

## Small random perturbations for dynamical systems with reflecting boundary in Besov-Orlicz space

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### Abstract

This paper is devoted to prove Freidlin & Wentzell estimations for diffusion processes with reflecting boundary using a modification of Azencott's method in Besov-Orlicz space defined by the Young function  $M_2(x) = \exp(x^2) - 1$ .

**Keywords:** Large deviations principle, perturbed diffusion processes with reflecting boundary, Besov-Orlicz space.

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### 1 Introduction

Let  $(X^\varepsilon, a^\varepsilon)$  be the solution of the equation defined by :

$$X_t^\varepsilon = x + \sqrt{\varepsilon} \int_0^t \sigma(X_s^\varepsilon) dW_s + \int_0^t b_\varepsilon(X_s^\varepsilon) ds + \int_0^t \nu(X_s^\varepsilon) da_s^\varepsilon \quad (1.1)$$

with  $X_t^\varepsilon \in \bar{D}$  ( $D$  is a regular connected open set of  $\mathbb{R}^d$ ),  $a_t^\varepsilon$  is a continuous increasing process increasing only on  $\{s; X_s^\varepsilon \in \partial D\}$  and  $\nu(x)$  is a non-tangent vector field on  $\partial D$ .

We suppose that  $b_\varepsilon, b$  are vector fields defined on  $\mathbb{R}^d$  and  $\sigma$  a matrix field  $d \times l$  on  $\mathbb{R}^d$  satisfying the following conditions:

- i)  $\sigma$  et  $b_\varepsilon$  are bounded lipschitzian.
- ii)  $b_\varepsilon \rightarrow b$  uniformly when  $\varepsilon \rightarrow 0$ .

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$W$  is a brownian motion with value in  $\mathbb{R}^l$  defined on the filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ .

In this paper, we want to establish the LDP of the equation (1.1) on the Besov-Orlicz space. The case  $\partial D = \emptyset$  has been studied by Freidlin, M. and Wentzell, A. (1984) see also Varadhan, S.R.S (1984), Azencott, R. (1980) and Stroock, D.W. (1984) for the uniform norm, then in Ben Arous, G. and Ledoux, M. (1994) see also Baldi, P. and Caramellino, L. (2011), they have shown the LDP of this equation for the Hölderian norm and Eddahbi, M., Nzi, M. and Ouknine, Y. (1999) on the Besov-Orlicz space. The general case has been studied by Anderson, R. and Orey, S. (1976) and Doss, H. and Priouret, P. (1983) for a regular domain, then Dupuis, P. (1987) for a convex domain and Sheu, S.S. (1998) for a non regular domain that is normally reflected.

Our goal is to prove that we can extend the result of Rajaonarison, L.I. and Rabehimanana, T.J. (2014) and Doss, H. and Priouret, P. (1983) and Anderson, R. and Orey, S. (1976) on the large deviations of the law of  $X^\varepsilon$  solution of (1.1) to the topology of the Besov-Orlicz space defined by the Young function

$$M_\beta(x) = \begin{cases} \exp(|x|^\beta) - 1 & \text{si } 1 \leq \beta \leq 2 \\ E_\beta(x) - E_\beta(0) & \text{si } 0 < \beta < 1 \end{cases}$$

where  $E_\beta(-x) = E_\beta(x)$  is the extension of the convex part of  $\exp(x^\beta)$  on  $]x_\beta, +\infty[$  by its tangent at  $x_\beta > 0$  and the continuity modulus  $w_{\alpha,\lambda}(t) = t^\alpha(1 + \log \frac{1}{t})^\lambda$ .

It is useful in section 1 to present some notions and results on the topology of Besov-Orlicz space. The section 2 will be devoted to the general results of this work. The last section is devoted to the proof of the main theorem, the theorem (4.3) and (4.7).

## 2 Preliminaries

In this section we recall useful facts about the Besov-Orlicz space functions.

Let  $B_{M_\beta, w_\alpha}$  denote the Besov-Orlicz space of continuous functions  $f : [0, 1] \rightarrow \mathbb{R}^d$  such that  $\|f\|_{M_\beta, w_\alpha} < \infty$ . For all  $\alpha > 0$ , put

$$\|f\|_{M_\beta, w_\alpha} = \|f\|_{M_\beta} + \sup_{0 \leq t \leq 1} \frac{w_{M_\beta}(f, t)}{w_{\alpha,\lambda}(t)}$$

where  $w_{\alpha,\lambda}(t) = t^\alpha(1 + \log \frac{1}{t})^\lambda, \forall \alpha > 0, \|f\|_{M_\beta} = \inf\{\tau > 0, \frac{1}{\tau}[1 + \int_0^1 M_\beta(\tau|f(t)|)dt]\}$  and  $w_{M_\beta}(f, t) = \sup_{0 \leq h \leq t} \|\Delta_h f\|_{M_\beta}$  with

$$\Delta_h f(x) = 1_{[0, 1-h]}(x)(f(x+h) - f(x)), \forall h \in [0, 1].$$

We will use the equivalent of Ciesleski, Z., Kerkycharian, G. and Roynette, B. (1993). Let  $\chi_1, \chi_{j,k}, j = 0, 1, \dots, k = 1 \dots 2^j, \text{supp} \chi_{j,k} = [(k-1)/2^j, k/2^j]$  be the set of Haar functions over the interval  $[0, 1]$ , and let  $\varphi_0(t) = 1, \varphi_1(t) = t, \varphi_{j,k}(t) = \int_0^t \chi_{j,k}(s)ds$  be the set of Schauder functions. For all continuous functions  $f : [0, 1] \rightarrow \mathbb{R}^d$ , its development in series of Schauder is given by

$$f(t) = f_0 \varphi_0(t) + f_1 \varphi_1(t) + \sum_{j,k} f_{j,k} \varphi_{j,k}(t)$$

where  $f_0 = f(0), f_1 = f(1) - f(0)$  and

$$f_{j,k} = 2^{\frac{j}{2}} \left[ \left( f\left(\frac{2k-1}{2^{j+1}}\right) - f\left(\frac{2k-2}{2^{j+1}}\right) \right) - \left( f\left(\frac{2k}{2^{j+1}}\right) - f\left(\frac{2k-1}{2^{j+1}}\right) \right) \right]$$

are the coefficients of  $f$  in this base.

Let  $B_{M_\beta, w_\alpha}^0$  be the subspace of  $B_{M_\beta, w_\alpha}$  corresponding to the sequences  $f_{j,k}$  such that

$B_{M_\beta, w_\alpha}^0 = \{f \in \mathcal{C}([0, 1], \mathbb{R}^d); \|f\|_{M_2, w_\alpha} < \infty, \lim_{j \wedge p \rightarrow \infty} 2^{-j(\frac{1}{2} - \alpha + \frac{1}{p})} p^{-\gamma} (1+j)^{-\lambda} \|f_{j,\cdot}\|_p = 0\}$   
where

$$\|f_{j,\cdot}\|_p = \left( \sum_{k=1}^{2^j} |f_{j,k}|^p \right)^{\frac{1}{p}} \text{ et } \beta\gamma = 1$$

$B_{M_\beta, w_\alpha}^0$  is a separable Banach space.

For all  $p_0 \geq 1$ , the norms  $\|f\|_{M_\beta, w_\alpha}$  and  $\|f\|^*$  are equivalent (cf. Ciesleski, Z., Kerkycharian, G. and Roynette, B. (1993)) where

$$\|f\|^* = \max(|f_0|, |f_1|, \sup_{p \geq p_0} \sup_{j \geq 0} 2^{-j(\frac{1}{2} - \alpha + \frac{1}{p})} p^{-\gamma} (1+j)^{-\lambda} \|f_{j,\cdot}\|_p)$$

for all  $\frac{1}{p_0} < \alpha \leq \frac{1}{2}$  and  $\alpha > 0$ .

