
Oscillation of Second Order Difference Equation With a Superlinear Neutral Term

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Abstract

This paper deals with oscillation of certain class of second order difference equation with a superlinear neutral term. Using comparison method some new oscillation criteria are obtained. Examples are included to illustrate the main results.

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1 Introduction

In this paper, we are concerned with the oscillatory behavior of nonlinear difference equation with a superlinear neutral term

$$\Delta(a_n \Delta(x_n + p_n x_{n-k}^\alpha)) + q_n x_{n+1-l}^\beta = 0, \quad n \geq n_0 \quad (1.1)$$

where n_0 is a nonnegative integer, subject to the following conditions:

- (H₁) $\alpha \in [1, \infty)$ and β are ratios of odd positive integers;
- (H₂) $\{a_n\}$, $\{p_n\}$ and $\{q_n\}$ are positive real sequences for all $n \geq n_0$;
- (H₃) k is a positive integer and l is a nonnegative integer.

Let $\theta = \max\{k, l-1\}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and satisfies equation (1.1) for all $n \geq n_0$. A solution of equation (1.1) is called oscillatory if its terms are neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. If all solutions of the equation are oscillatory then the equation itself called oscillatory.

In the last few years, there has been a great interest in investigating the oscillatory and asymptotic behavior of neutral type difference equations, see Agarwal, R.P. et al. (2005, 2000); Gyori, I. and Ladas, G. (1991); Yang, J. et al. (1997) and the references cited therein.

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In Tang, XH. et al. (2001), X.Liu considered the equation of the form

$$\Delta(x_n - p_n x_{n-k}^\alpha) + q_n x_{n-l}^\beta = 0, \quad n \geq n_0 \quad (1.2)$$

and studied the oscillatory behavior of equation (1.2).

In Thandapani, E. et al. (2004a), Thandapani et.al investigated the oscillation of all solutions of the equation

$$\Delta(a_n \Delta(x_n - p_n x_{n-k}^\alpha)) + q_n x_{n+1-l}^\beta = 0, \quad n \geq n_0 \quad (1.3)$$

where p is a real number, k and l are positive integers, $0 < \alpha \leq 1$ and β are ratios of odd positive integers.

In Yildiz, MK. and H.Ogurmez (2014), Yildiz and Ogunmez considered equation (1.1) with $m = 2$ which has the form

$$\Delta^2(x_n + p_n x_{n-k}^\alpha) + q_n x_{n-l}^\beta = 0, \quad n \geq n_0 \quad (1.4)$$

where $\{p_n\}$ is a real sequence, $\{q_n\}$ is a nonnegative real sequence, and $\alpha > 1$ and β are ratios of odd positive integers. They also investigated the oscillatory behavior of solutions of equation (1.4).

In Thandapani, E. et al. (2004b), the authors considered equation (1.1) and obtained criteria for the oscillation of solutions when $0 < \alpha \leq 1$ and either

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty, \quad (1.5)$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty \quad (1.6)$$

using some inequalities and Riccati type transformation.

Thus, we see that many oscillation results known for the equation (1.1) when the neutral term is sublinear and very few results available for the equation (1.1) when the neutral term is superlinear, that is $\alpha > 1$. Therefore in this paper we obtain some new oscillation criteria for the equation (1.1) when the neutral term is superlinear.

In Yildiz, MK. and H.Ogurmez (2014), the authors discussed the oscillatory behavior of equation (1.1) when $a_n \equiv 1$ and $\lim_{n \rightarrow \infty} \inf q_n > 0$. But in this paper we derived conditions for the oscillation of all solutions of equation (1.1) without such assumptions. Hence our results improve and complement to that of in Thandapani, E. et al. (2004a,b); Vidhyaa, KS. et al. (2017); Yildiz, MK. and H.Ogurmez (2014); Zhang, Z. et al. (2001). Example are included to illustrate the main results.

2 Oscillation Results

In this section, we obtain sufficient conditions for the oscillation of all solutions of the equation (1.1). We set

$$z_n = x_n + p_n x_{n-k}^\alpha, \quad R_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s}$$

and

$$B_n = (M^{1-\alpha} R_n^{1-\alpha} - p_n) > 0 \text{ for all constant } M > 0.$$

From the nature of equation, we only need to give proofs for the case of eventually positive solutions since the proofs for the eventually negative solutions would be similar. We begin with the following theorem.

