_0 y^{iv} + \beta_1 y''' + \beta_2 y'' + \beta_3 y' + \beta_4 y = \gamma, \quad y = y(t), \quad a \leq t \leq b \quad (2.1)$$

subject to $y(a) = \alpha_0, y(b) = \varpi_0, y'(a) = \alpha_1, y'(b) = \varpi_1$, where $\beta_k = \beta_k(t), k = 0, 1, 2, 3$ are continuous functions defined on the given closed interval, and $\alpha_0, \alpha_1, \varpi_0, \varpi_1$ are finite real constant

Generally, the collocation method belongs to a family of projection methods. It approximate the solution of a differential problem by a finite linear combination of basis fuction. This is done by defining the state variables and the initial/boundary conditions by the basis function. We approximate the solution to equation (2.1) as

$$y(t) = \sum_{j=1}^{n+1} \xi_j B_j(t_i) \quad (2.2)$$

where ξ_j are coefficents to be determined and $t_i \in [a, b]$. We look for a vector of coefficents that guarantees a solution satisfying equation (2.1) on equally spaced grid values t_i (collocation points). Nevertheless, collocation point does not have to always be equally spaced. Optimal grid points selection is not straight forward for it affect and greatly influences the appropriateness of CBSCM. And the computational cost of the method depends on the number of collocation points. The more the points the better the approximate solution and the better equation (2.1) is satisfied. Many authors have reported on how the collocation points can be selected. Some has favoured Gauss-Legendre scheme. Another alternative is to select the points in correspondence to the maximum value of the basis function. That is the so called 'smoothest collocation proceedure'. In general, the point selection should be done in order to obtain a relatively dense coverage of the given domain. Equation (2.1) becomes

$$\beta_0 \sum_j \xi_j B_j^{iv}(t_i, p) + \beta_1 \sum_j \xi_j B_j'''(t_i, p) + \beta_2 \sum_j \xi_j B_j''(t_i, p) + \beta_3 \sum_j \xi_j B_j'(t_i, p) + \Psi = e_i \quad (2.3)$$

where $\Psi = \beta_4 \sum_j \xi_j B_j(t_i, p) - \varphi(t_i)$. And the boundary conditions represented as $\sum_j \xi_j B_j(a, p) = \alpha_0$, $\sum_j \xi_j B_j(b, p) = \varpi_0$, $\sum_j \xi_j B_j'(a, p) = \alpha_1$ and $\sum_j \xi_j B_j'(b, p) = \varpi_1$. e_i is the approximate error evaluated on the i th collocation point and p is the degree of the spline function with $p = 3$ in this paper (cubic spline). The coefficients ξ are approximated under the constraints represented by the given mixed initial and boundary values. We introduce additional six knots, $t_{-3} < t_{-2} < t_{-1} < \dots < t_{n+1} < t_{n+2} < t_{n+3}$ out of $[a, b]$ with $t_0 = a$ and $t_n = b$. For details on the concept CBSCM see [Viswanadham and Raju (2012)].

2.2 Concept of ADM

The concept of ADM has been widely reported in literature this include Adomian himself in [Adomian (1994)], [Agom et al (2017)], [Agom et al (2016)] and [Agom and Ogunfiditimi (2015)]. It is well known that in the vicinity of regular point any ordinary differential equation or system of differential equations may be represented by a normal system of ordinary differential equations of the first order in respect to the unknown function, $y_i = y_i(t)$, $i \in \mathbb{Z}$. On the interval $\Omega : t \in [0, 1]$ with the boundary conditions on the $\delta\Omega : t = (0, 1)$. In compact form, equation (2.1) is reported from the concept ADM as

$$Ly_i + R_i(t, y_1, y_2, \dots, y_n) + N_i(t, y_1, y_2, \dots, y_n) = G_i(t) \quad (2.4)$$

