

Original Research Article

Super convergent method for solution of higher order boundary value problems

ABSTRACT

A new $O(h^{10})$ super convergent method based on B-spline of degree eight has been developed for solution of higher order boundary value problems. Our presented collocation method leads to optimal approximation, we describe the mathematical procedure in detail also analyze the convergence of the method. The obtained numerical results have been compared with results obtained by recent existing methods to verify the applicability and super convergent properties of the presented method numerically.

Keywords: Boundary value problem; B-spline collocation method; super convergence; Green's function.

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1. INTRODUCTION

Higher order boundary value problems occur in the study of fluid dynamics, astrophysics, astronomy, beam and long wave theory, quantum mechanics, induction motors, engineering and applied physics. Many researchers have attempt to developed numerical solution of higher order boundary value problems. Many mathematical models arising in various applications can be written as boundary value problems, eighth and seventh order boundary value problems arise in the mathematical modeling of the fluid dynamics and induction motors respectively, of the fluid dynamics and induction motors respectively [1,2]. The solution of eighth order BVPs by differential quadrature rule is given by Liu and Wu [3] without any convergence analysis. Adomian decomposition and homotopy perturbation method have been given by [4,5] without any convergence analysis. Non-polynomial spline technique proposed by Siddiqi and Akram [6], but they obtained second order convergence. Reproducing kernel space method proposed by Akram and Rehman [7,8]. Variational iteration and differential transformation method are given by Siddiqi et al [9,10]. Many researchers applied collocation methods for solution of BVPs [11, 12,13]. The cubic spline has been used to solve BVPs by pioneers [14, 15, 16, 17] , but their methods have second order convergent. An $O(h^4)$ optimal cubic spline collocation method was developed by Danial and Swatz in [18]. Another optimal collocation method on mid-points based on quadratic spline to approximate the second order BVPs was proposed by Houstis etal [19]. Irodoutou-Ellina and Houstis in [20] developed an $O(h^6)$ optimal collocation method based on quintic spline for solving linear fourth order two point BVPs. In [21] Rashidinia and Ghasemi developed an optimal method based on sextic spline at the grid points for solving of

42 nonlinear fifth order two point BVPs, after that they developed collocation method based on
43 B-spline at the mid-points for the numerical solution of nonlinear sixth order BVPs [22].

44 In the present work, we will focus on developing a collocation method based on B-spline of
45 degree eight to approximate the solution of the following nonlinear two point boundary value
46 problems:

$$47 \quad L_p u \equiv u^{(p)}(x) - \phi(x, u(x), u'(x), \dots, u^{(p-1)}(x)) = 0, 1 \leq p \leq 8, a \leq x \leq b, \quad (1.1)$$

48 with the boundary conditions,

$$49 \quad B_p u \equiv \sum_{j=0}^{p-1} (\alpha_j u^{(j)}(a) + \beta_j u^{(j)}(b)) = \gamma_i, 0 \leq i \leq p-1, \quad (1.2)$$

50 where α_{ij} , β_{ij} and γ_i are given real constants, ϕ is a continuous function, $u(x)$ is a unknown
51 function, and L_p and B_p are differential operators. In this paper we will derive super
52 convergent approximations of order $O(h^{10})$ at mid-points and grid points of the partition Δ on
53 the interval $[a, b]$. In section 2, we obtained the consistency relations for spline of degree
54 eight at mid-points and grid points of the partition Δ to construct higher order approximation.
55 Section 3, is devoted to deriving the new super convergence collocation method based on
56 spline of degree eight. The convergence analysis of the presented method is given in detail,
57 in section 4. In section 5 numerical experiments are given to demonstrate the efficiency of
58 the proposed method, we compared our numerical results with the results reported in [5-8,
59 22, 25-28]. The paper ends with conclusion.

60 2. SPLINE INTERPOLATION

61
62 In this section we define spline interpolant $S(x)$ of degree eight that satisfies certain end
63 conditions and then derive several relations that are required in the formulation of the
64 collocation method. Now let $\Delta \equiv \{a = x_0 < x_1 < \dots < x_n = b\}$ be a uniform partition of the
65 interval $[a, b]$ with step size $h = \frac{b-a}{n}$. Suppose that τ is the set of mid-points of the partition
66 Δ include the boundary points as follows:

$$67 \quad \tau \equiv \{t_0 = x_0, t_1 = \frac{x_0 + x_1}{2}, \dots, t_i = \frac{x_{i-1} + x_i}{2}, \dots, t_n = \frac{x_{n-1} + x_n}{2}, t_{n+1} = x_n\}$$

68 Consider a smooth spline of degree eight $S(x)$ that is an element of
69 $SP_8(\Delta) \equiv \{v(x) \mid v(x) \in [a, b]\}$, and $v(x)$ is a polynomial of degree at most 8 on the
70 partition Δ . By making use of a B-spline function

$$\mathbf{B}_{k+1}(x) = \frac{1}{k!} \sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (x-j)_+^k, (x-j)_+^k = \begin{cases} (x-j)^k & x > j, \\ 0 & x \leq j, \end{cases}, S(x) = \sum_{j=-3}^{n+4} c_j \mathbf{B}_{k+1}\left(\frac{x-a}{h} - j + 5\right)$$

we consider a polynomial spline of degree eight $S(x)$ of the above form that **that** satisfies the following interpolatory conditions:

$$S(t_i) = u(t_i), \quad 1 \leq i \leq n, \quad (2.1)$$

associated with the end conditions:

$$S^{(8)}(t_i) = u^{(8)}(t_i) - \frac{h^2}{24} u^{(10)}(t_i) + \frac{7h^4}{5760} u^{(12)}(t_i) - \frac{31h^6}{967680} u^{(14)}(t_i) + \frac{127h^8}{154828800} u^{(16)}(t_i),$$

$$i = 1, 2, 3, 4, n-3, n-2, n-1, n. \quad (2.2)$$

In the following, for sake of convenience we will denote that, $u_i^j \equiv u^j(t_i)$, $S_i^j \equiv S^j(t_i)$, $i = 0(1)n+1$, $j = 0(1)p$ and $f^{(j)} \equiv D^{(j)}f$. By using [23] we have the following consistency relations for spline of degree eight S , at the mid-points for $5 \leq i \leq n-4$:

$$\Upsilon S_i^{(8)} = \frac{10321920}{h^8} (S_{i\pm4} - 8S_{i\pm3} + 28S_{i\pm2} - 56S_{i\pm1} + 70S_i), (a)$$

$$\Upsilon S_i^{(7)} = \frac{5160960}{h^7} (\mp S_{i\pm4} \pm 6S_{i\pm3} \mp 14S_{i\pm2} \pm 14S_{i\pm1}), (b)$$

$$\Upsilon S_i^{(6)} = \frac{1290240}{h^6} (S_{i\pm4} - 20S_{i\pm2} + 64S_{i\pm1} - 90S_i), (c)$$

$$\Upsilon S_i^{(5)} = \frac{215040}{h^5} (\mp S_{i\pm4} \mp 18S_{i\pm3} \pm 82S_{i\pm2} \mp 106S_{i\pm1}), (d)$$

$$\Upsilon S_i^{(4)} = \frac{26880}{h^4} (S_{i\pm4} + 72S_{i\pm3} - 68S_{i\pm2} - 392S_{i\pm1} + 774S_i), (e)$$

$$\Upsilon S_i^{(3)} = \frac{2688}{h^3} (\mp S_{i\pm4} \mp 234S_{i\pm3} \mp 974S_{i\pm2} \pm 2654S_{i\pm1}), (f)$$

$$\Upsilon S_i^{(2)} = \frac{224}{h^2} (S_{i\pm4} + 720S_{i\pm3} + 9100S_{i\pm2} + 3184S_{i\pm1} - 26010S_i), (g)$$

$$\Upsilon S_i^{(1)} = \frac{16}{h} (\mp S_{i\pm4} \mp 2178S_{i\pm3} \mp 58478S_{i\pm2} \mp 199066S_{i\pm1}), (h)$$

$$\Upsilon f_i \equiv f_{i\pm4} + 6552f_{i\pm3} + 331612f_{i\pm2} + 2485288f_{i\pm1} + 4675014f_i, (i) \quad (2.3)$$

where the discrete operator Υ is defined for any function f on the interval $[a, b]$. Now we will prove the following theorem to obtain the error bounds for spline $S(x)$ of degree eight and its derivatives up to order eight at the mid-points t_i , $i = 1(1)n$ of the partition Δ .

Theorem 2.1: Let $S(x)$ be the unique spline of degree eight satisfying (2.1)-(2.2) and interpolating the function $u \in C^{18}[a, b]$, then for $i=1(1)n$ the following relations hold:

$$\begin{aligned}
 S_i^{(1)} &= u_i^{(1)} - \frac{127h^8}{154828800} u_i^{(9)} + O(h^{10}), (a) \\
 S_i^{(2)} &= u_i^{(2)} + \frac{127h^8}{22118400} u_i^{(10)} + O(h^{10}), (b) \\
 S_i^{(3)} &= u_i^{(3)} + \frac{31h^6}{967680} u_i^{(9)} - \frac{127h^8}{5529600} u_i^{(11)} + O(h^{10}), (c) \\
 S_i^{(4)} &= u_i^{(4)} - \frac{31h^6}{193536} u_i^{(10)} + \frac{127h^8}{4423680} u_i^{(12)} + O(h^{10}), (d) \\
 S_i^{(5)} &= u_i^{(5)} - \frac{7h^4}{5760} u_i^{(9)} + \frac{31h^6}{96768} u_i^{(11)} - \frac{127h^8}{4423680} u_i^{(13)} + O(h^{10}), (e) \\
 S_i^{(6)} &= u_i^{(6)} + \frac{7h^4}{1920} u_i^{(10)} - \frac{31h^6}{96768} u_i^{(12)} + \frac{127h^8}{7372800} u_i^{(14)} + O(h^{10}), (f) \\
 S_i^{(7)} &= u_i^{(7)} + \frac{h^2}{24} u_i^{(9)} - \frac{7h^4}{1920} u_i^{(11)} + \frac{31h^6}{193536} u_i^{(13)} - \frac{127h^8}{22118400} u_i^{(15)} + O(h^{10}), (g) \\
 S_i^{(8)} &= u_i^{(8)} - \frac{h^2}{24} u_i^{(10)} + \frac{7h^4}{5760} u_i^{(12)} - \frac{31h^6}{967680} u_i^{(14)} + \frac{127h^8}{154828800} u_i^{(16)} + O(h^{10}), (h)
 \end{aligned} \tag{2.4}$$

and further satisfying the following error bounds:

$$\|S - u\|^{(k)} = O(h^{9-k}), \quad k = 0, \dots, 7. \tag{2.5}$$

The above relations are held at the mid-points of the partition Δ too.