Theorem 2.1. Assume that $(H_1) - (H_3)$ and (1.5) hold. If

$$\sum_{n=n_1}^{\infty} q_n B_{n+1-l}^{\beta} = \infty \quad (2.1)$$

holds, then every solution of equation (1.1) is oscillatory.

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution $\{x_n\}$, say $x_n > 0$, $x_{n-k} > 0$, and $x_{n+1-l} > 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. From the definition of z_n and equation (1.1), we have

$$\Delta(a_n \Delta z_n) = -q_n x_{n+1-l}^{\beta} < 0, \quad n \geq n_1. \quad (2.2)$$

In view of condition (1.5), it is easy to see that $\Delta z_n > 0$ for all $n \geq n_1$. From (2.2), we have $a_n \Delta z_n$ is positive and decreasing, and therefore

$$z_n = z_{n_1} + \sum_{s=n_1}^{n-1} \frac{a_s \Delta z_s}{a_s} \geq R_n a_n \Delta z_n. \quad (2.3)$$

Now

$$\Delta \left(\frac{z_n}{R_n} \right) = \frac{R_n \Delta z_n - z_n \cdot \frac{1}{R_n}}{R_n R_{n+1}} \leq 0 \quad \text{for all } n \geq n_1, \quad (2.4)$$

where we have used (2.3). It follows from the definition of z_n , and using $\frac{z_n}{R_n} \leq M > 0$ for all $n \geq n_1$, we have

$$\begin{aligned} x_n &= z_n - p_n x_{n-k}^{\alpha} \geq (z_n^{1-\alpha} - p_n) z_n^{\alpha} \\ &\geq (M^{1-\alpha} R_n^{1-\alpha} - p_n) z_n^{\alpha}, \quad n \geq n_1. \end{aligned} \quad (2.5)$$

Substituting (2.5) into (2.2) yields

$$\Delta(a_n \Delta z_n) + q_n B_{n+1-l}^{\beta} z_{n+1-l}^{\alpha\beta} \leq 0, \quad n \geq n_1. \quad (2.6)$$

Summing equation (2.6) from n_1 to n , we obtain

$$\sum_{s=n_1}^n q_s B_{s+1-l}^{\beta} z_{s+1-l}^{\alpha\beta} \leq a_{n_1} \Delta z_{n_1}.$$

Since $z_n \geq K > 0$, the last inequality contradicts (2.1) as $n \rightarrow \infty$. This completes the proof of the theorem. $\square$

In the next theorem, we reduce the oscillation of equation (1.1) to that of a first order delay difference equation.

Theorem 2.2. Assume that $(H_1) - (H_3)$ and (1.5) hold. If the first order delay difference equation

$$\Delta w_n + q_n B_{n+1-l}^{\beta} R_{n+1-l}^{\alpha\beta} w_{n+1-l}^{\alpha\beta} = 0, \quad n \geq n_0 \quad (2.7)$$

is oscillatory, then every solution of equation (1.1) is oscillatory.

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution $\{x_n\}$, say $x_n > 0$, $x_{n-k} > 0$, and $x_{n+1-l} > 0$ for all $n \geq n_1 \geq n_0$. Proceeding as in proof of Theorem 2.1, we obtain (2.6). Now use (2.3) in (2.6), we have

$$\Delta(a_n \Delta z_n) + q_n B_{n+1-l}^{\beta} R_{n+1-l}^{\alpha\beta} (a_{n+1-l} \Delta z_{n+1-l})^{\alpha\beta} \leq 0, \quad n \geq n_1.$$

Set $w_n = a_n \Delta z_n$. Then $\{w_n\}$ is a positive solution of the inequality

$$\Delta w_n + q_n B_{n+1-l}^{\beta} R_{n+1-l}^{\alpha\beta} w_{n+1-l}^{\alpha\beta} \leq 0, \quad n \geq n_1.$$

It follows from Lemma 2.7 of Thandapani, E. et al. (2012), the corresponding difference equation (2.7) also has a positive solution, which is a contradiction. The proof is now completed. $\square$

Remark 2.1. Employing sufficient conditions for the oscillation of all solutions of equation (2.7), one can obtain easily verifiable criteria for the oscillation of all solutions of equation (1.1).