Consider the following norm that are crucial in the proof of our results :

$$\|f\|^{**} = \sup_{0 \leq s < t \leq 1} \frac{|f(t) - f(s)|}{w(t-s)}$$

this is dominated by the norm

$$\|f\|_* = \max(|f(1)|, \sup_{j \geq 0} \sup_{0 \leq k \leq 2^j} \frac{|f_{j,k}|}{\sqrt{1+j}}).$$

It is easy to see that there exist  $D_1 > 0$  and  $D_2 > 0$  such that  $\|f\|_{M_2, w} \leq D_1 \|f\|^{**} \leq D_2 \|f\|_*$ .

### 3 Definitions and general results

In this section, we recall some definitions and results.

**Definition 3.1.** A function  $I : E \rightarrow [0; +\infty]$  is called a rate function if it is semi-continuous lower (s.c.l) i.e.  $\forall f_n \rightarrow f$ , then  $I(f) \leq \lim_{n \rightarrow +\infty} \inf I(f_n)$ . Thus, if for any  $a < \infty$ ,  $\Gamma_a = \{x \in E, I(x) \leq a\}$  is compact, then  $I$  is a good rate function.

**Definition 3.2.** For a function  $I$ , the probability family  $\{P^\varepsilon\}_{\varepsilon > 0}$  satisfy the LDP with a good rate function  $I$  if we have:

i) For every open set  $O$  of  $E$

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log P_\varepsilon(O) \geq -I(O)$$

ii) For every closed set  $F$  of  $E$

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \log P_\varepsilon(F) \leq -I(F).$$

$$\text{where } I(A) = \inf_{x \in A} I(x)$$

Mellouk, M. and Millet, A. (1998) have proved the following Schilder theorem

**Theorem 3.1.** Let  $P^\varepsilon$  be the law of  $\varepsilon W$  on  $B_{M_2, w_\alpha}^0$  equipped with the norm  $\|\cdot\|_{M_2, w_\alpha}$ , then  $P^\varepsilon$  satisfy the LDP with the good rate function :

$$I(f) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{f}(s)|^2 ds & \text{if } f \in \mathcal{H} \\ +\infty & \text{outside} \end{cases}.$$

**Theorem 3.2.** Let  $Q^\varepsilon$  be the probability measure family on a Polish space  $E$  satisfying the LDP with a good rate function  $\lambda$  and let  $F : E \rightarrow E'$  be a continuous function. Denote by  $Q^\varepsilon = P^\varepsilon \circ F^{-1}$  the image measure family of  $P^\varepsilon$ , then  $\{Q^\varepsilon\}$  satisfy the LDP with the good rate function  $\tilde{\lambda}$  define by

$$\tilde{\lambda}(y) = \inf_{x:F(x)=y} \lambda(x).$$

Let us recall the two lemmas, cf Eddahbi, M., Nzi, M. and Oukinine, Y. (1999).

**Lemma 3.3.** There exist a constant  $C = C_l$  such that for all  $\lambda > 0$  and  $\mu > 0$  with  $\lambda > 4l\mu > 0$  and  $\lambda > 2\sqrt{\log 2}$  we have

$$P[\|W\|^{**} \geq \lambda, \|W\| \leq \mu] \leq C \max(1, l((\frac{\lambda}{4l\mu})^2) \exp(-\frac{\lambda^2}{C} \ln(\frac{\lambda}{4l\mu})).$$

**Lemma 3.4.** There exist a constant  $C = C_l$  such that for all  $u > 2\sqrt{\log 2}$  and all process  $K$  on  $[0, 1]$

$$P[\|\int_0^\cdot K_s dW_s\|^{**} \geq u, \|K\| \leq 1] \leq C \exp(-\frac{u^2}{C}).$$

## 4 Main results and discussion

### 1-Case of the half-space

We take for  $D = \mathbb{R}_p^+$  the set of  $(x_1, \dots, x_p), x_1 \geq 0; e_1 = (1, 0, \dots, 0)$ . Consider  $(X^\varepsilon, a^\varepsilon)$  the solution of

$$X_t^\varepsilon = x + \sqrt{\varepsilon} \int_0^t \sigma(X_s^\varepsilon) dw_s + \int_0^t b(X_s^\varepsilon) ds + e_1 a_t^\varepsilon \quad (4.1)$$

$(X_t^\varepsilon)^1 \geq 0, e_1 = (1, 0, \dots, 0)$

and  $a_t^\varepsilon$  a continuous increasing process verifying

$$a_0^\varepsilon = 0 \text{ and } a_t^\varepsilon = \int_0^t 1_{(x^1=0)}(X_s^\varepsilon) da_s^\varepsilon.$$

We give on  $\mathbb{R}_p^+$  a fields of matrix  $p \times d, \sigma(x)$  and fields of vectors  $b_\varepsilon(x), b(x)$  satisfying the set of conditions (L):

- (i)  $\sigma$  and  $b_\varepsilon$  are Lipschitz.
- (ii)  $b_\varepsilon$  converge uniformly to  $b$  when  $\varepsilon$  tends to 0.

Under (L), there is existence and uniqueness of solutions of (4.1). Recall the construction given by Anderson, R. & Orey, S. Anderson, R. and Orey, S. (1976). For  $w = (w^1, \dots, w^d) \in C(\mathbb{R}^+, \mathbb{R}^d)$ , denote

$$Ew = \tilde{w}^1 \text{ et } \Gamma(w) = (w^1 + \tilde{w}^1, \dots, w^d) \quad (4.2)$$

where  $\tilde{w}^1(t) = -\inf_{s \leq t} (w^1(s) \wedge 0)$ ,

with  $x \wedge y = \inf(x, y)$ .

Then,  $\Gamma$  verify

$$\sup_{s \leq t} |\Gamma_s w - \Gamma_s w'| \leq 2 \sup_{s \leq t} |w(s) - w'(s)|$$

and if  $\overline{X}_t^\varepsilon$  is solution of

$$\overline{X}_t^\varepsilon = x + \sqrt{\varepsilon} \int_0^t \sigma(\Gamma_s \overline{X}^\varepsilon) dw_s + \int_0^t b(\Gamma_s \overline{X}^\varepsilon) ds. \quad (4.3)$$

We have  $X_t^\varepsilon = \Gamma_t \bar{X}^\varepsilon$  and  $a_t^\varepsilon = E_t \bar{X}^\varepsilon$ , where we have noted  $(\Gamma_t w) = (\Gamma w)(t)$  and  $(E_t w) = (Ew)(t)$ . The existence and uniqueness of solutions of  $\bar{X}_t^\varepsilon$ , solution of (4.3) will result from the following assumptions:

for any  $w \in C(\mathbb{R}_+, \mathbb{R})$ , put

$$\begin{aligned}\bar{\sigma}_t(w) &= \sigma(\Gamma_t w) \\ \bar{b}_t(w) &= b(\Gamma_t w)\end{aligned}$$

We have:

- i)  $\bar{\sigma}_t$  is a non-anticipative functional with value in  $\mathcal{M}_{d \times k}$  and  $\bar{b}_t$  is a non-anticipative functional with value in  $\mathbb{R}^d$ , cf Anderson, R. and Orey, S. (1976) for the definition.
- ii) Input  $\bar{w}$  is jointly measurable in  $w$  and there exist a constant  $M > 0$  such that

$$\begin{aligned}\sup_{0 \leq s \leq t} |\bar{b}_s(w)| &\leq M(1 + 2 \sup_{0 \leq s \leq t} |w_s|) \\ \sup_{0 \leq s \leq t} |\bar{b}_s(w) - \bar{b}_s(w')| &\leq M(1 + 2 \sup_{0 \leq s \leq t} |w_s - w'_s|), \forall w, w' \in C(\mathbb{R}_+, \mathbb{R}).\end{aligned}$$