Proof. We will prove the relation (2.4)h, then by using this relation we can prove the rest of relations (2.4). Using Taylor's series expansion and taking into account that $S_i = u_i$, $i = 1(1)n$, in the relation (2.3)a we obtain,

$$\begin{aligned}
 \Upsilon S_i^{(8)} &= 10321920u_i^{(8)} + 3440640h^2u_i^{(10)} + 544768h^4u_i^{(12)} + \frac{163840}{3}h^6u_i^{(14)} + \frac{58496}{15}h^8u_i^{(16)} + O(h^{10}), \\
 5 \leq i \leq n-4.
 \end{aligned} \tag{2.6}$$

Further, using Taylor's series expansion for any function $f \in C^{18}[a, b]$, we conclude that

$$\Upsilon f_i = 10321920f_i + 3870720h^2f_i^{(2)} + 69350h^4f_i^{(4)} + 79136h^6f_i^{(6)} + \frac{32349}{5}h^8f_i^{(8)} + O(h^{10}),$$

$$98 \quad 5 \leq i \leq n-4. \quad (2.7)$$

$$99 \quad \text{Setting } f(x) = u_i^{(8)} - \frac{h^2}{24}u_i^{(10)} + \frac{7h^4}{5760}u_i^{(12)} - \frac{31h^6}{967680}u_i^{(14)} + \frac{127h^8}{154828800}u_i^{(16)}, \text{ we can obtain,}$$

$$100 \quad \Upsilon f_i = \Upsilon(u_i^{(8)} - \frac{h^2}{24}u_i^{(10)} + \frac{7h^4}{5760}u_i^{(12)} - \frac{31h^6}{967680}u_i^{(14)} + \frac{127h^8}{154828800}u_i^{(16)}) = 10321920u_i^{(8)} + \quad (2.8)$$

$$3440640h^2u_i^{(10)} + 544768h^4u_i^{(12)} + \frac{163840}{3}h^6u_i^{(14)} + \frac{58496}{15}h^8u_i^{(16)} + O(h^{10}), 5 \leq i \leq n-4.$$

101 By subtracting equation (2.6) from (2.8) we obtain:

$$102 \quad \Upsilon(S_i^{(8)} - u_i^{(8)} + \frac{h^2}{24}u_i^{(10)} - \frac{7h^4}{5760}u_i^{(12)} + \frac{31h^6}{967680}u_i^{(14)} - \frac{127h^8}{154828800}u_i^{(16)}) = O(h^{10}) \quad (2.9)$$

103 Denoting $d_i \equiv S_i^{(8)} - u_i^{(8)} + \frac{h^2}{24}u_i^{(10)} - \frac{7h^4}{5760}u_i^{(12)} + \frac{31h^6}{967680}u_i^{(14)} - \frac{127h^8}{154828800}u_i^{(16)}$, then by
104 using the end conditions (2.2) and consistency equation (2.9) we obtain,

$$105 \quad \Upsilon d_i = O(h^{10}) \quad \blacksquare u^{(18)} \quad \blacksquare 5 \leq i \leq n-4, d_1 = d_2 = d_3 = d_4 = d_{n-3} = d_{n-2} = d_{n-1} = d_n = 0. \quad (2.10)$$

106 Since the coefficient matrix of the above system is positive definite hence, it is nonsingular,
107 and its inverse has a finite norm, thus we have $\Upsilon d_i = O(h^{10}) \quad \blacksquare u^{(18)} \quad \blacksquare i = 1(1)n$, so the proof of
108 relation (2.4)h is completed. To prove relation (2.4)g we consider the following consistency
109 relations, which can be easily obtained (but long straightforward calculations) for any spline
110 of degree eight at the interior mid-points t_i ,

$$S_i^{(7)} = \frac{1}{10321920h^7} [h^8(S_i^{(8)} - 5154407S_{i+1}^{(8)} - 9983755S_{i+2}^{(8)} - 7498467S_{i+3}^{(8)} - 2823453S_{i+4}^{(8)} -$$

$$111 \quad 338165S_{i+5}^{(8)} - 6553S_{i+6}^{(8)} - S_{i+7}^{(8)}) \mp 10321920S_{i,i+7} \mp 72253440S_{i+1,i+6} \mp 216760320S_{i+2,i+5} \pm$$

$$361267200S_{i+3,i+4}], 1 \leq i \leq n-7,$$

$$S_i^{(7)} = \frac{1}{10321920h^7} [h^8(S_{i-7}^{(8)} + 6553S_{i-6}^{(8)} + 338165S_{i-5}^{(8)} + 2823453S_{i-4}^{(8)} + 7498467S_{i-3}^{(8)} + 9983755S_{i-2}^{(8)} +$$

$$112 \quad 10315367S_{i-1}^{(8)} + 5160959S_i^{(8)}) \pm 10321920S_{i,i-7} \mp 72253440S_{i-1,i-6} \pm 216760320S_{i-2,i-5} \mp$$

$$361267200S_{i-3,i-4}], 8 \leq i \leq n.$$

113 Using relation (2.4)h in the above relation and applying Taylor's series expansions of $u_{i \pm l}^{(k)}$
114 for $k = 0, 10, 12, 14, 16$, we can obtain,

$$115 \quad S_i^{(7)} = u_i^{(7)} + \frac{h^2}{24}u_i^{(9)} - \frac{7h^4}{1920}u_i^{(11)} + \frac{31h^6}{193536}u_i^{(13)} - \frac{127h^8}{22118400}u_i^{(15)} + O(h^{10}), 1 \leq i \leq n.$$

116 In a similar manner by using any appropriate consistency relations we can prove the other
117 relations in this Theorem. According to the relation (2.5), it is well known that if the spline of

118 degree eight has been applied for solving interpolating problems and boundary value
119 problems of order $k, k = 0(1)7$ has at most $O(h^9)$ and $O(h^{9-k})$ accuracy, respectively. $\square$

120 **Theorem 2.2:** Let $S(x)$ be the unique spline of degree eight satisfying (2.1)-(2.2) and
121 interpolating the function $u \in C^{18}[a, b]$, then for $i = 0(1)n$ the following relations hold:

$$\begin{aligned} S^{(1)}(x_i) &= u^{(1)}(x_i) + \frac{h^8}{1209600} u^{(9)}(x_i) + O(h^{10}), (a) \\ S^{(2)}(x_i) &= u^{(2)}(x_i) - \frac{h^8}{172800} u^{(10)}(x_i) + O(h^{10}), (b) \\ S^{(3)}(x_i) &= u^{(3)}(x_i) - \frac{h^6}{30240} u^{(9)}(x_i) + \frac{h^8}{57600} u^{(11)}(x_i) + O(h^{10}), (c) \\ 122 \quad S^{(4)}(x_i) &= u^{(4)}(x_i) + \frac{h^6}{6048} u^{(10)}(x_i) - \frac{h^8}{34560} u^{(12)}(x_i) + O(h^{10}), (d) \\ S^{(5)}(x_i) &= u^{(5)}(x_i) + \frac{h^4}{720} u^{(9)}(x_i) - \frac{h^6}{3024} u^{(11)}(x_i) + \frac{h^8}{34560} u^{(13)}(x_i) + O(h^{10}), (e) \\ S^{(6)}(x_i) &= u^{(6)}(x_i) - \frac{h^4}{240} u^{(10)}(x_i) + \frac{h^6}{3024} u^{(12)}(x_i) - \frac{h^8}{57600} u^{(14)}(x_i) + O(h^{10}), (f) \\ S^{(7)}(x_i) &= u^{(7)}(x_i) - \frac{h^2}{12} u^{(9)}(x_i) + \frac{h^4}{240} u^{(11)}(x_i) - \frac{h^6}{6048} u^{(13)}(x_i) + \frac{h^8}{172800} u^{(15)}(x_i) + O(h^{10}). (g) \end{aligned}$$

123 (2.11)

124 The above relations are held at the grid points of the partition Δ too.

125 **Proof.** The proof is similar to the proof of theorem 2.1.