Corollary 2.3. Assume that $(H_1) - (H_3)$ and (1.5) hold. If $\alpha\beta = 1$, $l \geq 2$, and

$$\liminf_{n \rightarrow \infty} \sum_{s=n-l+1}^{n-1} q_s B_{s+1-l}^\beta R_{s+1-l} > \left(\frac{l-1}{l}\right)^l \quad (2.8)$$

hold, then every solution of equation (1.1) is oscillatory.

Proof. Condition (2.8) and Theorem 7.5.1 of Gyori, I. and Ladas, G. (1991) implies oscillation of equation (2.7). The assertion now follows from Theorem 2.2. This completes the proof. $\square$

Corollary 2.4. Assume that $(H_1) - (H_3)$ and (1.5) hold. If $\alpha\beta < 1$, and

$$\sum_{n=n_1}^{\infty} q_n B_{n+1-l}^\beta R_{n+1-l}^{\alpha\beta} = \infty \quad (2.9)$$

hold, then every solution of equation (1.1) is oscillatory.

Proof. Condition (2.9) and Theorem 1 of Tang, XH. et al. (2001) implies oscillation of equation (2.7). The assertion now follows from Theorem 2.2. The proof is now completed. $\square$

Corollary 2.5. Assume that $(H_1) - (H_3)$ and (1.5) hold. If $\alpha\beta > 1$, $l \geq 2$, and there exists a $\lambda > \frac{1}{l-1} \log \alpha\beta$ such that

$$\liminf_{n \rightarrow \infty} \left[q_n B_{n+1-l}^\beta R_{n+1-l}^{\alpha\beta} \exp(-e^{\lambda n}) \right] > 0 \quad (2.10)$$

hold, then every solution of equation (1.1) is oscillatory.

Proof. Condition (2.10) and Theorem 2 of Tang, XH. et al. (2001) implies oscillation of equation (2.7). The conclusion now follows from Theorem 2.2, and the proof is completed. $\square$

Our next results are for the case where (1.6) holds in place of (1.5). We let

$$A_n = \sum_{s=n}^{\infty} \frac{1}{a_s}.$$

We will also need the condition

$$E_n = \left(K^{1-\alpha} - \frac{p_n A_{n-k}^\alpha}{A_n^\alpha} \right) > 0 \quad (2.11)$$

for all constant $K > 0$ and all $n \geq n_1 \geq n_0$.

Theorem 2.6. Let $\alpha\beta \geq 1$, and $(H_1) - (H_3)$, (1.6) and (2.11) hold. If there exists a positive nondecreasing real sequence $\{\rho_n\}$ such that

$$\limsup_{n \rightarrow \infty} \sum_{s=n_1}^n \left[\rho_s q_s B_{s+1-l}^\beta - \frac{a_{s-l} (\Delta \rho_s)^2}{4\alpha\beta \rho_s} \right] = \infty \quad (2.12)$$

and

$$\limsup_{n \rightarrow \infty} \sum_{s=n_1}^{n-1} \left[A_{s+1}^{\alpha\beta} q_s E_{s+1-l}^\beta - \frac{\alpha\beta K_1^{1-\alpha\beta} A_s^{\alpha\beta-1}}{4a_s A_{s+1}^{\alpha\beta}} \right] = \infty \quad (2.13)$$

hold for all constants $K_1 > 0$, then every solution of equation (1.1) is oscillatory.

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution such that $x_n > 0$, $x_{n-k} > 0$, and $x_{n+1-l} > 0$ for all $n \geq n_1 \geq n_0$. From equation (1.1) that (2.2) holds. We then have either $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If $\Delta z_n > 0$ holds, then we can proceed as in the proof of Theorem 2.1, we obtain (2.6). Define

$$w_n = \rho_n \frac{a_n \Delta z_n}{z_{n-l}^{\alpha\beta}}, \quad n \geq n_1. \quad (2.14)$$

Then $w_n > 0$ for $n \geq n_1$, and

$$\begin{aligned} \Delta w_n &= \rho_n \frac{\Delta(a_n \Delta z_n)}{z_{n+1-l}^{\alpha\beta}} + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta z_{n-l}^{\alpha\beta}}{z_{n-l}^{\alpha\beta}} \\ &\leq -\rho_n q_n B_{n+1-l}^\beta + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta z_{n-l}^{\alpha\beta}}{z_{n-l}^{\alpha\beta}}. \end{aligned} \quad (2.15)$$