- iii) Input  $\sigma_t(w)$  is jointly measurable in  $w$  and there exist a constant  $K > 0$  such that

$$\begin{aligned}\sup_{0 \leq s \leq t} |\bar{\sigma}_s(w)| &\leq M(1 + 2 \sup_{0 \leq s \leq t} |w_s|) \\ \sup_{0 \leq s \leq t} |\bar{\sigma}_s(w) - \bar{\sigma}_s(w')| &\leq M(1 + 2 \sup_{0 \leq s \leq t} |w - w'|), \forall w, w' \in C(\mathbb{R}_+, \mathbb{R}).\end{aligned}$$

We are interested in the following if, on the one hand,  $\bar{X}_t^\varepsilon$  is still a solution of the differential equation in the space  $B_{M_2, w_\alpha}^0$  and on the other hand, to establish the LDP associated with its law. For all  $\lambda \geq \frac{1}{2}$ , from the theorem 2 in Mellouk, M. (1994) and the remark I in Eddahbi, M., Nzi, M. and Ouknine, Y. (1999), we easily have the existence of solutions  $\bar{X}_t^\varepsilon$  in  $B_{M_2, w_\alpha}^0$ . For all  $0 \leq \lambda < \frac{1}{2}$ , the theorem 2 in Mellouk, M. (1994) shows that  $\bar{X}_t^\varepsilon$  almost certainly belongs to  $B_{M_2, w_\alpha}^0$ . Now, let  $h \in \mathcal{H}([0, 1], \mathbb{R}^d)$  be the absolutely continuous functions space with  $\int_0^1 |\dot{h}_s|^2 ds < \infty$  and define  $g = S_z(h)$  if and only if :

$$g_t = z + \int_0^t \{\sigma(g_s) \dot{h}_s + b(g_s)\} ds + e_1 a_t; (g_t)_1 \geq 0 \quad (4.4)$$

and  $a_s$  continuous increasing process,  $a_0 = 0$  et  $\int_0^1 1_{x_1=0}(g_s) da_s = a_t$ .

We have  $g_t = \Gamma_t k$  and  $a_t = E_t k$  where  $k$  is solution of

$$k_t = z + \int_0^t \{\sigma(\Gamma_s k) \dot{h}_s + b(\Gamma_s k)\} ds. \quad (4.5)$$

Put  $k = S_x(h)$ . Define  $\bar{\lambda} : B_{M_2, w_\alpha}^0 \rightarrow [0, +\infty]$  by

$$\bar{\lambda}(\bar{h}) = \inf\{\lambda(h); h \in \mathcal{H}([0, 1], \mathbb{R}^d) \text{ such that } \bar{h} = S(h)\}. \quad (4.6)$$

**Theorem 4.1.** Assume that (L) is satisfied. Then the family  $\bar{P}^\varepsilon = P \circ (\bar{X}^\varepsilon)^{-1}$  of laws of  $\bar{X}^\varepsilon$  satisfy a LDP with the good rate function  $\bar{\lambda}$ .

Consider the rates functions on  $B_{M_2, w_\alpha}^{0, d++}$  and  $B_{M_2, w_\alpha}^{0, d+}$

$$\begin{cases} \bar{\lambda}_+(g, a) = \inf\{\bar{\lambda}(k); \exists k \in B_{M_2, w_\alpha}^0 \text{ such that } g = \Gamma k \text{ and } a = E_k\} \\ \inf \emptyset = +\infty \end{cases} \quad (4.7)$$

and

$$\begin{cases} \bar{\lambda}_+(g) = \inf\{\bar{\lambda}(k); \exists k \in B_{M_2, w_\alpha}^0 \text{ such that } g = \Gamma k\} \\ \inf \emptyset = +\infty \end{cases} \quad (4.8)$$

where  $\gamma$ ,  $E$ ,  $\bar{\lambda}$  are define in (4.2), (4.6) and

$$B_{M_2, w_\alpha}^{0, d++} = \{f \in \mathcal{C}([0, 1], \mathbb{R}^{d,+} \times \mathbb{R}^+); \|f\|_{M_2, w_\alpha} < \infty, \lim_{j \wedge p \rightarrow \infty} 2^{-j(\frac{1}{2} - \alpha + \frac{1}{p})} p^{-\gamma} (1+j)^{-\lambda} \|f_{j..}\|_p = 0\}$$

$$B_{M_2, w_\alpha}^{0, d+} = \{f \in \mathcal{C}([0, 1], \mathbb{R}^{d,+}); \|f\|_{M_2, w_\alpha} < \infty, \lim_{j \wedge p \rightarrow \infty} 2^{-j(\frac{1}{2} - \alpha + \frac{1}{p})} p^{-\gamma} (1+j)^{-\lambda} \|f_{j..}\|_p = 0\}.$$

**Theorem 4.2.** i) Let  $\bar{Q}^\varepsilon$  be the law of  $(X^\varepsilon, a^\varepsilon)$  solution of (4.1),  $\bar{Q}^\varepsilon \in \mathcal{M}_1(B_{M_2, w_\alpha}^{0, d++})$  the probability measure space on  $B_{M_2, w_\alpha}^{0, d++}$ .

Then  $\bar{Q}^\varepsilon$  satisfy a LDP with the good rate function  $\bar{\lambda}_+$  define in (4.7).

ii) Let  $Q^\varepsilon$  be the law of  $X^\varepsilon$ , solution of (4.1),  $Q^\varepsilon \in \mathcal{M}_1(B_{M_2, w_\alpha}^{0, d+})$  the probability measure space on  $B_{M_2, w_\alpha}^{0, d+}$ .

Then  $Q^\varepsilon$  satisfy a LDP with the good rate function  $\bar{\lambda}_+$  define in (4.8).

*Proof.* In Anderson, R. and Orey, S. (1976), we have  $\Gamma\psi - \psi = (E\psi, 0, \dots, 0)$ , it suffices to show that  $\Gamma$  is continuous and to apply the principle of contractions.

Let  $\psi_1$  and  $\psi_2$ , two elements of  $\mathcal{C}([0, 1], \mathbb{R}^d)$

$$\|\Gamma\psi_1 - \Gamma\psi_2\|_{M_2, w_\alpha} = \|\Gamma\psi_1 - \Gamma\psi_2\|_{M_2} + \sup_{0 \leq s \leq 1} \frac{w_{M_2}(\Gamma\psi_1 - \Gamma\psi_2, s)}{w_{\alpha, \lambda}(s)}$$

but

$$\|\Gamma\psi_1 - \Gamma\psi_2\|_{M_2} = \inf \left\{ \tau > 0, \frac{1}{\tau} \left[ 1 + \int_0^1 M_2(\tau |\Gamma_s \psi_1 - \Gamma_s \psi_2|) ds \right] \right\}$$

with

$$\begin{aligned} M_2(\tau |(\Gamma_s \psi_1 - \Gamma_s \psi_2)(s)|) &= \exp \{ (\tau |\Gamma_s \psi_1 - \Gamma_s \psi_2|)^2 \} - 1 \\ &\leq \exp \{ (2\tau \sup_{s \leq t} |\psi_1(s) - \psi_2(s)|)^2 \} - 1 \\ &= \exp \{ (2\tau \|\psi_1 - \psi_2\|_\infty)^2 \} - 1 \\ &= M_2(2\tau \|\psi_1 - \psi_2\|_\infty) \\ &= M_2(\tau_1 \|\psi_1 - \psi_2\|_\infty) \end{aligned}$$

where  $\tau_1 = 2\tau$ , we have

$$\|\Gamma\psi_1 - \Gamma\psi_2\|_{M_2} \leq 2 \|\psi_1 - \psi_2\|_{M_2}$$

and

$$w_{M_2}(\Gamma\psi_1 - \Gamma\psi_2, s) = \sup_{0 \leq h \leq s} \|\Delta_h(\Gamma\psi_1 - \Gamma\psi_2)\|_{M_2}$$