126 To derive a super convergent method for the solution of equations (1.1) and (1.2) using
127 Theorems 2.1 and 2.2, we need to define some appropriate relations to connect $u^{(9)}, \dots, u^{(16)}$
128 with spline $S(x)$ and its first eighth derivatives $S', \dots, S^{(8)}$ at the mid-points as well as at grid
129 points of the partition Δ . For sake of convenience, we define the following discrete operators
130 for $5 \leq i \leq n-4$:

$$\begin{aligned} \mu_1 g_i &= g_{i \pm 4} - 8g_{i \pm 3} + 28g_{i \pm 2} - 56g_{i \pm 1} + 70g_i, \\ \mu_2 g_i &= \frac{1}{4}(-g_{i \pm 4} + 12g_{i \pm 3} - 52g_{i \pm 2} + 116g_{i \pm 1} - 150g_i), \\ 131 \quad \mu_3 g_i &= \frac{1}{24}(-7g_{i \pm 4} + 80g_{i \pm 3} - 340g_{i \pm 2} + 752g_{i \pm 1} - 970g_i), \\ \mu_4 g_i &= \frac{1}{24}(-5g_{i \pm 4} + 64g_{i \pm 3} - 284g_{i \pm 2} + 640g_{i \pm 1} - 830g_i), \\ \mu_5 g_i &= \frac{1}{1152}(32g_{i \pm 4} - 472g_{i \pm 3} + 328g_{i \pm 2} - 9448g_{i \pm 1} + 13202g_i), \end{aligned}$$

$$\begin{aligned}
 \mu_6 g_i &= \frac{1}{1920} (49 g_{i\pm 4} - 712 g_{i\pm 3} + 5212 g_{i\pm 2} - 15224 g_{i\pm 1} + 21350 g_i), \\
 \mu_7 g_i &= \frac{1}{5760} (259 g_{i\pm 4} - 3272 g_{i\pm 3} + 20212 g_{i\pm 2} - 55544 g_{i\pm 1} + 76690 g_i), \\
 132 \quad \mu_8 g_i &= \frac{1}{1920} (37 g_{i\pm 4} - 536 g_{i\pm 3} + 4396 g_{i\pm 2} - 13352 g_{i\pm 1} + 18910 g_i), \\
 \mu_9 g_i &= \frac{1}{1024} (-5 g_{i\pm 4} + 64 g_{i\pm 3} - 412 g_{i\pm 2} + 2176 g_{i\pm 1} - 3646 g_i), \\
 \mu_{10} g_i &= \frac{1}{107520} (-75 g_{i\pm 4} + 1104 g_{i\pm 3} - 9604 g_{i\pm 2} + 137200 g_{i\pm 1} - 257250 g_i),
 \end{aligned} \tag{2.12}$$

133 **Lemma 2.1:** If $u \in C^{18}[a, b]$ and $5 \leq i \leq n-4$, then using the above operators we have

$$\begin{aligned}
 u_i^{(r)} &= \frac{\mu_1 S_i^{(r-8)}}{h^8} + O(h^2), \quad 9 \leq r \leq 16, \quad u_i^{(r)} = \frac{\mu_2 S_i^{(r-8)}}{h^6} + O(h^4), \quad 9 \leq r \leq 12, \\
 134 \quad u_i^{(13)} &= \frac{\mu_3 S_i^{(7)}}{h^6} + O(h^4), \quad u_i^{(14)} = \frac{\mu_4 S_i^{(8)}}{h^6} + O(h^4), \quad u_i^{(9)} = \frac{\mu_5 S_i^{(5)}}{h^4} + O(h^6), \\
 u_i^{(10)} &= \frac{\mu_6 S_i^{(6)}}{h^4} + O(h^6), \quad u_i^{(11)} = \frac{\mu_7 S_i^{(7)}}{h^4} + O(h^6), \quad u_i^{(12)} = \frac{\mu_8 S_i^{(8)}}{h^4} + O(h^6), \\
 u_i^{(9)} &= \frac{\mu_9 S_i^{(7)}}{h^2} + O(h^8), \quad u_i^{(10)} = \frac{\mu_{10} S_i^{(8)}}{h^2} + O(h^8).
 \end{aligned}$$

135 **Proof.** Using relation (2.12) and the results of Theorem 2.1, these relations can be proved
 136 easily. $\square$

137 **Corollary 2.1:** If $S(x)$ be the unique spline of degree eight interpolating $u \in C^{18}[a, b]$, then
 138 for $i = 5(1)n-4$ the following relations hold:

$$\begin{aligned}
 u_i^{(8)} &= S_i^{(8)} + \frac{1}{24} \mu_{10} S_i^{(8)} - \frac{7}{5760} \mu_8 S_i^{(8)} + \frac{31}{967680} \mu_4 S_i^{(8)} - \frac{127}{154828800} \mu_1 S_i^{(8)} + O(h^{10}), \\
 u_i^{(7)} &= S_i^{(7)} - \frac{1}{24} \mu_9 S_i^{(7)} + \frac{7}{1920} \mu_7 S_i^{(7)} - \frac{31}{193536} \mu_3 S_i^{(7)} + \frac{127}{22118400} \mu_1 S_i^{(7)} + O(h^{10}), \\
 139 \quad u_i^{(6)} &= S_i^{(6)} - \frac{7}{1920} \mu_6 S_i^{(6)} + \frac{31}{96768} \mu_2 S_i^{(6)} - \frac{127}{7372800} \mu_1 S_i^{(6)} + O(h^{10}), \\
 u_i^{(5)} &= S_i^{(5)} + \frac{7}{5760} \mu_5 S_i^{(5)} - \frac{31}{96768} \mu_2 S_i^{(5)} + \frac{127}{4423680} \mu_1 S_i^{(5)} + O(h^{10}), \\
 u_i^{(4)} &= S_i^{(4)} + \frac{31}{193536} \mu_2 S_i^{(4)} - \frac{127}{4423680} \mu_1 S_i^{(4)} + O(h^{10}), \\
 u_i^{(3)} &= S_i^{(3)} - \frac{31}{967680} \mu_2 S_i^{(3)} + \frac{127}{5529600} \mu_1 S_i^{(3)} + O(h^{10}),
 \end{aligned}$$

$$u_i^{(2)} = S_i^{(2)} - \frac{127}{22118400} \mu_1 S_i^{(2)} + O(h^{10}),$$

$$u_i^{(1)} = S_i^{(1)} + \frac{127}{154828800} \mu_1 S_i^{(1)} + O(h^{10}).$$

Remark: The Corollary 2.1 gives the approximation for $u^{(p)}$, $p = 1(1)8$ at the interior mid-points of the partition Δ which is the conclusion of Lemma 2.1 and Theorem 2.1. Now we need to obtain some similar relations at the boundaries and its neighboring points. In order to obtain approximations for derivatives at the boundaries and its neighboring points $\{x_0, t_1, t_2, t_3, t_4, t_{n-3}, t_{n-2}, t_{n-1}, t_n, x_n\}$ we should use Taylor's series expansions.

Lemma 2.2: If $u \in C^{18}[a, b]$ and $\sigma_i = i, i = 0, 1, 2, 3, 4$ denoting the grid points, near the left end point x_0 and $\sigma_i = n - i, i = n - 3, n - 2, n - 1, n, n + 1$ denoting the grid points, near the right end point x_n , that σ_i is the index of τ , then the following $O(h^2)$ approximations to the higher order derivatives of u hold at the boundaries and its neighboring points,

$$u_{\sigma_0}^{(r)} = \frac{1}{2h^8} \mu_1 (23S_{\sigma_5}^{(r-8)} - 19S_{\sigma_6}^{(r-8)}) + O(h^2), u_{\sigma_1}^{(r)} = \frac{1}{h^8} \mu_1 (5S_{\sigma_5}^{(r-8)} - 4S_{\sigma_6}^{(r-8)}) + O(h^2),$$

$$u_{\sigma_2}^{(r)} = \frac{1}{h^8} \mu_1 (4S_{\sigma_5}^{(r-8)} - 3S_{\sigma_6}^{(r-8)}) + O(h^2), u_{\sigma_3}^{(r)} = \frac{1}{h^8} \mu_1 (3S_{\sigma_5}^{(r-8)} - 2S_{\sigma_6}^{(r-8)}) + O(h^2),$$

$$u_{\sigma_4}^{(r)} = \frac{1}{h^8} \mu_1 (2S_{\sigma_5}^{(r-8)} - S_{\sigma_6}^{(r-8)}) + O(h^2), 9 \leq r \leq 16.$$

Proof. To prove Lemma 2.2, we consider the following relations, which can be easily obtained by finite difference,

$$u_{\sigma_0}^{(r)} = \frac{1}{2} (23u_{\sigma_5}^{(r)} - 19u_{\sigma_6}^{(r)}) + O(h^2), u_{\sigma_1}^{(r)} = 5u_{\sigma_5}^{(r)} - 4u_{\sigma_6}^{(r)} + O(h^2),$$

$$u_{\sigma_2}^{(r)} = 4u_{\sigma_5}^{(r)} - 3u_{\sigma_6}^{(r)} + O(h^2), u_{\sigma_3}^{(r)} = 3u_{\sigma_5}^{(r)} - 2u_{\sigma_6}^{(r)} + O(h^2),$$

$$u_{\sigma_4}^{(r)} = 2u_{\sigma_5}^{(r)} - u_{\sigma_6}^{(r)} + O(h^2), 9 \leq r \leq 16.$$

By using the results of Lemma 2.1 and above relations, the proof is completed. $\square$

Lemma 2.3: Under the assumptions of Lemma 2.2, we have the following $O(h^4)$ approximations to the higher order derivatives of u hold at the boundaries and its neighboring points,

$$u_{\sigma_0}^{(r)} = \frac{1}{16h^6} \mu_2 (715S_{\sigma_5}^{(r-6)} - 1755S_{\sigma_6}^{(r-6)} + 1485S_{\sigma_7}^{(r-6)} - 429S_{\sigma_8}^{(r-6)}) + O(h^4),$$

$$u_{\sigma_1}^{(r)} = \frac{1}{h^6} \mu_2 (35S_{\sigma_5}^{(r-6)} - 84S_{\sigma_6}^{(r-6)} + 70S_{\sigma_7}^{(r-6)} - 20S_{\sigma_8}^{(r-6)}) + O(h^4),$$

$$u_{\sigma_2}^{(r)} = \frac{1}{h^6} \mu_2 (20S_{\sigma_5}^{(r-6)} - 45S_{\sigma_6}^{(r-6)} + 36S_{\sigma_7}^{(r-6)} - 10S_{\sigma_8}^{(r-6)}) + O(h^4),$$

$$u_{\sigma_3}^{(r)} = \frac{1}{h^6} \mu_2 (10S_{\sigma_5}^{(r-6)} - 20S_{\sigma_6}^{(r-6)} + 15S_{\sigma_7}^{(r-6)} - 4S_{\sigma_8}^{(r-6)}) + O(h^4),$$

$$u_{\sigma_4}^{(r)} = \frac{1}{h^6} \mu_2 (4S_{\sigma_5}^{(r-6)} - 6S_{\sigma_6}^{(r-6)} + 4S_{\sigma_7}^{(r-6)} - S_{\sigma_8}^{(r-6)}) + O(h^4), 9 \leq r \leq 12,$$

$$u_{\sigma_0}^{(13)} = \frac{1}{16h^6} \mu_3 (715S_{\sigma_5}^{(7)} - 1755S_{\sigma_6}^{(7)} + 1485S_{\sigma_7}^{(7)} - 429S_{\sigma_8}^{(7)}) + O(h^4),$$