where $G_i(y_1, y_2, \dots, y_n)$ at $\delta\Omega$ is zero. And in this paper $L = \frac{d^4}{dt^4}$. L , R_i are the linear operators and N_i, G_i are the nonlinear operators. We assume that the point $t_0 = 0$ belongs to the closure Ω while R_i, N_i and G_i are the holomorphic functions for $\{y_i\}_{i=1}^n$. According to [Igor (2010)] the solution for equation (2.1) is represented in the form

$$Ly_i = A_{i,j} \quad (2.5)$$

$$y_i = \sum_{j=0}^{\infty} y_{i,j}^A \quad (2.6)$$

where $i = \overline{1, n}, j = \overline{0, \infty}$. $A_{i,j}$ are the Adomian polynomials widely reported in Adomian himself [], [agom poly], [lane emden]. Two important observation can be made in the Adomian Polynomial, A_0 depend y_0 alone, A_1 depend y_0 and y_1 . In general, A_n depends on $y_0, y_1, y_2, \dots, y_{n-2}, y_{n-1}, y_n$. This has also been reported in [Agom et al (2017)] that the solution of equation (2.1) is

$$y(t) - \sum_{p=0}^3 \frac{t^p}{p!} \Psi_p = L^{-1}(\eta_1 - \eta_2 y''' - \eta_3 y'' - \eta_4 y' - \eta_5 y) \quad (2.7)$$

where $\eta_1 = \frac{\gamma}{\beta_0}, \eta_2 = \frac{\beta_1}{\beta_0}, \eta_3 = \frac{\beta_2}{\beta_0}, \eta_4 = \frac{\beta_3}{\beta_0}$ and $\eta_5 = \frac{\beta_4}{\beta_0}$. Ψ_p are determined by using the given boundary conditions.

Generally, ADM is equivalent to the perturbation of the governing equations and its solution in respect of a grouping parameter λ which is introduced in calculating the Adomian polynomials. For extensive details and its calculations see [Agom et al (2017)], [Agom and Ogunfiditimi (2015)], [Agom et al (2016)] and [Agom et al (2016)].

3 Numerical Illustrations

In this section, we give numerical illustrations in form of examples using CBSCM and ADM. The results of the two methods are compared with the analytical solutions and the absolute errors are calculated to determine the accuracy of the two methods and the error norm $L_{\infty} = \max |y_{exact} - y_{approx}|$ clearly seen. We report our findings in Tables 1 to 4. The following examples are adapted from [Viswanadham and Raju (2012)]

3.1 Example 1

Consider the equation

$$y^{(iv)} = y + y'' + e^t(t-3), 0 < t < 1$$

$y(0) = 1, y'(0) = 0, y(1) = 0, y'(1) = -e$. The exact solution is $y = (1-t)e^t$. Numerical results for this problem using the concept of the two methods are shown in Table 1

Table 1: Numerical results for Example 1

t	Exact Solution	Absolute error by CBSCM [Viswanadham and Raju (2012)]	Absolute error by ADM
0.1	9.946538E-01	1.013279E-06	0
0.2	9.771222E-01	3.457069E-06	0
0.3	9.449012E-01	6.496906E-06	0
0.4	8.950948E-01	9.477139E-06	0
0.5	8.243606E-01	1.132488E-05	0
0.6	7.288475E-01	1.102686E-05	0
0.7	6.041259E-01	9.298325E-06	0
0.8	4.451082E-01	5.841255E-06	0
0.9	2.459602E-01	2.205372E-06	0

From Table 1, L_∞ error of CBSCM is 1.132488×10^{-5} while that of ADM is zero.

3.2 Example 2

Consider the equation

$$y^{(iv)} = -4y + 1, -1 < t < 1$$

$$y(-1) = y(1) = 0, y'(-1) = -y'(1) = \frac{\sinh 2 - \sin 2}{4(\cosh 2 - \cos 2)}$$

The exact solution is $y = \frac{1}{4} \left[1 - 2 \frac{\sinh 1 \sin 1 \sinh t \sin t + \cosh 1 \cos 1 \cosh t \cos t}{\cosh 2 + \cos 2} \right]$. Numerical results for this problem using the concept of the two methods in section 2 are shown in Table 2.