with

$$\|\Delta_h \Gamma\psi_1 - \Gamma\psi_2\|_{M_2} = \inf \left\{ \tau > 0, \frac{1}{\tau} \left[ 1 + \int_0^1 M_2(\tau |\Delta_h \Gamma_s \psi_1 - \Gamma_s \psi_2|) ds \right] \right\},$$

$$\begin{aligned} |\Delta_h \Gamma_s \psi_1 - \Gamma_s \psi_2| &= |1_{[0, 1-h]}(s) ((\Gamma_{s+h} \psi_1 - \Gamma_{s+h} \psi_2) - (\Gamma_s \psi_1 - \Gamma_s \psi_2))| \\ &\leq 4 \sup_{s \leq t} |\psi_1(s) - \psi_2(s)| \\ &= 4 \|\psi_1 - \psi_2\|_\infty \end{aligned}$$

then

$$\begin{aligned} M_2(\tau|\Delta_h\Gamma_s\psi_1 - \Gamma_s\psi_2|) &\leq M_2(4\tau \|\psi_1 - \psi_2\|_\infty) \\ &\leq M_2(\tau_2 \|\psi_1 - \psi_2\|_\infty) \end{aligned}$$

where  $\tau_2 = 4\tau$ , we have

$$\|\Delta_h\Gamma\psi_1 - \Gamma\psi_2\|_{M_2} \leq 4 \|\psi_1 - \psi_2\|_{M_2}$$

then

$$\sup_{0 \leq s \leq 1} \frac{w_{M_2}(\Gamma\psi_1 - \Gamma\psi_2, s)}{w_{\alpha, \lambda}(s)} \leq C \|\psi_1 - \psi_2\|_{M_2}.$$

Finally, we have

$$\|\Gamma\psi_1 - \Gamma\psi_2\|_{M_2, w_\alpha} \leq K \|\psi_1 - \psi_2\|_{M_2, w_\alpha}.$$

□

**Theorem 4.3.** Assume  $L$ , for any  $r > 0$ ,  $A > 0$  and  $\rho > 0$ , there exist  $\varepsilon_0$ ,  $\alpha$ ,  $\tilde{r} > 0$  such that for all  $\|h\|_{\mathcal{H}} \leq A$ , if  $\varepsilon \leq \varepsilon_0$ ,  $|z - x| < \tilde{r}$ , we have:

$$P(\|X^\varepsilon - g\|_{M_2, w} + \|a^\varepsilon - a\|_{M_2, w} > \rho; \|\sqrt{\varepsilon}W - h\| < \alpha) \leq \exp(-\frac{r}{\varepsilon}).$$

*Proof.* Given the Lipschitz character of  $\Gamma$  and  $E$ , it suffices to show the following proposition:

**Proposition 4.1.** For all  $r > 0$ ,  $\rho > 0$  and for all  $A > 0$ , there exist  $\varepsilon_0$ ,  $\alpha$ ,  $\tilde{r} > 0$  such that for all  $\|h\|_{\mathcal{H}} \leq A$ , if  $\varepsilon \leq \varepsilon_0$ ,  $|z - x| < \tilde{r}$ , we have:

$$P(\|\bar{X}^\varepsilon - k\|_{M_2, w_\alpha} > \rho, \|\sqrt{\varepsilon}W\| < \alpha) \leq \exp(-\frac{r}{\varepsilon}).$$

Let us first deal with the case  $h = 0$ . The important step of the proof then lies in the following property: for all  $r, a > 0$  and for all  $\rho > 0$ , there exist  $\alpha > 0$  such that if  $\|h\|_{\mathcal{H}} \leq a$  and for all  $\varepsilon$  small enough,

$$P\{\|\sqrt{\varepsilon} \int_0^\cdot \sigma(\Gamma_s \bar{X}^\varepsilon) dW_s\|_{M_2, w} \geq \rho, \|\sqrt{\varepsilon}W\| \leq \alpha\} \leq \exp(-\frac{r}{\varepsilon}).$$

For any  $n \in \mathbb{N}$ , we consider the approximation sequence of the process  $\bar{X}_t^\varepsilon$  defined by  $\bar{X}_t^{\varepsilon, n} = \bar{X}_{t_k}^\varepsilon$  if  $t \in [\frac{k}{n}, \frac{k+1}{n}]$  for all  $k \in \{0, 1, \dots, n-1\}$ . Note that the application  $\Gamma$  extends to the functions r.c.l.l. so, in particular, to the processes  $\bar{X}_t^{\varepsilon, n}$  with the same property :

$$\|\Gamma\psi_1 - \Gamma\psi_2\| \leq K \|\psi_1 - \psi_2\|_{M_2, w}.$$

We have for any  $\gamma > 0$ ,  $n \in \mathbb{N}$

$$A = \{\|\sqrt{\varepsilon} \int_0^\cdot \sigma(\Gamma_s \bar{X}^\varepsilon) dW_s\|_{M_2, w} \geq \rho, \|\sqrt{\varepsilon}W\| \leq \alpha\} \subset A_1 \cup A_2 \cup A_3$$

with  $A_1 = \{\|\sqrt{\varepsilon} \int_0^\cdot [\sigma(\Gamma_s \bar{X}^\varepsilon) - \sigma(\Gamma_s \bar{X}^{\varepsilon, n})] dB_s\|_{M_2, w} \geq \frac{\rho}{2}, \|\bar{X}^\varepsilon - \bar{X}^{\varepsilon, n}\| \leq \gamma\}$ ,  $A_2 = \{\|\bar{X}^\varepsilon - \bar{X}^{\varepsilon, n}\| \geq \gamma\}$  and  $A_3 = \{\|\sqrt{\varepsilon} \int_0^\cdot \sigma(\Gamma_s \bar{X}^{\varepsilon, n}) dW_s\|_{M_2, w} \geq \frac{\rho}{2}, \|\sqrt{\varepsilon}W\| \leq \alpha\}$ .

In Doss, H. and Priouret, P. (1983), we have for all  $r > 0$  and for all  $\gamma > 0$  there exist  $\varepsilon_0$  and  $n$  such that for all  $0 < \varepsilon < \varepsilon_0$ , we have:

$$P(A_2) \leq \exp(-\frac{r}{\varepsilon})$$

As on  $\|\bar{X}^\varepsilon - \bar{X}^{\varepsilon, n}\| \leq \gamma$ ,  $\|\sqrt{\varepsilon}[\sigma(\Gamma_s \bar{X}^\varepsilon) - \sigma(\Gamma_s \bar{X}^{\varepsilon, n})]\|^{**2} \leq 4\varepsilon M^2 \gamma^2$ . Consequently by lemma (3.4), we get

$$P(A_1) \leq C \exp\left(-\frac{\rho^2}{C\gamma^2\varepsilon}\right)$$

To estimate the last term  $P(A_3)$ , we have

$$\begin{aligned} \|\sqrt{\varepsilon} \int_0^\cdot \sigma(\Gamma_s \bar{X}^{\varepsilon,n}) dW_s\|_{M_2,w} &= \sqrt{\varepsilon} \left\| \sum_{j=0}^n \sigma(\bar{X}_{t_j}^{\varepsilon,n}) [W(t_{j+1} \wedge \cdot) - W(t_j \wedge \cdot)] \right\|_{M_2,w} \\ &\leq \sqrt{\varepsilon} \sum_{j=0}^n \|\sigma(\bar{X}_{t_j}^{\varepsilon,n}) [W(t_{j+1} \wedge \cdot) - W(t_j \wedge \cdot)]\|_{M_2,w} \\ &\leq 2\sqrt{\varepsilon} M n \|W\|^{**} \end{aligned}$$

because the norm  $\|\cdot\|_{M_2,w}$  is dominated by the norm  $\|\cdot\|^{**}$ .