$$u_{\sigma_1}^{(13)} = \frac{1}{h^6} \mu_3 (35S_{\sigma_5}^{(7)} - 84S_{\sigma_6}^{(7)} + 70S_{\sigma_7}^{(7)} - 20S_{\sigma_8}^{(7)}) + O(h^4),$$

$$u_{\sigma_2}^{(13)} = \frac{1}{h^6} \mu_3 (20S_{\sigma_5}^{(7)} - 45S_{\sigma_6}^{(7)} + 36S_{\sigma_7}^{(7)} - 10S_{\sigma_8}^{(7)}) + O(h^4),$$

$$u_{\sigma_3}^{(13)} = \frac{1}{h^6} \mu_3 (10S_{\sigma_5}^{(7)} - 20S_{\sigma_6}^{(7)} + 15S_{\sigma_7}^{(7)} - 4S_{\sigma_8}^{(7)}) + O(h^4),$$

$$u_{\sigma_4}^{(13)} = \frac{1}{h^6} \mu_3 (4S_{\sigma_5}^{(7)} - 6S_{\sigma_6}^{(7)} + 4S_{\sigma_7}^{(7)} - S_{\sigma_8}^{(7)}) + O(h^4),$$

$$u_{\sigma_0}^{(14)} = \frac{1}{16h^6} \mu_4 (715S_{\sigma_5}^{(8)} - 1755S_{\sigma_6}^{(8)} + 1485S_{\sigma_7}^{(8)} - 429S_{\sigma_8}^{(8)}) + O(h^4),$$

$$u_{\sigma_1}^{(14)} = \frac{1}{h^6} \mu_4 (35S_{\sigma_5}^{(8)} - 84S_{\sigma_6}^{(8)} + 70S_{\sigma_7}^{(8)} - 20S_{\sigma_8}^{(8)}) + O(h^4)$$

$$159 \quad u_{\sigma_2}^{(14)} = \frac{1}{h^6} \mu_4 (20S_{\sigma_5}^{(8)} - 45S_{\sigma_6}^{(8)} + 36S_{\sigma_7}^{(8)} - 10S_{\sigma_8}^{(8)}) + O(h^4),$$

$$u_{\sigma_3}^{(14)} = \frac{1}{h^6} \mu_4 (10S_{\sigma_5}^{(8)} - 20S_{\sigma_6}^{(8)} + 15S_{\sigma_7}^{(8)} - 4S_{\sigma_8}^{(8)}) + O(h^4),$$

$$u_{\sigma_4}^{(14)} = \frac{1}{h^6} \mu_4 (4S_{\sigma_5}^{(8)} - 6S_{\sigma_6}^{(8)} + 4S_{\sigma_7}^{(8)} - S_{\sigma_8}^{(8)}) + O(h^4).$$

160 **Proof.** The proof is similar to the proof of Lemma 2.2.

161 **Lemma 2.4:** Under the assumptions of Lemma 2.2, we have the following $O(h^6)$
 162 approximations hold at the boundaries and its neighboring points,

$$u_{\sigma_0}^{(r)} = \frac{1}{256h^4} \mu_l (46189S_{\sigma_5}^{(l)} - 188955S_{\sigma_6}^{(l)} + 159885S_{\sigma_7}^{(l)} - 138567S_{\sigma_8}^{(l)} + 122265S_{\sigma_9}^{(l)} - 21879S_{\sigma_{10}}^{(l)}) + O(h^6), r = 9, 10, l = 5, 6$$

$$163 \quad u_{\sigma_1}^{(r)} = \frac{1}{h^4} \mu_l (126S_{\sigma_5}^{(l)} - 504S_{\sigma_6}^{(l)} + 840S_{\sigma_7}^{(l)} - 720S_{\sigma_8}^{(l)} + 315S_{\sigma_9}^{(l)} - 56S_{\sigma_{10}}^{(l)}) + O(h^6), r = 9, 10, l = 5, 6$$

$$u_{\sigma_2}^{(r)} = \frac{1}{h^4} \mu_l (56S_{\sigma_5}^{(l)} - 210S_{\sigma_6}^{(l)} + 336S_{\sigma_7}^{(l)} - 280S_{\sigma_8}^{(l)} + 120S_{\sigma_9}^{(l)} - 21S_{\sigma_{10}}^{(l)}) + O(h^6), r = 9, 10, l = 5, 6$$

$$u_{\sigma_2}^{(r)} = \frac{1}{h^4} \mu_l (56S_{\sigma_5}^{(l)} - 210S_{\sigma_6}^{(l)} + 336S_{\sigma_7}^{(l)} - 280S_{\sigma_8}^{(l)} + 120S_{\sigma_9}^{(l)} - 21S_{\sigma_{10}}^{(l)}) + O(h^6), r = 9, 10, l = 5, 6$$

$$u_{\sigma_3}^{(r)} = \frac{1}{h^4} \mu_l (21S_{\sigma_5}^{(l)} - 70S_{\sigma_6}^{(l)} + 105S_{\sigma_7}^{(l)} - 84S_{\sigma_8}^{(l)} + 35S_{\sigma_9}^{(l)} - 6S_{\sigma_{10}}^{(l)}) + O(h^6), r = 9, 10, l = 5, 6$$

$$u_{\sigma_4}^{(r)} = \frac{1}{h^4} \mu_l (6S_{\sigma_5}^{(l)} - 15S_{\sigma_6}^{(l)} + 20S_{\sigma_7}^{(l)} - 15S_{\sigma_8}^{(l)} + 6S_{\sigma_9}^{(l)} - S_{\sigma_{10}}^{(l)}) + O(h^6), r = 9, 10, l = 5, 6$$

$$u_{\sigma_0}^{(r)} = \frac{1}{256h^4} \mu_l (46189S_{\sigma_5}^{(l)} - 188955S_{\sigma_6}^{(l)} + 159885S_{\sigma_7}^{(l)} - 138567S_{\sigma_8}^{(l)} + 122265S_{\sigma_9}^{(l)} - 21879S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11, 12, l = 7, 8,$$

$$u_{\sigma_1}^{(r)} = \frac{1}{h^4} \mu_l (126S_{\sigma_5}^{(l)} - 504S_{\sigma_6}^{(l)} + 840S_{\sigma_7}^{(l)} - 720S_{\sigma_8}^{(l)} + 122265S_{\sigma_9}^{(l)} - 21879S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11, 12, l = 7, 8,$$

$$u_{\sigma_1}^{(r)} = \frac{1}{h^4} \mu_l (126S_{\sigma_5}^{(l)} - 504S_{\sigma_6}^{(l)} + 840S_{\sigma_7}^{(l)} - 720S_{\sigma_8}^{(l)} + 315S_{\sigma_9}^{(l)} - 56S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11, 12, l = 7, 8,$$

$$u_{\sigma_2}^{(r)} = \frac{1}{h^4} \mu_l (56S_{\sigma_5}^{(l)} - 210S_{\sigma_6}^{(l)} + 336S_{\sigma_7}^{(l)} - 280S_{\sigma_8}^{(l)} + 120S_{\sigma_9}^{(l)} - 21S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11, 12, l = 7, 8,$$

$$u_{\sigma_3}^{(r)} = \frac{1}{h^4} \mu_l (21S_{\sigma_5}^{(l)} - 70S_{\sigma_6}^{(l)} + 105S_{\sigma_7}^{(l)} - 84S_{\sigma_8}^{(l)} + 35S_{\sigma_9}^{(l)} - 6S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11, 12, l = 7, 8,$$

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$$u_{\sigma_4}^{(r)} = \frac{1}{h^4} \mu_l (6S_{\sigma_5}^{(l)} - 15S_{\sigma_6}^{(l)} + 20S_{\sigma_7}^{(l)} - 15S_{\sigma_8}^{(l)} + 6S_{\sigma_9}^{(l)} - S_{\sigma_{10}}^{(l)}) + O(h^6), r = 11, 12, l = 7, 8.$$

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Proof. The proof is similar to the proof of Lemma 2.2.

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Lemma 2.5: Under the assumptions of Lemma2.2, we have the following $O(h^8)$

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approximations to the higher order derivatives of u hold at the boundaries and its

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neighboring points,

$$u_{\sigma_0}^{(r)} = \frac{\mu_r}{2048h^2} (1062347S_{\sigma_5}^{(l)} - 6084351S_{\sigma_6}^{(l)} + 15444891S_{\sigma_7}^{(l)} - 22309287S_{\sigma_8}^{(l)} + 1984665S_{\sigma_9}^{(l)} - 10567557S_{\sigma_{10}}^{(l)} + 3187041S_{\sigma_{11}}^{(l)} - 415701S_{\sigma_{12}}^{(l)}) + O(h^8), r = 9, 10, l = 7, 8.$$

$$u_{\sigma_1}^{(r)} = \frac{\mu_r}{h^2} (330S_{\sigma_5}^{(l)} - 1848S_{\sigma_6}^{(l)} + 4620S_{\sigma_7}^{(l)} - 6600S_{\sigma_8}^{(l)} + 5775S_{\sigma_9}^{(l)} - 3080S_{\sigma_{10}}^{(l)} + 924S_{\sigma_{11}}^{(l)} - 120S_{\sigma_{12}}^{(l)}) + O(h^8), r = 9, 10, l = 7, 8.$$

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$$u_{\sigma_2}^{(r)} = \frac{\mu_r}{h^2} (120S_{\sigma_5}^{(l)} - 630S_{\sigma_6}^{(l)} + 1512S_{\sigma_7}^{(l)} - 2100S_{\sigma_8}^{(l)} + 1800S_{\sigma_9}^{(l)} - 945S_{\sigma_{10}}^{(l)} + 280S_{\sigma_{11}}^{(l)} - 36S_{\sigma_{12}}^{(l)}) + O(h^8), r = 9, 10, l = 7, 8.$$

$$u_{\sigma_3}^{(r)} = \frac{\mu_r}{h^2} (36S_{\sigma_5}^{(l)} - 168S_{\sigma_6}^{(l)} + 378S_{\sigma_7}^{(l)} - 504S_{\sigma_8}^{(l)} + 420S_{\sigma_9}^{(l)} - 216S_{\sigma_{10}}^{(l)} + 634S_{\sigma_{11}}^{(l)} - 8S_{\sigma_{12}}^{(l)}) + O(h^8), r = 9, 10, l = 7, 8.$$

$$u_{\sigma_4}^{(r)} = \frac{\mu_r}{h^2} (8S_{\sigma_5}^{(l)} - 28S_{\sigma_6}^{(l)} + 56S_{\sigma_7}^{(l)} - 70S_{\sigma_8}^{(l)} + 56S_{\sigma_9}^{(l)} - 28S_{\sigma_{10}}^{(l)} + 8S_{\sigma_{11}}^{(l)} - S_{\sigma_{12}}^{(l)}) + O(h^8), r = 9, 10, l = 7, 8.$$

Proof. The proof is similar to the proof of Lemma 2.2.