By Mean value theorem, we have

$$\Delta z_{n-l}^{\alpha\beta} \leq \alpha\beta z_{n+1-l}^{\alpha\beta-1} \Delta z_{n-l}$$

and using this in (2.15) gives

$$\Delta w_n \leq -\rho_n q_n B_{n+1-l}^\beta + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \alpha\beta \frac{\rho_n}{\rho_{n+1}^2 a_{n-l}} w_{n+1}^2, \quad n \geq n_1 \quad (2.16)$$

where we have used $a_n \Delta z_n$ is positive and decreasing and z_n is increasing. Using the completing the square in (2.16) yields

$$\Delta w_n \leq -\rho_n q_n B_{n+1-l}^\beta + \frac{a_{n-l}(\Delta \rho_n)^2}{4\alpha\beta \rho_n}, \quad n \geq n_1.$$

Summing the last inequality from n_1 to n , we obtain

$$\sum_{s=n_1}^n \left[\rho_s q_s B_{s+1-l}^\beta - \frac{a_{s-l}(\Delta \rho_s)^2}{4\alpha\beta \rho_s} \right] \leq w_{n_1}$$

which contradicts (2.12) as $n \rightarrow \infty$. Next assume that $\Delta z_n < 0$ for all $n \geq n_1$. Define

$$u_n = \frac{a_n \Delta z_n}{z_n^{\alpha\beta}}, \quad n \geq n_1. \quad (2.17)$$

Thus $u_n < 0$ for all $n \geq n_1$, and from (2.2) we have

$$\Delta z_s \leq \frac{a_n \Delta z_n}{a_s}, \quad s \geq n.$$

Summing the last inequality from n to j , we obtain

$$z_{j+1} - z_n \leq a_n \Delta z_n \sum_{s=n}^j \frac{1}{a_s};$$

and then letting $j \rightarrow \infty$ gives

$$\frac{a_n \Delta z_n A_n}{z_n} \geq -1, \quad n \geq n_1. \quad (2.18)$$

Thus

$$\frac{-a_n \Delta z_n (-a_n \Delta z_n)^{\alpha\beta-1} A_n^{\alpha\beta}}{z_n^{\alpha\beta}} \leq 1$$

for $n \geq n_1$. Since $-a_n \Delta z_n > 0$ and (2.2) and (2.17) hold, we have

$$-\frac{1}{L^{\alpha\beta-1}} \leq u_n A_n^{\alpha\beta} \leq 0, \quad (2.19)$$

where $L = -a_{n_1} \Delta z_{n_1}$. On the other hand from (2.18), we obtain

$$\Delta \left(\frac{z_n}{A_n} \right) \geq 0, \quad n \geq n_1. \quad (2.20)$$

From the definition of z_n and (2.20), we have

$$\begin{aligned} x_n &= z_n - p_n x_{n-k}^{\alpha} \geq z_n - p_n z_{n-k}^{\alpha} \geq z_n - p_n \frac{A_{n-k}^{\alpha}}{A_n^{\alpha}} z_n^{\alpha} \\ &\geq \left(K^{1-\alpha} - \frac{p_n A_{n-k}^{\alpha}}{A_n^{\alpha}} \right) z_n^{\alpha}, \quad n \geq n_1, \end{aligned} \quad (2.21)$$

where we have used $z_n \leq K$ for $K > 0$ and all $n \geq n_1$. From (2.21) and (2.2), we have

$$\Delta(a_n \Delta z_n) + q_n E_{n+1-l}^{\beta} z_{n+1-l}^{\alpha\beta} \leq 0, \quad n \geq n_1. \quad (2.22)$$

From (2.14), we have

$$\Delta u_n = \frac{\Delta(a_n \Delta z_n)}{z_{n+1}^{\alpha\beta}} - u_n \frac{\Delta z_n^{\alpha\beta}}{z_{n+1}^{\alpha\beta}}, \quad n \geq n_1. \quad (2.23)$$