Table 2: Numerical results for Example 2

t	Exact Solution	Absolute error by CBSCM [Viswanadham and Raju (2012)]	Absolute error by ADM
-0.8	3.976926E-02	1.206994E-06	0
-0.6	7.498498E-02	2.093613E-06	2.0E-11
-0.4	1.023106E-01	2.659857E-06	0
-0.2	1.195382E-01	2.957880E-06	0
0.0	1.254157E-01	3.099442E-06	0
0.2	1.195382E-01	3.047287E-06	0
0.4	1.023106E-01	2.719462E-06	0
0.6	7.498498E-02	2.190471E-06	2.0E-11
0.8	3.976926E-02	1.288950E-06	0

From Table 2, L_∞ error of CBSCM is 3.099442×10^{-6} while that of ADM, whose result was evaluated to thirty decimal places, is 2.0×10^{-11} .

3.3 Example 3

Consider the equation

$$y^{(iv)} - y = -4(2t \cos t + 3 \sin t), -1 < t < 1$$

$y(-1) = y(1) = 0, y'(-1) = y'(1) = 2 \sin 1$. The exact solution is $y = (t^2 - 1) \sin t$. Numerical results for this problem using the concept of the two methods in section 2 are shown in Table 3.

Table 3: Numerical results for Example 3

t	Exact Solution	Absolute error by CBSCM [Viswanadham and Raju (2012)]	Absolute error by ADM
-0.8	2.582482E-01	2.741814E-06	0
-0.6	3.613712E-01	3.486872E-06	0
-0.4	3.271114E-01	2.622604E-06	0
-0.2	1.907225E-01	9.089708E-07	0
0.0	0	9.540817E-07	0
0.2	-1.907226E-01	2.682209E-06	0
0.4	-3.271114E-01	3.755093E-06	0
0.6	-3.613712E-01	4.321337E-06	0
0.8	-2.582482E-01	3.248453E-06	0

From Table 3, L_∞ error of CBSCM is 4.321337×10^{-6} while that of ADM, whose result was evaluated to thirty decimal places, is zero

3.4 Example 4

Consider the equation

$$y^{(iv)} - y^2 = -t^{10} + 4t^9 - 4t^8 - 4t^7 + 8t^6 - 4t^4 + 120t - 48, 0 < t < 1$$

$y(0) = y'(0) = 0, y(1) = y'(1) = 1$. The exact solution is $y = t^5 - 2t^4 + 2t^2$. Numerical results for this problem using the concept of the two methods in section 2 are shown in Table 4.

Table 4: Numerical results for Example 4

t	Exact Solution	Absolute error by CBSCM [Viswanadham and Raju (2012)]	Absolute error by ADM
0.1	1.981000E-02	4.921108E-06	0
0.2	7.712000E-02	6.802380E-06	0
0.3	1.662300E-01	6.362796E-06	0
0.4	2.760400E-01	4.351139E-06	0
0.5	4.062500E-01	1.370907E-06	0
0.6	5.385600E-01	2.145767E-06	0
0.7	6.678700E-01	4.649162E-06	0
0.8	7.884800E-01	6.258488E-06	0
0.9	8.982900E-01	4.827976E-06	1.0E-10

From Table 4, L_∞ error of CBSCM is 6.802380×10^{-6} while that of ADM, whose result was evaluated to thirty decimal places, is 1.0×10^{-10}

4 CONCLUSIONS

We have, in this paper, shown that results of 4th order linear and nonlinear differential equations by ADM are better than those obtain by CBSCM on the same class of equations in all the illustrative example considered. We began this article by giving the abstract, introducing the topic and referring to some related works done that can be found in literature. The concept of ADM and CBSCM were outlined in section two. In the subsequent section we gave illustrative examples to demonstrate the use of the two methods. In each example we gave the absolute error in relation to the analytical solutions in tabular form and the maximum absolute error for each example indicated. In general, ADM results were to a large extent, in total agreement with the exact solutions while CBSCM results were to a lesser degree in agreement with analytical solutions.

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