By using lemma (3.3), we have:

$$P(A_3) \leq C \max \left( 1, \left( \frac{\rho}{16lMn\alpha} \right)^2 \right) \exp \left( - \frac{\rho^2}{C\varepsilon 16M^2n^2} \log \left( \frac{\rho}{16lMn\alpha} \right) \right) \quad (4.9)$$

where  $C$  is a constant which depends only on  $l$  and  $M$ .

For a given  $r > 0$  and  $\rho > 0$ , we choose  $\gamma > 0$  small enough such that  $\frac{\rho}{C\gamma^2} > r$ , then we choose  $n$  large enough such that

$$P(A_1) \leq C \exp \left( - \frac{r}{\varepsilon} \right)$$

and then we choose  $\left( \frac{\rho^2}{16M^2n^2} \log \left( \frac{\rho}{16lMn\alpha} \right) \right) > Cr$  in (4.9), which completes the proof of proposition.  $\square$

In the general case, the following lemma allows us to reduce the proof to that of case  $h = 0$ . In this operation, the fields  $b_\varepsilon, \varepsilon > 0$ , take the form

$$b_\varepsilon(s, x) = b_\varepsilon(x) + \sigma(x) \dot{h}(s)$$

$$b(s, x) = b(x) + \sigma(x) \dot{h}(s).$$

**Lemma 4.4.** For all  $h \in \mathcal{H}$ , let  $\bar{X}^\varepsilon$  and  $k$  be processes such that : for all  $t \in [0, 1]$

$$\bar{X}_t^\varepsilon = x + \sqrt{\varepsilon} \int_0^t \sigma(\Gamma_s \bar{X}^\varepsilon) dW_s^\varepsilon + \int_0^t (b_\varepsilon(\Gamma_s \bar{X}^\varepsilon) + \sigma(\Gamma_s \bar{X}^\varepsilon) \dot{h}(s)) ds,$$

where  $W^\varepsilon$  of the same law as  $W$

$$k_t = z + \int_0^t (\sigma(\Gamma_s k) \dot{h}(s) + b(\Gamma_s k)) ds$$

then, for all  $a, r, \rho > 0$ , there exist  $\varepsilon_0, \alpha, \tilde{r} > 0$  such that if  $\int_0^1 |\dot{h}|^2 ds \leq a, \varepsilon \leq \varepsilon_0, |z - x| < \tilde{r}$ , we have

$$P\{\|\bar{X}^\varepsilon - k\|_{M_2,w} \geq \rho, \|\sqrt{\varepsilon} W^\varepsilon\| \leq \alpha\} \leq \exp\left(-\frac{r}{\varepsilon}\right)$$

*Proof.* We have:

$$\begin{aligned} \bar{X}_t^\varepsilon - k_t &= x - z + \int_0^t (b_\varepsilon(\Gamma_s \bar{X}^\varepsilon) - b_\varepsilon(\Gamma_s k)) ds + \int_0^t (b_\varepsilon(\Gamma_s k) - b(\Gamma_s k)) ds \\ &\quad + \int_0^t (\sigma(\Gamma_s \bar{X}^\varepsilon) - \sigma(\Gamma_s k)) \dot{h}_s ds + \int_0^t \sqrt{\varepsilon} \sigma(\Gamma_s \bar{X}^\varepsilon) dW_s. \end{aligned}$$

Put  $I_t^\varepsilon = \int_0^t \sqrt{\varepsilon} \sigma(\Gamma_s \bar{X}^\varepsilon) dW_s$ , for  $\delta > 0$  such that  $\|I_t^\varepsilon\| \leq \|I_t^\varepsilon\|^{**} \leq \delta$  and  $\sup_{x \in \mathbb{R}^d} |b_\varepsilon(x) - b(x)| \leq \delta$ . Then

$$\|\bar{X}_t^\varepsilon - k_t\| \leq |x - z| + \sup_{x \in \mathbb{R}^d} |b_\varepsilon(x) - b(x)| + \|I_t^\varepsilon\| + M \int_0^t (1 + |\dot{h}_s|) \sup_{u \in \mathbb{R}^d} |\bar{X}_u^\varepsilon - k_u| ds.$$

By Gronwall's lemma, if  $\varepsilon \leq \varepsilon_0$  et  $|x - z| < \tilde{r} < \delta$ , we have

$$\| \overline{X}_t^\varepsilon - k_t \| \leq (|x - z| + \sup_{x \in \mathbb{R}^d} |b_\varepsilon(x) - b(x)| + \| I_t^\varepsilon \|) \exp \left( M \int_0^t (1 + |\dot{h}_s|) ds \right)$$

then

$$\begin{aligned} \| \overline{X}_t^\varepsilon - k_t \| &\leq 3\delta \exp(M(1 + \| h \|_{\mathcal{H}})) \\ &= \delta C. \end{aligned}$$

And then,

$$\begin{aligned} \| \overline{X}_t^\varepsilon - k_t \|^{**} &\leq |x - z| + \| I_t^\varepsilon \|^{**} \\ &+ \left\| \int_0^t (b_\varepsilon(\Gamma_s \overline{X}^\varepsilon) + \sigma(\Gamma_s \overline{X}^\varepsilon)) \dot{h}_s - (b(\Gamma_s k) + \sigma(\Gamma_s k)) \dot{h}_s ds \right\|^{**} \\ &\leq |x - z| + \| I_t^\varepsilon \|^{**} + \sup_{x \in \mathbb{R}^d} |b_\varepsilon(x) - b(x)| \\ &+ \sup_{0 \leq u \leq v \leq 1} \int_u^v \frac{M(1 + |\dot{h}_s|) |\overline{X}_s^\varepsilon - k_s|}{w(v - u)} ds \\ &\leq 3\delta + \sup_{0 \leq u \leq v \leq 1} \frac{M}{w(v - u)} \int_u^v (1 + |\dot{h}_s|) |\overline{X}_s^\varepsilon - k_s| ds. \end{aligned}$$

From Cauchy-Schwartz inequality and expression (4.10), we have

$$\begin{aligned} \| \overline{X}_t^\varepsilon - k_t \|^{**} &\leq 3\delta + 2M(1 + \| h \|_{\mathcal{H}}) \delta C \sup_{0 \leq u \leq v \leq 1} \frac{\sqrt{v - u}}{w(v - u)} \\ &\leq \delta C \end{aligned}$$

where  $C$  is a constant wich depends only on of  $h$ ,  $M$  and  $l$ .

□

**Lemma 4.5.** For all  $A \geq 0$ , the mapping

$$\overline{\mathbb{R}^{d,+}} \times (B_{M_2, w_\alpha}^0 \cap \{h \in \mathcal{H}([0, 1], \mathbb{R}^k); \frac{1}{2} \int_0^1 |\dot{h}_s|^2 ds \leq A\}) \rightarrow B_{M_2, w_\alpha}^{0, d++}$$

wich to  $(x, h) \rightarrow (g, a)$  solution of (4.4) is continuous.