3. SUPERCONVERGENT SPLINE COLLOCATION

3.1 FORMULATION OF THE METHOD

Suppose that $u \in C^{10+p}[a, b]$, $p = 1(1)8$, be the exact solution of the given BVP (1.1) and (1.2) and $S(x) \in Sp_8(\Delta)$ be the spline to approximate the solution of BVP. By discretization the BVP (1.1) and (1.2) at the points of τ we have:

$$L_p u_i \equiv [u^{(p)}(x) - \phi(x, u(x), u'(x), \dots, u^{(p-1)}(x))]_{t_i}, \quad 1 \leq i \leq n, \\ B_p u_i \equiv \sum_{j=0}^{p-1} (\alpha_{ij} u^{(j)}(a) + \beta_{ij} u^{(j)}(b)) = \gamma_i, \quad 0 \leq i \leq p-1, \quad (3.1)$$

Replacing u by S and using relations in the Lemmas 2.2-2.5, we can obtain a system of $n+8$ nonlinear equations for each $p = 1(1)8$. For example, hence we consider this system for the case of $p = 8$ and the other cases are similar,

$$L_{8,\sigma_1}^s S = S_{\sigma_1}^{(8)} + \mu_{10} \xi^{(8)} - \mu_8 \psi^{(8)} + \mu_4 \chi^{(8)} - \eta^{(8)} - \phi(t_{\sigma_1}, S_{\sigma_1}, S'_{\sigma_1} + \eta^{(1)}, S''_{\sigma_1} - 7\eta^{(2)}, \\ S_{\sigma_1}^{(3)} - \mu_2 \chi^{(3)} + 28\eta^{(3)}, S_{\sigma_1}^{(4)} + 5\mu_2 \chi^{(4)} - 35\eta^{(4)}, S_{\sigma_1}^{(5)} + \mu_5 \psi^{(5)} - 10\mu_2 \chi^{(5)} + 35\eta^{(5)}, \\ S_{\sigma_1}^{(6)} - 3\mu_6 \psi^{(6)} + 10\mu_2 \chi^{(6)} - 21\eta^{(6)}, S_{\sigma_1}^{(7)} - \mu_9 \xi^{(7)} + 3\mu_7 \psi^{(7)} - 5\mu_3 \chi^{(7)} + 7\eta^{(7)}) + O(h^{10}), i = 1, n, \quad (3.2)$$

In the above system, just replacing ξ, ψ, χ and η by $\bar{\xi}, \bar{\psi}, \bar{\chi}$ and $\bar{\eta}$ we can obtain $L_{8,\sigma_2}^s S$ for $i = 2, n-1$ and by $\bar{\xi}, \bar{\psi}, \bar{\chi}$ and $\bar{\eta}$ we can obtain $L_{8,\sigma_3}^s S$ for $i = 3, n-2$. By $\bar{\xi}, \bar{\psi}, \bar{\chi}$ and $\bar{\eta}$ we can obtain $L_{8,\sigma_4}^s S$ for $i = 4, n-3$. By using relations of Corollary 2.1, we can obtain $L_{8,i}^s S$ for $5 \leq i \leq n-4$:

$$L_{8,i}^s S = S_i^{(8)} + \frac{1}{24} \mu_{10} S_i^{(8)} - \frac{7}{5760} \mu_8 S_i^{(8)} + \frac{31}{967680} \mu_4 S_i^{(8)} - \frac{127}{154828800} \mu_1 S_i^{(8)} - \\ \phi(t_i, S_i, S'_i + \frac{127}{154828800} \mu_1 S'_i, S''_i - \frac{127}{22118400} \mu_1 S''_i, S_i^{(3)} - \frac{31}{967680} \mu_2 S_i^{(3)} + \frac{127}{5529600} \mu_1 S_i^{(3)}, \\ S_i^{(4)} + \frac{31}{193536} \mu_2 S_i^{(4)} - \frac{127}{4423680} \mu_1 S_i^{(4)}, S_i^{(5)} + \frac{7}{5760} \mu_5 S_i^{(5)} - \frac{31}{96768} \mu_2 S_i^{(5)} + \frac{127}{4423680} \mu_1 S_i^{(5)},$$

$$\begin{aligned}
 & S_i^{(6)} - \frac{7}{1920} \mu_6 S_i^{(6)} + \frac{31}{96768} \mu_2 S_i^{(6)} - \frac{127}{7372800} \mu_1 S_i^{(6)}, S_i^{(7)} - \frac{1}{24} \mu_9 S_i^{(7)} + \frac{7}{1920} \mu_7 S_i^{(7)} - \\
 & \frac{31}{193536} \mu_3 S_i^{(7)} + \frac{127}{22118400} \mu_1 S_i^{(7)} + O(h^{10}), 5 \leq i \leq n-4,
 \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
 \xi^{(k)} &= \frac{1}{24} (330S_{\sigma_5}^{(k)} - 1848S_{\sigma_6}^{(k)} + 4620S_{\sigma_7}^{(k)} - 6600S_{\sigma_8}^{(k)} + 5775S_{\sigma_9}^{(k)} - 3080S_{\sigma_{10}}^{(k)} + 924S_{\sigma_{11}}^{(k)} - 120S_{\sigma_{12}}^{(k)}), k = 7, 8, \\
 \bar{\xi}^{(k)} &= \frac{1}{24} (120S_{\sigma_5}^{(k)} - 630S_{\sigma_6}^{(k)} + 1512S_{\sigma_7}^{(k)} - 2100S_{\sigma_8}^{(k)} + 1800S_{\sigma_9}^{(k)} - 945S_{\sigma_{10}}^{(k)} + 280S_{\sigma_{11}}^{(k)} - 36S_{\sigma_{12}}^{(k)}), k = 7, 8, \\
 \bar{\bar{\xi}}^{(k)} &= \frac{1}{24} (36S_{\sigma_5}^{(k)} - 168S_{\sigma_6}^{(k)} + 378S_{\sigma_7}^{(k)} - 504S_{\sigma_8}^{(k)} + 420S_{\sigma_9}^{(k)} - 216S_{\sigma_{10}}^{(k)} + 63S_{\sigma_{11}}^{(k)} - 8S_{\sigma_{12}}^{(k)}), k = 7, 8, \\
 \tilde{\xi}^{(k)} &= \frac{1}{24} (8S_{\sigma_5}^{(k)} - 28S_{\sigma_6}^{(k)} + 56S_{\sigma_7}^{(k)} - 70S_{\sigma_8}^{(k)} + 56S_{\sigma_9}^{(k)} - 28S_{\sigma_{10}}^{(k)} + 8S_{\sigma_{11}}^{(k)} - S_{\sigma_{12}}^{(k)}), k = 7, 8, \\
 \psi^{(k)} &= \frac{7}{5760} (126S_{\sigma_5}^{(k)} - 504S_{\sigma_6}^{(k)} + 840S_{\sigma_7}^{(k)} - 720S_{\sigma_8}^{(k)} + 315S_{\sigma_9}^{(k)} - 56S_{\sigma_{10}}^{(k)}), k = 5, \dots, 8, \\
 \bar{\psi}^{(k)} &= \frac{7}{5760} (56S_{\sigma_5}^{(k)} - 210S_{\sigma_6}^{(k)} + 336S_{\sigma_7}^{(k)} - 280S_{\sigma_8}^{(k)} + 120S_{\sigma_9}^{(k)} - 21S_{\sigma_{10}}^{(k)}), k = 7, 8, \\
 \eta^{(k)} &= \frac{127}{154828800} \mu_1 (5S_{\sigma_5}^{(k)} - 4S_{\sigma_6}^{(k)}), \bar{\eta}^{(k)} = \frac{127}{154828800} \mu_1 (4S_{\sigma_5}^{(k)} - 3S_{\sigma_6}^{(k)}), k = 1, \dots, 8, \\
 \bar{\bar{\eta}}^{(k)} &= \frac{127}{154828800} \mu_1 (3S_{\sigma_5}^{(k)} - 2S_{\sigma_6}^{(k)}), \tilde{\eta}^{(k)} = \frac{127}{154828800} \mu_1 (2S_{\sigma_5}^{(k)} - S_{\sigma_6}^{(k)}), k = 1, \dots, 8, \\
 \bar{\bar{\psi}}^{(k)} &= \frac{7}{5760} (21S_{\sigma_5}^{(k)} - 70S_{\sigma_6}^{(k)} + 105S_{\sigma_7}^{(k)} - 84S_{\sigma_8}^{(k)} + 35S_{\sigma_9}^{(k)} - 6S_{\sigma_{10}}^{(k)}), k = 5, \dots, 8, \\
 \tilde{\psi}^{(k)} &= \frac{7}{5760} (6S_{\sigma_5}^{(k)} - 15S_{\sigma_6}^{(k)} + 20S_{\sigma_7}^{(k)} - 15S_{\sigma_8}^{(k)} + 6S_{\sigma_9}^{(k)} - S_{\sigma_{10}}^{(k)}), k = 5, \dots, 8, \\
 \chi^{(k)} &= \frac{31}{967680} (35S_{\sigma_5}^{(k)} - 84S_{\sigma_6}^{(k)} + 70S_{\sigma_7}^{(k)} - 20S_{\sigma_8}^{(k)}), k = 3, \dots, 8, \\
 \bar{\chi}^{(k)} &= \frac{31}{967680} (20S_{\sigma_5}^{(k)} - 45S_{\sigma_6}^{(k)} + 36S_{\sigma_7}^{(k)} - 10S_{\sigma_8}^{(k)}), k = 3, \dots, 8, \\
 \bar{\bar{\chi}}^{(k)} &= \frac{31}{967680} (10S_{\sigma_5}^{(k)} - 20S_{\sigma_6}^{(k)} + 15S_{\sigma_7}^{(k)} - 4S_{\sigma_8}^{(k)}), k = 3, \dots, 8, \\
 \tilde{\chi}^{(k)} &= \frac{31}{967680} (4S_{\sigma_5}^{(k)} - 6S_{\sigma_6}^{(k)} + 4S_{\sigma_7}^{(k)} - S_{\sigma_8}^{(k)}), k = 3, \dots, 8,
 \end{aligned} \tag{3.4}$$

and the boundary formulas,

$$B_8^s S \equiv \alpha_{i,0} S_0 + \beta_{i,0} S_{n+1} + \sum_{i,0}^7 \alpha_{i,j} \theta_j + \sum_{i,0}^7 \beta_{i,j} \bar{\theta}_j = \gamma_i + O(h^{10}), 0 \leq i \leq 7, \quad (3.5)$$