By Mean value theorem

$$\Delta z_n^{\alpha\beta} \leq \begin{cases} \alpha\beta z_{n+1}^{\alpha\beta-1} \Delta z_n, & \text{if } \alpha\beta > 1; \\ \alpha\beta z_n^{\alpha\beta-1} \Delta z_n, & \text{if } \alpha\beta < 1, \end{cases} \quad (2.24)$$

and so combining (2.23) with (2.24) and then using the fact that $\Delta z_n < 0$ gives

$$\Delta u_n \leq -q_n E_{n+1-l}^{\beta} - \alpha\beta K_1^{\alpha\beta-1} A_n^{\alpha\beta-1} \frac{u_n^2}{a_n}, \quad n \geq n_1, \quad (2.25)$$

since $\frac{z_n}{z_{n+1}} \geq 1$ and $\frac{z_n}{A_n} \geq K > 0$ for all $n \geq n_1$. Multiplying (2.25) by $A_{n+1}^{\alpha\beta}$ and then summing it from n_1 to $n-1$, we have

$$\sum_{s=n_1}^{n-1} A_{s+1}^{\alpha\beta} \Delta u_s + \sum_{s=n_1}^{n-1} A_{s+1}^{\alpha\beta} q_s E_{s+1-l}^{\beta} + \sum_{s=n_1}^{n-1} \alpha\beta K_1^{\alpha\beta-1} A_s^{\alpha\beta-1} A_{s+1}^{\alpha\beta} \frac{u_s^2}{a_s} \leq 0. \quad (2.26)$$

Using summation by parts formula and Mean value theorem, we have

$$\sum_{s=n_1}^{n-1} A_{s+1}^{\alpha\beta} \Delta u_s \geq A_n^{\alpha\beta} u_n - A_{n_1}^{\alpha\beta} u_{n_1} + \sum_{s=n_1}^{n-1} \alpha\beta A_s^{\alpha\beta-1} \frac{u_s^2}{a_s}. \quad (2.27)$$

From (2.26) and (2.27), we obtain

$$\begin{aligned} A_n^{\alpha\beta} u_n^{\alpha\beta} - A_{n_1}^{\alpha\beta} u_{n_1}^{\alpha\beta} + \sum_{s=n_1}^{n-1} A_{s+1}^{\alpha\beta-1} q_s E_{s+1-l}^{\beta} &+ \sum_{s=n_1}^{n-1} \alpha\beta A_s^{\alpha\beta-1} \frac{u_n}{a_n} \\ &+ \sum_{s=n_1}^{n-1} \alpha\beta K_1^{\alpha\beta-1} A_s^{\alpha\beta-1} A_{s+1}^{\alpha\beta} \frac{u_s^2}{a_s} \leq 0 \end{aligned}$$

which on using completing the square yields

$$\sum_{s=n_1}^{n-1} \left[A_{s+1}^{\alpha\beta} q_s E_{s+1-l}^{\beta} - \frac{\alpha\beta K_1^{1-\alpha\beta} A_s^{\alpha\beta-1}}{4a_s A_{s+1}^{\alpha\beta}} \right] \leq \frac{1}{L^{\alpha\beta-1}} + A_{n_1}^{\alpha\beta} u_{n_1}$$

when using (2.19). This contradicts (2.13) as $n \rightarrow \infty$, and the proof is now completed. $\square$

Corollary 2.7. Let $\alpha\beta > 1$ and $(H_1) - (H_3)$, (1.6) and (2.11) hold. If condition (2.13) and $l \geq 2$ hold, and there exists a $\lambda > \frac{1}{(l-1)} \log \alpha\beta$ such that (2.10) holds then every solution of equation (1.1) is oscillatory.

Proof. The proof follows from Corollary 2.5 and Theorem 2.6. This completes the proof. $\square$

Theorem 2.8. Let $0 < \alpha\beta < 1$ and $(H_1) - (H_3)$, (1.6) and (2.11) hold. If condition (2.9) holds, and

$$\lim_{n \rightarrow \infty} \sup \sum_{s=n_1}^{n-1} \left[K^{\alpha\beta-1} A_{s+1} q_s E_{s+1-l}^\beta - \frac{1}{4a_s A_{s+1}} \right] = \infty \quad (2.28)$$

for every constant $K > 0$, then every solution of equation (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.6, we see that $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If $\Delta z_n > 0$ then by Corollary 2.4, we obtain a contradiction to (2.9). Next, we assume that $\Delta z_n < 0$ for all $n \geq n_1$. Proceeding as in the proof of Theorem 2.6 we obtain (2.22). Define

$$u_n = \frac{a_n \Delta z_n}{z_n}, \quad n \geq n_1. \quad (2.29)$$

Thus $u_n < 0$ for all $n \geq n_1$. From (2.29) and (2.22), we see that

$$\Delta u_n \leq \frac{-q_n E_{n+1-l}^\beta}{K^{1-\alpha\beta}} - \frac{u_n^2}{a_n}, \quad n \geq n_1. \quad (2.30)$$