*Proof.* Let  $x_n \rightarrow x$  in  $\overline{\mathbb{R}^{d,+}}$ ,  $h_n \rightarrow h$  in  $B_{M_2, w_\alpha}^0 \cap \{h \in \mathcal{H}([0, 1], \mathbb{R}^k); \frac{1}{2} \int_0^1 |\dot{h}_s|^2 ds \leq A\}$ . Let  $k_t^n$  be the solution of (4.5) relative  $x_n$  and  $h_n$ .

$$\| k_t^n - k_t \|_{M_2, w_\alpha} = \| k_t^n - k_t \|_{M_2} + \sup_{0 \leq t \leq 1} \frac{w_{M_2}(k_t^n - k_t, t)}{w_{\alpha, \lambda}(t)}.$$

But  $\| k_t^n - k_t \|_{M_2} = \inf \left\{ \tau > 0, \frac{1}{\tau} \left[ 1 + \int_0^1 M_2(\tau |k_t^n - k_t|) dt \right] \right\}$  with

$$|k_t^n - k_t| = \left| \int_0^t \{ (\sigma(\Gamma_s k^n) - \sigma(\Gamma_s k)) \dot{h}_s^n + b(\Gamma_s k^n) - b(\Gamma_s k) + \sigma(\Gamma_s k) (\dot{h}_s^n - \dot{h}_s) \} ds \right|.$$

By the triangular inequality, using the hypothesis (L), we have :

$$|k_t^n - k_t| \leq M \left| \int_0^t (\dot{h}_s^n - \dot{h}_s) ds \right| + M \left| \int_0^t (\Gamma_s k^n - \Gamma_s k) ds \right| + M \left| \int_0^t (\Gamma_s k^n - \Gamma_s k) \dot{h}_s^n ds \right|.$$

By using Schwartz's inequality, we obtain

$$|k_t^n - k_t| \leq M \left( \int_0^t |\dot{h}_s^n - \dot{h}_s|^2 ds \right)^{\frac{1}{2}} + 2M \int_0^t \sup_{s \leq t} |(k_s^n - k_s)| (1 + |\dot{h}_s|) ds.$$

Therefore, Gronwall's lemma implies

$$\begin{aligned} |k_t^n - k_t| &\leq (M \| h^n - h \|_{\mathcal{H}([0,1], \mathbb{R}^k)}) \exp \{ 2M 2^{\frac{1}{2}} (1 + 2A)^{\frac{1}{2}} \} \\ &= K \| h^n - h \|_{\mathcal{H}([0,1], \mathbb{R}^k)} \end{aligned}$$

where  $K$  is a constant depends on  $A$ .

Then

$$\begin{aligned} M_2(\tau |k_t^n - k_t|) &\leq M_2(\tau K \| h^n - h \|_{\mathcal{H}([0,1], \mathbb{R}^k)}) \\ &= M_2(\tau_3 \| h^n - h \|_{\mathcal{H}([0,1], \mathbb{R}^k)}) \end{aligned}$$

where  $\tau_3 = \tau K$ , we have

$$\| k_t^n - k_t \|_{M_2} \leq K \| h^n - h \|_{M_2}.$$

And then,

$$w_{M_2}(k_t^n - k_t, t) = \sup_{0 \leq h \leq t} \| \Delta_h(k_t^n - k_t) \|_{M_2}$$

where

$$\| \Delta_h(k_t^n - k_t) \|_{M_2} = \inf \left\{ \tau > 0, \frac{1}{\tau} \left[ 1 + \int_0^1 M_2(\tau |\Delta(k_t^n - k_t)| dt) \right] \right\}$$

with

$$\begin{aligned} |\Delta_h k_t^n - k_t| &\leq |k_{t+s}^n - k_{t+s}| + |k_t^n - k_t| \\ &\leq (2M \| h^n - h \|_{\mathcal{H}([0,1], \mathbb{R}^k)}) \exp \{ 2M 2^{\frac{1}{2}} (1 + 2A)^{\frac{1}{2}} \} \\ &= K' \| h^n - h \|_{\mathcal{H}([0,1], \mathbb{R}^k)} \end{aligned}$$

where  $K'$  is a constant wich depends on  $A$ . Then,

$$\begin{aligned} M_2(\tau |\Delta(k_t^n - k_t)|) &\leq M_2(\tau K' \| \Delta(h_t^n - h) \|_{\mathcal{H}([0,1], \mathbb{R}^k)}) \\ &= M_2(\tau_4 \| \Delta(h_t^n - h) \|_{\mathcal{H}([0,1], \mathbb{R}^k)}) \end{aligned}$$

where  $\tau_4 = \tau K'$ . We obtain,

$$\| \Delta_h(k_t^n - k_t) \|_{M_2} \leq K' \| h^n - h \|_{M_2}.$$

Finally, we have:

$$\begin{aligned} \| k_t^n - k_t \|_{M_2, w_\alpha} &= \| k_t^n - k_t \|_{M_2} + \sup_{0 \leq t \leq 1} \frac{w_{M_2}(k_t^n - k_t, t)}{w_{\alpha, \lambda}(t)} \\ &\leq C \| h^n - h \|_{M_2, w_\alpha} \end{aligned}$$

where  $C$  constant depending only on of  $A$ . □

## 2-Case of an open of $\mathbb{R}^p$

Let  $D$  be an open conex of  $\mathbb{R}^p$ . We suppose that there exist a function  $u$  of class  $\mathcal{C}_b^2$  such that

$$D = \{u > 0\}, \partial D = \{u = 0\}, \nabla u(x) \neq 0 \quad \forall x \in \partial D \quad (4.10)$$

We consider on  $\partial D$  a vector field  $\nu(x)$  of class  $\mathcal{C}_b^2$  satisfying

$$\langle \nabla u(x), \nu(x) \rangle \geq \beta > 0, \quad \forall x \in \partial D \quad (4.11)$$

Under these assumptions, there exist, for all  $\forall x \in \partial D$  a local map  $(U, \phi)$  containing  $x$  with  $\phi = (\phi_1, \phi_2, \dots, \phi_p)$  of class  $\mathcal{C}^2$ ,  $U \cap \bar{D} = \{\phi_1 \geq 0\}$  and  $\phi^* \nu = \frac{\partial}{\partial \phi_1} / \phi(x)$ . A map will be canonical at the

edge.

We then consider the equation (1.1) and also the set of hypotheses  $(L^D)$  for the coefficients  $\sigma$  and  $b_\varepsilon(x)$  tends to  $b$  uniformly over  $\bar{D}$  satisfying, for all  $x, y \in \bar{D}$ :

$$|\sigma(x)| \leq M, |b(x)| \leq M, |\sigma(x) - \sigma(y)| \leq K|x - y|, |b_\varepsilon(x) - b_\varepsilon(y)| \leq K|x - y|.$$

Under these assumptions, there is existence and uniqueness of the solutions  $(X^\varepsilon, a^\varepsilon)$  of the equation (1.1) with  $a_t^\varepsilon$  a continuous increasing process defined by

$$a_0^\varepsilon = 0 \text{ et } a_t^\varepsilon = \int_0^t 1_{\partial D}(X_s^\varepsilon) da_s^\varepsilon$$

and they are defined on  $\mathbb{R}^+$ , (see Doss, H. and Priouret, P. (1983)).

Let  $(V, \varphi)$  be a canonical map at the edge and  $U$  an open of  $\bar{D}$  such that  $\bar{U} \subset V$ , we define on  $\mathbb{R}^p$

$$\begin{aligned} \bar{\sigma}(z) &= J(\varphi)^* \sigma(\cdot, z) \varphi^{-1}(z), z \in \varphi(U), J(\varphi) \text{ Jacobien de } \varphi, \tilde{b}^i(z) = L^\varepsilon \varphi^i(\varphi^{-1}(z)), \\ z &\in \varphi(U) \end{aligned} \quad (4.12)$$

$$L^\varepsilon \varphi^i = \frac{1}{2} \varepsilon (\nabla \varphi_i)^* \sigma \sigma^* \nabla \varphi_i + \nabla \varphi_i b_\varepsilon \quad (4.13)$$

that one extends over all the space  $\mathbb{R}^p$  to functions satisfying  $(L)$ .