With

$$\begin{aligned} \theta_k &= \bar{\theta}_k = S_{\sigma_0}^{(k)} + \mu_k^* (23S_{\sigma_5}^{(k)} - 19S_{\sigma_6}^{(k)}), k = 1, 2, \mu_1^* = \frac{-\mu_1}{2419200}, \mu_2^* = \frac{\mu_1}{345600}, \\ \theta_k &= \bar{\theta}_k = S_{\sigma_0}^{(k)} + \mu_k^* (715S_{\sigma_5}^{(k)} - 1755S_{\sigma_6}^{(k)} + 1485S_{\sigma_7}^{(k)} - 429S_{\sigma_8}^{(k)}) + \mu_k^{**} (23S_{\sigma_5}^{(k)} - 19S_{\sigma_6}^{(k)}), \\ k &= 3, 4, \mu_3^* = \frac{\mu_2}{483840}, \mu_3^{**} = \frac{-\mu_1}{115200}, \mu_4^* = \frac{-\mu_2}{96768}, \mu_4^{**} = \frac{\mu_1}{69120}, \\ \theta_k &= \bar{\theta}_k = S_{\sigma_0}^{(k)} + \mu_k^* (46189S_{\sigma_5}^{(k)} - 188955S_{\sigma_6}^{(k)} + 159885S_{\sigma_7}^{(k)} - 138567S_{\sigma_8}^{(k)} + 122265S_{\sigma_9}^{(k)} - 21879S_{\sigma_{10}}^{(k)}) \\ &+ \bar{\mu} (715S_{\sigma_5}^{(k)} - 1755S_{\sigma_6}^{(k)} + 1485S_{\sigma_7}^{(k)} - 429S_{\sigma_8}^{(k)}) + \bar{\mu}_k (23S_{\sigma_5}^{(k)} - 19S_{\sigma_6}^{(k)}), k = 5, 6, \\ \mu_5^* &= \frac{-\mu_5}{184320}, \mu_6^* = \frac{\mu_6}{61440}, \bar{\mu} = \frac{\mu_2}{48384}, \bar{\mu}_5 = \frac{-\mu_1}{69120}, \bar{\mu}_6 = \frac{\mu_1}{115200}, \\ \theta_7 &= \bar{\theta}_7 = S_{\sigma_0}^{(7)} + \frac{\mu_9}{24576} (1062347S_{\sigma_5}^{(7)} - 6084351S_{\sigma_6}^{(7)} + 15444891S_{\sigma_7}^{(7)} - 22309287S_{\sigma_8}^{(7)} \\ &+ 1984665S_{\sigma_9}^{(7)} - 10567557S_{\sigma_{10}}^{(7)} + 3187041S_{\sigma_{11}}^{(7)} - 415701S_{\sigma_{12}}^{(7)}) - \frac{\mu_8}{61440} (46189S_{\sigma_5}^{(7)} \\ &- 188955S_{\sigma_6}^{(7)} + 159885S_{\sigma_7}^{(7)} - 138567S_{\sigma_8}^{(7)} + 122265S_{\sigma_9}^{(7)} - 21879S_{\sigma_{10}}^{(7)}) \\ &+ \frac{\mu_2}{96768} (715S_{\sigma_5}^{(7)} - 1755S_{\sigma_6}^{(7)} + 1485S_{\sigma_7}^{(7)} - 429S_{\sigma_8}^{(7)}) - \frac{\mu_1}{345600} (23S_{\sigma_5}^{(7)} - 19S_{\sigma_6}^{(7)}). \end{aligned}$$

In relation (3.5), we have $\sigma_i = i$ for $\alpha_{i,j} S$ and $\sigma_i = n+1-i$ for $\beta_{i,j} S$.

3.2 EXTRA BOUNDARY FORMULAS

The space $Sp_8(\Delta)$ has $n+8$ dimensions, but the collocation equation (3.1) gives $n+p$, $1 \leq p \leq 8$, nonlinear equations, thus we need $8-p$ extra equations to determine spline $S(x)$ uniquely. To obtain these extra boundary conditions we use the collocation equation at the boundary or its neighbor grid points. Let $E_x = \{x_0, x_1, x_2, x_3, x_{n-3}, x_{n-2}, x_{n-1}, x_n\}$, then we have the following system of $n+8$ equations for $p=1(1)8$.

$$\begin{cases} (L_p u = u^{(p)} - \phi)_{t_i}, & 1 \leq i \leq n, \\ (L_p u = u^{(p)} - \phi)_{x_r}, & x_r \in E_x, \\ (B_p u = \gamma_i)_{x_0, x_n}, & 0 \leq i \leq p-1. \end{cases} \quad (3.2.1)$$

3.3 ERROR ESTIMATION

208 Let $L_{p,i}^s$ and B_p^s be the perturbations of the operators L_p and B_p , respectively. In the case
 209 of $p = 8$, $L_{8,i}^s$ and B_8^s are those given above in equations (3.2)-(3.5). If $S(x)$ is the unique
 210 spline of degree eight to approximate the solution of problem (1.1)-(1.2) and satisfying (2.1)-
 211 (2.2), then the following relations hold:

$$212 \quad \begin{cases} L_{p,i}^s S = O(h^{10}), & 1 \leq i \leq n, \\ L_{p,r}^s S = O(h^{10}), \\ B_p^s S = O(h^{10}). \end{cases} \quad (3.3.1)$$

213 4. CONVERGENCE ANALYSIS

214
 215 The proof of convergence analysis of the method is based on Green's function approach.
 216 The exact and spline solutions of the problem (1.1) which satisfy the boundary conditions
 217 $B_p u = \gamma$ are defined by $u^{(p)} = \psi_p$ and $\hat{S}^{(p)} = \nu_p$ respectively. Suppose that the boundary
 218 value problem $u^{(p)} = 0, p = 1(1)8$ subjected to boundary conditions $B_p u = 0$ has the unique
 219 solution. This means that there is a Green's function $G_p(x, t)$ for which $u(x)$ and $\hat{S}(x)$ can
 220 be obtained in the following forms:

$$221 \quad u^{(i)}(x) = \int_a^b \frac{\partial^i G_p(x, t)}{\partial x^i} \psi_p(t) dt, \hat{S}^{(i)}(x) = \int_a^b \frac{\partial^i G_p(x, t)}{\partial x^i} \nu_p(t) dt, i = 0, \dots, p-1.$$

222 We need to introduce the following operators:

$$223 \quad \Lambda_n : C[a, b] \rightarrow R^n, \Lambda_n g = [g(t_1), \dots, g(t_n)]^T, \mathbf{M}_n : R^n \rightarrow C[a, b], \\ \mathbf{k}_p : C[a, b] \rightarrow C[a, b], \mathbf{k}_p g = \phi(x, \bar{G}_{p,0}(x), \bar{G}_{p,1}(x), \dots, \bar{G}_{p,j}(x)), j = p-1,$$

224 where $g \in C[a, b]$ and $\bar{G}_{p,i}(x) = \int_a^b \frac{\partial^i G_p(x, t)}{\partial x^i} g(t) dt, i = 0(1)p-1$. Also we define an operator
 225 $\mathbf{R}_p$ with respect to the spline solution of the problem in the following form:

$$226 \quad \mathbf{R}_p : C[a, b] \rightarrow C[a, b], \mathbf{R}_p g = \phi(x, Q_0 \Lambda_n \bar{G}_{p,0}(x), \dots, Q_j \Lambda_n \bar{G}_{p,j}(x)), j = p-1,$$

227 where $g \in C[a, b], Q_0 = I_{n \times n}$ and $Q_p, p = 1(1)8$, are the coefficients matrices of $S_i^{(p)}$ in equation
 228 (3.3.1).

229 **Lemma 4.1:** If $p = \{p_{ij}\}$ be an $m \times m$ matrix and $p_{ii} \geq \sum_{j=1, i \neq j}^m |p_{ij}| + \delta$, for $i = 1(1)m$,
 230 for $\delta > 0$, then we have $\|p^{-1}\|_{\infty} \leq \delta^{-1}$.

231 **Proof.** See Lucas [24], Lemma 4. $\square$

Lemma 4.2: If coefficient matrix of $S_i^{(p)}$ in the system of equations (3.3.1) is denoted by $Q_p, p = 1(1)8$, is nonsingular and $\|Q_p^{-1}\|_\infty$ is bounded.