where we have used $z_n \leq K$ for $K > 0$ and for all $n \geq n_1$. Multiplying (2.30) by A_{n+1} and then summing it from n_1 to $n-1$, we obtain

$$\sum_{s=n_1}^{n-1} A_{s+1} \Delta u_s + \sum_{s=n_1}^{n-1} K^{\alpha\beta-1} A_{s+1} q_s E_{s+1-l}^\beta + \sum_{s=n_1}^{n-1} \frac{A_{s+1} u_s^2}{a_s} \leq 0.$$

Using the summation by parts formula in the first term of the last inequality and rearranging, we obtain

$$A_n u_n - A_{n_1} u_{n_1} + \sum_{s=n_1}^{n-1} K^{\alpha\beta-1} A_{s+1} q_s E_{s+1-l}^\beta + \sum_{s=n_1}^{n-1} \left(\frac{u_s}{a_s} + A_{s+1} \frac{u_s^2}{a_s} \right) \leq 0$$

which on using completing the square yields

$$\sum_{s=n_1}^{n-1} \left[K^{\alpha\beta-1} A_{s+1} q_s E_{s+1-l}^\beta - \frac{1}{4a_s A_{s+1}} \right] \leq 1 + A_{n_1} u_{n_1}$$

when using (2.18). This contradicts (2.28) as $n \rightarrow \infty$, and the proof is now complete. $\square$

3 Examples

In this section, we present some examples to illustrate our main results.

Example 3.1. Consider the neutral difference equation

$$\Delta \left(\frac{1}{2n+1} \Delta \left(x_n + \frac{1}{n} x_{n-2}^3 \right) \right) + \left(\frac{4(n+1)}{(2n+1)(2n+3)} + \frac{2}{n(n+2)} \right) x_{n-1}^{1/3} = 0, \quad n \geq 1. \quad (3.1)$$

Here $a_n = \frac{1}{2n+1}$, $p_n = \frac{1}{n}$, $q_n = \frac{4(n+1)}{(2n+1)(2n+3)} + \frac{2}{n(n+2)}$, $\alpha = 3$, $\beta = \frac{1}{3}$, $k = 2$ and $l = 2$. Simple calculation shows that $R_n = n^2 - 1$ and we see that all conditions of Corollary 2.3 are satisfied. Hence every solution of equation (3.1) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such solution of equation (3.1).

Example 3.2. Consider the neutral difference equation

$$\Delta \left(n \Delta \left(x_n + \frac{1}{n} x_{n-2}^3 \right) \right) + \frac{(4n^3 + 18n^2 + 24n + 9)}{(n+1)(n+2)} x_{n-1}^{1/5} = 0, \quad n \geq 1. \quad (3.2)$$

Here $a_n = n$, $p_n = \frac{1}{n}$, $q_n = \frac{(4n^3 + 18n^2 + 24n + 9)}{(n+1)(n+2)}$, $\alpha = 3$, $\beta = \frac{1}{5}$, $k = 1$ and $l = 2$. It is easy to see that all conditions of Corollary 2.4 are satisfied, and hence every solution of equation (3.2) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such solution of equation (3.2).

Example 3.3. Consider the neutral difference equation

$$\Delta \left((n+1)(n+2) \Delta \left(x_n + \frac{1}{n(n+1)} x_{n-1}^3 \right) \right) + 2 \left(2n^2 + 8n + 8 - \frac{2n^2 + 4n + 1}{n(n+1)} \right) x_{n-1}^3 = 0, \quad n \geq 1. \quad (3.3)$$

Here $a_n = (n+1)(n+2)$, $p_n = \frac{1}{n(n+1)}$, $q_n = 2 \left(2n^2 + 8n + 8 - \frac{2n^2 + 4n + 1}{n(n+1)} \right)$, $\alpha = 3$, $\beta = 3$, $k = 1$ and $l = 2$. Simple calculation shows that $A_n = \frac{1}{n+1}$, $\alpha\beta = 9$ and $E_n = \left(K^{-2} - \frac{(n+1)^2}{n^4} \right) > 0$. Further the conditions (2.12) and (2.13) are satisfied with $\rho_n = 1$. Therefore by Theorem 2.6, every solution of equation (3.3) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such solution of equation (3.3).

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