We then denote  $\tilde{X}_t^\varepsilon(t_0, \tilde{x}_0)$ ,  $\tilde{a}_t^\varepsilon(t_0, \tilde{x}_0)$  the solutions of

$$\tilde{X}_t^\varepsilon = \tilde{x}_0 + \sqrt{\varepsilon} \int_{t_0}^t \bar{\sigma}(X_s^\varepsilon) dw_s + \int_{t_0}^t \tilde{b}(X_s^\varepsilon) ds + e_1 \tilde{a}_t^\varepsilon, (\tilde{X}_t^\varepsilon)_1 \geq 0. \quad (4.14)$$

$\tilde{a}_t^\varepsilon$  continuous increasing process,  $\tilde{a}_{t_0}^\varepsilon = 0$ ,

$$\tilde{a}_t^\varepsilon = \int_{t_0}^t 1_{(X_1=0)}(\tilde{X}_s^\varepsilon) d\tilde{a}_s^\varepsilon$$

Then for  $u < v$ , we have a.s. on  $\{X_s^\varepsilon \in U\} \forall s \in [u, v]$ ,

$$\varphi(X_s^\varepsilon) = \tilde{X}_{s-u}^\varepsilon(u, \varphi(X_u^\varepsilon)), a_s^\varepsilon - a_u^\varepsilon = \tilde{a}_{s-u}^\varepsilon(u, \varphi(X_u^\varepsilon)) \forall s \in [u, v].$$

For  $h \in \mathcal{H}([0, 1], \mathbb{R}^d)$ , with  $\frac{1}{2} \int_0^1 |\dot{h}|^2 ds \leq a$ , we define  $(h_t, a_t)$  as solution of

$$g_t = z + \int_0^t \{\sigma(g_s) \cdot \dot{h}_s + b(g_s)\} ds + \int_0^t \nu(g_s) da_s, \text{ avec } g_t \in \bar{D}, \quad (4.15)$$

$a_t$  continuous increasing process,  $a_0 = 0$  and  $\int_0^1 1_{x_1=0}(g_s) da_s = a_t$ .

Under the assumption  $L^D$ , it is easily to establish the existence and uniqueness of the solution of (4.15) on  $\mathbb{R}^+$ . Put

$$\tilde{g}_s = \sigma(g_s) \dot{h}_s + b(g_s), \quad (4.16)$$

$$u(g_t) - u(z) = \int_0^t \nabla u(g_s) \tilde{g}_s ds + \int_0^t \nabla u(g_s) da_s \quad (4.17)$$

We remark if  $t_0, t_1$  are such that  $g_{t_0} \in \partial D$ ,  $g_{t_1} \in \partial D$  and  $g_t \in \partial D$  if  $t_0 < t < t_1$ , we have

$$\int_{t_0}^{t_1} \nabla u(g_s) \tilde{g}_s ds = 0 \text{ since } a_s \text{ is flat on } ]t_0, t_1[.$$

By using the same arguments that Doss, H. and Priouret, P. (1983), we have the formulas:

$$c_t = \int_0^t \nabla u(g_s) \nu(g_s) da_s = - \int_0^t \nabla u(g_s) \tilde{g}_s 1_{\partial D} g_s ds \quad (4.18)$$

$$a_t = - \int_0^t \frac{\nabla u(g_s) \tilde{g}_s}{\nabla u(g_s) \nu(g_s)} 1_{\partial D} g_s ds \quad (4.19)$$

where  $\tilde{g}_s = \sigma(g_s) \dot{h}_s + b(g_s)$ .

By (4.11), on a:

**Lemma 4.6.** *Doss, H. and Priouret, P. (1983) Let  $T, A > 0$  and  $K$  a compact and let  $(g_t, a_t)$  the solutions of (4.15) with  $z \in K$  et  $\|h\|_{\mathcal{H}} \leq A$ , then*

- i)  $g_t$  and  $a_t$  are absolutely continues with  $\dot{g}_t$  and  $\dot{a}_t$  in  $L^2([0, T])$ .
- ii) there exist of constants  $C_1(T, A, K)$  and  $C_2(T, A, K)$  such that:

$$\sup_{t \leq T} |g_t| \leq C_1(T, A, K), \quad \int_0^T |\dot{g}_s|^2 ds \leq C_2(T, A).$$

We have,  $(X^\varepsilon, a^\varepsilon)$  and  $(g, a)$  denoting the solutions of (1.1).

**Theorem 4.7.** *For all  $r, A, \rho > 0$  and  $K$  compact of  $\overline{D}$ , there exist  $\varepsilon_0, \alpha, \tilde{r} > 0$  such that if  $\|h\|_{\mathcal{H}} \leq A$ ,  $\varepsilon \leq \varepsilon_0$ ,  $|x - z| \leq \tilde{r}$ ,  $z \in K$ , we have :*

$$P(\|X^\varepsilon - g\|_{M_2, w} + \|a^\varepsilon - a\|_{M_2, w} > \rho, \|\sqrt{\varepsilon}W - h\| < \alpha) \leq \exp(-\frac{r}{\varepsilon})$$

**Theorem 4.8.** *For all  $r, A, \rho > 0$  and  $K$  compact of  $\overline{D}$ , there exist  $\varepsilon_0, \alpha, \tilde{r} > 0$  such that  $\|h\|_{\mathcal{H}} \leq A$ ,  $\varepsilon \leq \varepsilon_0$ ,  $|x - z| \leq \tilde{r}$ ,  $z \in K$ , we have :*

$$P(\|X^\varepsilon - g\|_{M_2, w} + \|a^\varepsilon - a\|_{M_2, w} > \rho, \|\sqrt{\varepsilon}W - h\| < \alpha) \leq \exp(-\frac{r}{\varepsilon})$$

*Proof.* By lemma (4.6)

$$\begin{aligned} \|g\|_{M_2, w} &= \|g\|_{M_2} + \sup_{0 \leq t \leq 1} \frac{w_{M_2}(g, t)}{w_{\alpha, \lambda}(t)} \\ &\leq C(T, A, K) \end{aligned}$$

because

$$\|g\|_{M_2} = \inf \left\{ \tau > 0, \frac{1}{\tau} \left[ 1 + \int_0^1 M_2(\tau |g_t|) dt \right] \right\}$$

with

$$\begin{aligned} M_2(\tau |g_t|) &= \exp\{(\tau |g_t|)^2\} - 1 \\ &\leq \exp\{(\tau C_1(T, A, K))^2\} - 1 \end{aligned}$$

then

$$\|g\|_{M_2} \leq C'_1(T, A, K), \quad \forall \tau > 0$$

and

$$w_{M_2}(g, t) = \sup_{0 \leq h \leq t} \|\Delta_h g\|_{M_2}$$

with

$$\| \Delta_h g_t \|_{M_2} = \inf \left\{ \tau > 0, \frac{1}{\tau} \left[ 1 + \int_0^1 M_2(\tau |\Delta_h g_t|) dt \right] \right\},$$

$$\begin{aligned} |\Delta_h g_t| &= |1_{[0,1-h]}(t)(g_{t+h} - g_t)| \\ &\leq 2C_1(T, A, K) \end{aligned}$$

thus

$$M_2(\tau |\Delta_h g_t|) \leq \exp \left\{ (2\tau C_1(T, A, K))^2 \right\} - 1$$

then

$$w_{M_2}(g, t) \leq C_1''(T, A, K)$$

and then

$$\begin{aligned} \sup_{0 \leq t \leq 1} \frac{w_{M_2}(g, t)}{w_{\alpha, \lambda}(t)} &\leq \sup_{0 \leq t \leq 1} \frac{C_1''(T, A, K)}{w_{\alpha, \lambda}(t)} \\ &= C_1'''(T, A, K). \end{aligned}$$

We can build  $\delta > 0, 0 < \bar{\alpha} < \bar{\beta}$  and of maps  $(U_1, \varphi_1), \dots, (U_q, \varphi_q)$  such that :

- i) The  $U_i$  are open of  $\mathbb{R}^d$ , canonical maps at the edge.
- ii) The  $U_i$  cover  $B(0, C)$ , closed ball with center 0, of radius C.
- iii) For all  $x \in \bar{D} \cap \bar{B}(0, C)$ ,  $B(0, \delta) \subset U_i$  for some i.
- iv) For all i and  $x, y \in U_i$ ,  $\bar{\alpha} \leq \frac{|\varphi(x) - \varphi(y)|}{|x - y|} \leq \bar{\beta}$ .