Proof. For $p = 1, 2, 3$, matrix Q_p is strictly diagonally dominant and thus invertible. For $p = 4, 5, 6, 7, 8$ we can use elementary row operations, then Q_p can be converted to strictly diagonally dominant matrix, thus it is nonsingular, finally using Lemma 4.1 we can conclude that $\|Q_p^{-1}\|_\infty, p = 1(1)8$ is finite. $\square$

By using the above notations in equations (1.1) and (3.3.1) we have:

$$\psi_p - \mathbf{k}_p \psi_p = (I - \mathbf{k}_p) \psi_p = 0, \quad (4.1)$$

and

$$Q_p \Lambda_n \hat{S}^{(p)} - \Lambda_n \mathbf{R}_p \nu_p = 0, \quad (4.2)$$

According to Lemma 4.2, Q_p is nonsingular, therefore we have

$$\mathbf{M}_n \Lambda_n \hat{S}^{(p)} - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p \nu_p = 0.$$

Since $\hat{S}^{(p)}(x)$ is a polynomial of order $8 - p$, $\mathbf{M}_n \Lambda_n \hat{S}^{(p)}(x) \equiv \hat{S}^{(p)}(x)$ we have

$$\hat{S}^{(p)} - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p \nu_p = (I - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p) \nu_p = (I - p_n \mathbf{R}_p) \nu_p = 0, \quad (4.3)$$

where $p_n = \mathbf{M}_n Q_p^{-1} \Lambda_n$ is an operator from $C[a, b]$ into the continuous piecewise functions of order $8 - p$ with break points t_i .

Lemma 4.3: Let $\{\Delta\}$ be a sequence of partitions of the interval $[a, b]$ with step size h . If $h \rightarrow 0$ as n increases then $p_n = \mathbf{M}_n Q_p^{-1} \Lambda_n$ converges to the identity operator uniformly.

Proof. Let $f \in C[a, b]$, we need to show that $\|p_n f - f\| \rightarrow 0$. We have

$$\|p_n f - f\| \leq \|\mathbf{M}_n Q_p^{-1} \Lambda_n f - \mathbf{M}_n \Lambda_n f\| + \|\mathbf{M}_n \Lambda_n f - f\|.$$

Since the second term is of order $O(h^{9-p})$, $p = 1(1)8$ thus $\|\mathbf{M}_n \Lambda_n f - f\| \rightarrow 0$, so we have

$$\|p_n f - f\| \leq \|\mathbf{M}_n Q_p^{-1} \Lambda_n f - \mathbf{M}_n \Lambda_n f\| \leq \|\mathbf{M}_n\| \|Q_p^{-1}\| \|\Lambda_n f - Q_p \Lambda_n f\| \leq C^* \|\Lambda_n f - Q_p \Lambda_n f\|,$$

where C^* is a finite constant thus we have $\|p_n f - f\| \leq C^* \|\Lambda_n f - Q_p \Lambda_n f\| \leq C^{**} \omega(f, 13h)$,

256 Where $\omega(f, \delta) = \sup\{|f(x + \delta) - f(x)| : x, x + \delta \in [a, b], |\delta| \leq \delta\}$ is the modulus of continuity
 257 of $f(x)$. Since $f(x)$ is continuous function and $h \rightarrow 0$ we have $\omega(f, 13h) \rightarrow 0$. So this
 258 completes the proof. $\square$

259 **Lemma 4.4:** Under the hypotheses of Lemma 4.3, the operator sequence $p_n \mathbf{R}_p$ converges
 260 to $\mathbf{k}_p$ uniformly.

261 **Proof.** Let $f \in C[a, b]$ thus we have

$$\begin{aligned} & \|p_n \mathbf{R}_p f - \mathbf{k}_p f\| = \| \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p f - \mathbf{k}_p f \| \\ 262 & \leq \| \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p f - \mathbf{M}_n \Lambda_n \mathbf{k}_p f \| + \| \mathbf{M}_n \Lambda_n \mathbf{k}_p f - \mathbf{k}_p f \|, \quad (4.4) \\ & \leq \| \mathbf{M}_n Q_p^{-1} \| \| \Lambda_n \mathbf{R}_p f - Q_p \Lambda_n \mathbf{k}_p f \| + O(h^{9-p}), \end{aligned}$$

263 Since $\|\mathbf{M}_n\|$ and $\|Q_p^{-1}\|$ are bounded thus for some finite constant C^{**} we have

$$264 \quad \|(p_n \mathbf{R}_p - \mathbf{k}_p)f\| \leq C^{**} \|\Lambda_n \mathbf{R}_p f - Q_p \Lambda_n \mathbf{k}_p f\| \leq C^{**} \omega(f, \delta),$$

265 Where

$$266 \quad \delta = \max\{13h, \omega(\bar{G}_{p,0}(x), 21h), \omega(\bar{G}_{p,1}(x), 21h), \dots, \omega(\bar{G}_{p,p-1}(x), 21h)\}. \quad (4.5)$$

267 Since $\bar{G}_{p,j}(x), 0 \leq i \leq p-1$ are continuous, $\omega(\bar{G}_{p,j}(x), 21h) \rightarrow 0, 0 \leq i \leq p-1$ as $h \rightarrow 0$. Now
 268 from (4.5) $\delta \rightarrow 0$. Finally since f is a continuous function and $\delta \rightarrow 0$ thus we have
 269 $\omega(f, \delta) \rightarrow 0$. To guarantee the uniqueness of the solution of (1.1)-(1.2) at least in a small
 270 neighborhood of an isolated solution, we need to state the following Theorem that is treated
 271 in [13].

272 **Theorem 4.1:** Suppose that $u(x)$ is a solution BVPs (1.1)-(1.2) and the continuous functions

$$273 \quad \phi(x, z_0, z_1, \dots, z_{p-1}) \text{ and } \frac{\partial^i}{\partial z_i} \phi(x, z_0, z_1, \dots, z_{p-1}), (0 \leq i \leq p-1), \text{ are defined and continuous}$$

274 in the following region: $a \leq x \leq b, |z_i - u^{(i)}(x)| \leq \delta^*, (0 \leq i \leq p-1, \delta^* > 0)$. also suppose
 275 that the homogeneous equation $u^{(p)} = 0$, subjected to boundary conditions (1.2) has only the
 276 trivial solution. Consider a sequence of partition $\{\Delta\}$ of $[a, b]$ such that $h \rightarrow 0$. If the linear
 277 homogenous equation,

$$278 \quad u^{(p)}(x) - \sum_{i=0}^{p-1} \frac{\partial^i}{\partial z_i} \phi(x, z_0, z_1, \dots, z_{p-1}) u^{(i)}(x) = 0,$$

279 subjected to the boundary conditions (1.2) has only a trivial solution, then there exists
 280 a $\sigma > 0$, such that $u(x)$ is the unique solution of (1.1)-(1.2) in the sphere $\|w - u^{(p)}\| \leq \sigma$,

281 further for sufficiently large n there exists a unique spline $\hat{S} \in Sp_8(\Delta)$ satisfying (1.1)-(1.2)
 282 such that $\|\hat{S}^{(p)} - u^{(p)}\| \leq \sigma$, and $\hat{S}(x)$ and its derivatives through $p-1$, converges to
 283 $u(x)$ and its derivatives of corresponding orders.

284 **Proof:** The proof is similar to that in Russell and Shampine, ([13], Theorem 5). $\square$

285 Now we want to verify and prove the main convergence theorem which gives the error
 286 bounds and the orders of convergence of the purposed method theoretically.

287 **Theorem 4.2:** Consider a sequence of partition $\{\Delta\}$ of interval $[a, b]$ such that the step
 288 size $h \rightarrow 0$. Suppose that $\hat{S}(x)$ is a spline of degree eight approximation to problems (1.1)-
 289 (1.2) then under the assumptions of Theorem 4.1, the following error bounds hold for the
 290 presented method,

$$\begin{aligned} & \| (u - \hat{S})^{(j)} \| = O(h^{9-j}), j = 0, \dots, 7, \quad | (u - \hat{S})_i^{(j)} | = O(h^{10}), j = 0, \\ 291 \quad & | (u - \hat{S})_i^{(j)} | = O(h^8), j = 1, 2, \quad | (u - \hat{S})_i^{(j)} | = O(h^6), j = 3, 4, \\ & | (u - \hat{S})_i^{(j)} | = O(h^4), j = 5, 6, \quad | (u - \hat{S})_i^{(j)} | = O(h^2), j = 7. \end{aligned}$$

292 **Proof.** We assume that $S(x) \in Sp_8(\Delta)$, be the unique spline interpolant of u as in Theorem
 293 2.1 and ψ_p and v_p are the exact and spline solutions of the problem respectively. We
 294 consider the following problem $S^{(p)} = \chi$, $B_p^s S = O(h^{10})$. According to the hypotheses there
 295 is a unique solution to the problem $u^{(p)} = 0, B_p u = 0$ thus there exists a polynomial $\xi(x)$ of
 296 order $p-1$ such that

$$297 \quad B_p^s \xi = B_p^s S = O(h^{10}), \quad \|\xi^{(k)}\| = O(h^{10}), k = 0, 1, \dots, p-1. \quad (4.6)$$

298 From solvability of $(S - \xi)^{(p)} = \chi, B_p^s (S - \xi) = 0$ we have

$$299 \quad (I - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p)(S^{(p)} - \xi^{(p)}) = \mathbf{M}_n Q_p^{-1} (Q_p \Lambda_n - \Lambda_n \mathbf{R}_p)(S - \xi)^{(p)},$$

300 then using (3.3.1) and the boundness of $\|\mathbf{M}_n\|$ and $\|Q_p^{-1}\|$ we get

$$301 \quad (I - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p)(S^{(p)} - \xi^{(p)}) = \mathbf{M}_n Q_p^{-1} (O(h^{10})) = O(h^{10}). \quad (4.7)$$

302 Subtracting equations (4.3) and (4.7) we have

$$303 \quad (I - \mathbf{M}_n Q_p^{-1} \Lambda_n \mathbf{R}_p)(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) = O(h^{10}),$$

304 and equivalently

$$(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) = p_n \mathbf{R}_p (S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) = O(h^{10}). \quad (4.8)$$