Let  $x_1, x_2, \dots, x_q \in x \in \bar{D} \cap \bar{B}(0, C)$  be such that the balls  $B(x_i, \frac{\delta}{4})$  cover  $\bar{D} \cap \bar{B}(0, C)$  and let  $V_i = B(x_i, \frac{\delta}{2})$ . We fix  $A, T, r$ , then for all  $\rho > 0$ , there exist  $\tilde{r}(\rho), \varepsilon_0(\rho), \alpha(\rho)$  such that if  $\|h\|_{\mathcal{H}} \leq A$ ,  $0 \leq s \leq t \leq T$  and if for some i  $g([s, t]) \subset V_i$ , we have, for  $\varepsilon \leq \varepsilon_0(\rho)$

$$\begin{aligned} P(\|X^\varepsilon - g\|_{M_2, w, [s, t]} + \|a^\varepsilon - a\|_{M_2, w, [s, t]} > \rho, |X_s^\varepsilon - g_s| + |a_s^\varepsilon - a_s| \leq \tilde{r}(\rho), \\ \|\sqrt{\varepsilon}W - h\| < \alpha) \leq \exp\left(-\frac{r+1}{\varepsilon}\right) \end{aligned}$$

This results from theorem (4.3) because thanks to (iii) and (iv), such a probability is evaluated in a map  $(U_i, \varphi_i)$  of the  $\rho < \frac{\delta}{2}$ . Let us define  $t_0 = 0, t_1 = \inf\{t > t_0; |g_t - z| > \frac{\delta}{4}\}, \dots, t_{i+1} = \inf\{t > t_i; |g_t - g_{t_i}| > \frac{\delta}{4}\}, \dots$  as  $\|g\|_{M_2, w} \leq C(T, A, K)$ , we thus build a sequence  $t_0 = 0 < \dots < t_m = T$  with  $m \leq N = N(T, a, K)$  and  $g([t_i, t_{i+1}]) \subset V_i = B(x_i, \frac{\delta}{2})$  for some i. Then

$$\begin{aligned} &P(\|X^\varepsilon - g\|_{M_2, w, [0, T]} + \|a^\varepsilon - a\|_{M_2, w, [0, T]} > \rho, \|\sqrt{\varepsilon}W - h\| < \alpha) \\ &\leq P(\|X^\varepsilon - g\|_{M_2, w, [0, T]} + \|a^\varepsilon - a\|_{M_2, w, [0, T]} > \frac{\rho}{2}, \|\sqrt{\varepsilon}W - h\| < \alpha) \\ &\leq P(\|X^\varepsilon - g\|_{M_2, w, [0, t_1]} + \|a^\varepsilon - a\|_{M_2, w, [0, t_1]} > \frac{\rho}{2}, \|\sqrt{\varepsilon}W - h\| < \alpha) \\ &\quad + \sum_{i=1}^{m-1} P(\|X^\varepsilon - g\|_{M_2, w, [t_i, t_{i+1}]} + \|a^\varepsilon - a\|_{M_2, w, [t_i, t_{i+1}]} > \rho_{m-i}, \\ &\quad |X_s^\varepsilon - g_s| + |a_s^\varepsilon - a_s| \leq \rho_{m-i+1}, \|\sqrt{\varepsilon}W - h\| < \alpha) \end{aligned}$$

where  $\rho_1 = \frac{\rho}{2}, \rho_{i+1} = \inf\{\tilde{r}(\rho_i), \rho_i\}$ .

Put

$$\alpha = \inf\{\alpha(\rho_1), \dots, \alpha(\rho_n)\}, \varepsilon_0 = \inf\{\varepsilon(\rho_1), \dots, \varepsilon(\rho_n)\}, \tilde{r} = \inf\{\tilde{r}(\rho_1), \dots, \tilde{r}(\rho_n)\};$$

if  $\varepsilon \leq \varepsilon_0, |x - z| \leq \tilde{r}$ ,

$$P(\|X^\varepsilon - g\|_{M_2, w} + \|a^\varepsilon - a\|_{M_2, w} > \rho, \|\sqrt{\varepsilon}W - h\| < \alpha) \leq N \exp\left(-\frac{r+1}{\varepsilon}\right),$$

it suffices to choose  $\varepsilon \leq \varepsilon_1$  for that  $N \exp\left(-\frac{r+1}{\varepsilon}\right) \leq \exp\left(-\frac{r}{\varepsilon}\right)$  □

Put  $g = T(h)$  when  $(g, a)$  is the solution of (4.15). Define  $\lambda_1^D : B_{M_2, w}^{0, D} \rightarrow [0, +\infty[$  by

$$\lambda_1^D(g, a) = \inf\{\lambda(h); h \in \mathcal{H}([0, 1], \mathbb{R}^k) \text{ such that } g = T(h)\} \quad (4.20)$$

where

$$\lambda(h) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{h}(s)|^2 ds & \text{si } h \in \mathcal{H} \\ +\infty & \text{outside} \end{cases}$$

and then, put  $g = T_A(h)$ . Note that  $\lambda_2^D : B_{M_2, w}^{0, D} \rightarrow [0, +\infty[$  by

$$\lambda_2^D(g) = \inf\{\lambda(h); h \in \mathcal{H}([0, 1], \mathbb{R}^k) \text{ such that } g = T_A(h)\}$$

If  $\lambda_2^D(g) < \infty$ ,  $\exists h \in \mathcal{H}([0, 1], \mathbb{R}^k)$  with  $\lambda(h) < \infty$  and  $a_s$  only increasing on  $\{s : g_s \in \partial D\}$  and satisfying  $\int_0^1 \dot{a}_s^2 1_{\partial D}(g_s) ds < \infty$  such that (4.15). We have

$$\lambda_2^D(g) = \inf\{\lambda_1^D(g, a), a(\cdot) = \int_0^\cdot \beta_s 1_{\partial D}(g_s) ds; \beta_s \text{ mesurable } \geq 0, \int_0^1 \beta_s^2 1_{\partial D}(g_s) ds < \infty\} \quad (4.21)$$

**Theorem 4.9.** i) Let  $\bar{Q}^{\varepsilon, D}$  be the law of  $(X^\varepsilon, a^\varepsilon)$  solution of (4.1),  $\bar{Q}^{\varepsilon, D} \in \mathcal{M}_1(B_{M_2, w_\alpha}^{0, D+})$  the probability measure space on  $B_{M_2, w_\alpha}^{0, D+}$ .

Then  $\bar{Q}^{\varepsilon, D}$  satisfy a LDP with the good rate function  $\lambda_1^D$  define in (4.20).

ii) Let  $Q^{\varepsilon, D}$  be the law of  $X^\varepsilon$  solution of (4.1),  $Q^{\varepsilon, D} \in \mathcal{M}_1(B_{M_2, w_\alpha}^{0, D})$  the probability measure space on  $B_{M_2, w_\alpha}^{0, D}$ .

Then  $Q^{\varepsilon, D}$  satisfy a LDP with the good rate function  $\lambda_2^D$  define in (4.21).

*Proof.* This theorem results from theorem (4.8). □

**Proposition 4.2.** For all  $A \geq 0$ , the mapping

$$\bar{D} \times (B_{M_2, w_\alpha}^0 \cap \{h \in \mathcal{H}([0, 1], \mathbb{R}^k); \frac{1}{2} \int_0^1 |\dot{h}_s|^2 ds \leq A\}) \rightarrow B_{M_2, w_\alpha}^{0, D}$$

wich to  $(x, h) \rightarrow (g, a)$  solution of (4.15) is continuous.

*Proof.* This proposition results from lemma (4.5) and the collations of the theorem (4.8). □

## 5 CONCLUSIONS

The first section established the aim of the work which prove the large deviations of small random perturbation for dynamical systems with reflecting boundary in Besov-Orlicz space. The second section concerned the topology of Besov-Orlicz space and some definitions and results are given in the third section to establish the main theorems wich extend the result of Rajaonarison, L.I. and Rabeherimanana, T.J. (2014) on the Besov-Orlicz space for the case  $Y = Z = 0$ .

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