It is known that the operator $\mathbf{R}_p$ is continuously differentiable in an area about u (see [13]) thus we can write (4.8) in the following integral equation form

$$(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) = p_n \left(\int_0^1 (\mathbf{R}'_p [\hat{S}^{(p)} + t(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)})]) dt \right) (S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) = O(h^{10}), \quad (4.9)$$

where $\{\vartheta_n\} = p_n \left(\int_0^1 (\mathbf{R}'_p [\hat{S}^{(p)} + t(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)})]) dt \right)$ is a sequence of linear operators converging to $\mathbf{R}'_p(u^{(p)})$. Thus we have

$$(S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) = \vartheta_n (S^{(p)} - \xi^{(p)} - \hat{S}^{(p)}) + O(h^{10}).$$

Finally since $(I - \vartheta_n)^{-1}$ exists and its norm is uniformly bounded we have

$$\|(S - \xi - \hat{S})^{(p)}\|_\infty = O(h^{10}). \quad (4.10)$$

Now according to the hypotheses the problem $(S - \xi - \hat{S})^{(p)} = r, B_p^s(S - \xi - \hat{S}) = 0$ is uniquely solvable thus we have via Green's function notation

$$(S - \xi - \hat{S})^{(i)} = \int \frac{\partial^i G_p(x, t)}{\partial x^i} (S^{(p)} - \xi^{(p)} - \hat{S}^{(p)})(t) dt, \quad i = 0, \dots, p-1, \quad (4.11)$$

which implies

$$\|(S - \xi - \hat{S})^{(i)}\|_\infty = O(h^{10}), \quad i = 0, \dots, p-1. \quad (4.12)$$

Now using the triangular inequality we have

$$\|(u - S)^{(i)}\| \leq \|(u - \hat{S})^{(i)}\| + \|(S - \hat{S})^{(i)}\| + \|\xi^{(i)}\|, \quad i = 0, \dots, p-1,$$

then using equations (3.3.1) and (4.6) and Theorem 2.1, and the proof is completed. $\square$

5. ILLUSTRATIVE EXAMPLES

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In this section we consider five boundary value problems of various orders with appropriate boundary conditions, to demonstrate the efficiency and applicability of the presented method. We compared our numerical results with the results in [5-8, 22, 25-28] and the results are tabulated in Tables 1-7. The computational costs are tabulated. All programs run in software mathematica 10. The initial guess that we use for solving the arising algebraic systems is zero.

331 **Example 1.** Consider the following linear eighth order boundary value problem

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$$u^{(8)}(x) = -xu(x) - e^x (48 + 15x + 2x^3), \quad 0 \leq x \leq 1$$

$$u(0) = u(1) = 0, u'(0) = 1, u'(1) = -e, u''(0) = 0, u''(1) = -4e, u^{(3)}(0) = -3, u^{(3)}(1) = -9e.$$

333 The exact solution for this problem is $u(x) = x(1-x)e^x$. First we solve this problem with
 334 step size $h = \frac{1}{30}$ and compared the errors in those given points in [5-7, 25]. These results
 335 are tabulated in Table 1, which show that the maximum absolute errors in the solutions of
 336 our method are considerable less than those methods in [5-7, 25]. Moreover we solve this
 337 problem for various values of $h = \frac{1}{18}, \frac{1}{36}, \frac{1}{72}, \frac{1}{144}$ and the maximum absolute errors in the
 338 solutions. CPU times are listed in Table 2. In this table E_i means,
 339 $E_i = \|u^{(i)} - \hat{S}^{(i)}\|_{\infty}, 1 \leq i \leq 7$, and O_i means the order of convergence of i th derivatives of
 340 u . This table also verified the accuracy nature of our method.

341 **Table 1. Numerical results for Example 1 with $h = \frac{1}{30}$**

x	Our method	Method in [25]	Method in [7]	Method in [6]	Method in [5]
0.1	4.16(-17)	3.89(-15)	1.63(-10)	5.62(-10)	3.73(-9)
0.2	2.81(-16)	9.45(-14)	1.63(-9)	4.88(-9)	6.61(-9)
0.3	1.92(-16)	7.04(-14)	4.90(-9)	1.37(-8)	2.33(-8)
0.4	3.31(-15)	4.37(-13)	8.46(-9)	2.29(-8)	5.17(-8)
0.5	2.84(-15)	1.20(-13)	1.01(-8)	2.71(-8)	9.76(-8)
0.6	1.63(-14)	4.02(-13)	8.68(-9)	2.38(-8)	1.78(-6)
0.7	2.47(-13)	2.07(-12)	5.15(-9)	1.49(-8)	4.12(-6)
0.8	1.43(-13)	2.66(-12)	1.76(-9)	5.54(-9)	1.83(-4)

342 **Table 2. Numerical results for Example 1 with various values of h**

h	1/18	1/36	1/72	1/144
E_0, O_0	2.2(-16),-	1.9(-19),10.2	1.8(-22),10.1	1.7(-25),10.1
E_1, O_1	8.7(-16),-	8.5(-19),9.9	8.4(-22),9.9	8.3(-25),9.9
E_2, O_2	6.7(-15),-	4(-17),6.7	6.2(-19),6.6	6.0(-21),6.6
E_3, O_3	7.8(-12),-	7.5(-14),6.7	7.3(-16),6.6	7.1(-21),6.6
E_4, O_4	4.3(-11),-	4.1(-13),6.7	3.9(-15),6.7	3.7(-17),6.7
E_5, O_5	1.3(-7),-	1.1(-9),6.8	1.0(-11),6.7	1.1(-13),6.5
E_6, O_6	4.3(-7),-	4.1(-9),6.7	3.8(-11),6.7	3.6(-13),6.7
E_7, O_7	3.1(-7),-	3.0(-8),3.4	2.8(-9),3.4	3.1(-10),3.2
CPU times	0.157	0.221	0.314	0.561

343 **Example 2.** Consider the following nonlinear eighth order boundary value problem

344 $u^{(8)}(x) = u^2(x)e^{-x}, \quad 0 \leq x \leq 1$

345 along with the boundary conditions,

346 $u(0) = u'(0) = u''(0) = u^{(3)}(0) = u^{(4)}(0) = 1, \quad u(1) = u'(1) = u''(1) = e.$

347 With the exact solution $u(x) = e^x$ We solve this problem with step size $h = \frac{1}{30}$. The results
 348 are compared with those mentioned in [5,26]. The maximum absolute errors in the solutions
 349 are tabulated in Table 3. Our results are compared with the results obtained by Adomian
 350 decomposition method and Variational iteration decomposition method, which show the
 351 applicability of our method computationally.

352 **Table 3. Numerical results for Example 2 with $h = \frac{1}{30}$**

x	Our method	Method in [5]	Method in [26]
0.1	5.21(-13)	2.34(-4)	1.27(-5)
0.2	2.62(-14)	2.54(-5)	2.43(-5)
0.3	7.04(-14)	3.02(-5)	3.35(-5)
0.4	1.87(-11)	5.26(-5)	3.94(-5)
0.5	3.54(-11)	8.39(-5)	4.16(-5)

353 **Example 3.** Consider the following nonlinear eighth order boundary value problem

354 $u^{(8)}(x) + 3u^{(7)} + u^{(6)}(x) + u'^2(x)e^{4u(x)} - 4u''(x)u^2(x) + u'''(x)e^{2x} = -36e^{-2x}, \quad 0 \leq x \leq 1$

355 $u(0) = 1, u(1) = e^{-2}, u'(0) = -2, u'(1) = -2e^{-2},$

$u''(0) = 4, u''(1) = 4e^{-2}, u^{(3)}(0) = -8, u^{(3)}(1) = -8e^{-2}.$

356 The exact solution of this problem is $u(x) = e^{-2x}$. We solve this problem with step size

357 $h = \frac{1}{30}$ for various of points and compared the errors in those special points given in [28].

358 Results are tabulated in Table 4. Our results are compared with the results obtained by
 359 Galerkin method. The data in table verified that our method is more accurate. Finally we

360 solve this problem with $h = \frac{1}{18}, \frac{1}{36}, \frac{1}{72}, \frac{1}{144}$, the maximum absolute errors in the solutions

361 and CPU times are tabulated in Table 5. This table shows that the orders of convergence in
 362 applications agree with those we obtained theoretically.

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Table 4. Numerical results for Example 3 with $h = \frac{1}{30}$

x	Our method	Method in [28]
0.1	4.32(-14)	7.98(-6)
0.2	7.83(-12)	2.18(-5)
0.3	4.22(-11)	2.09(-5)
0.4	6.64(-9)	2.87(-5)
0.5	5.42(-9)	2.68(-5)
0.6	9.27(-8)	1.69(-5)
0.7	6.53(-8)	1.15(-5)
0.8	4.37(-7)	4.79(-6)
0.9	5.23(-6)	1.82(-6)

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Table 5. Numerical results for Example 3 with various values of h

h	1/18	1/36	1/72	1/144
E_0, O_0	5.1(-16),-	4.9(-19),10	4.8(-22),9.9	4.7(-25),9.9
E_1, O_1	6.9(-15),-	6.7(-18),10	6.5(-21),10	6.3(-24),10
E_2, O_2	2.4(-14),-	2.3(-17),10	2.4(-20),9.9	2.7(-23),9.7
E_3, O_3	7.6(-13),-	7.3(-15),6.7	7.1(-17),6.6	6.9(-19),6.6
E_4, O_4	3.2(-11),-	3.1(-13),6.6	2.8(-15),6.7	2.7(-17),6.7
E_5, O_5	9.7(-9),-	8.8(-11),6.8	8.6(-13),6.7	8.1(-15),6.7
E_6, O_6	3.2(-9),-	3.4(-10),3.2	3.0(-11),3.5	2.6(-12),3.5
E_7, O_7	1.7(-8),-	1.6(-9),3.4	1.5(-10),3.4	1.1(-11),3.4
CPU times	0.266	0.325	0.500	0.926

367 **Example 4.** Consider the following nonlinear seventh order boundary value problem

368 $u^{(7)}(x) + u^{(4)}(x) - u(x)e^{u(x)} = e^x((-4(-3+x) + e^{(-e^x(x-1)\cos x)}(x-1))\cos x - 8(5+x)\sin x), 0 \leq x \leq 1$
 369 subject to the boundary conditions,

370 $u(0) = 1, u'(0) = u(1) = 0, u'(1) = -e \cos 1,$
 $u''(0) = u^{(3)}(0) = -2, u''(1) = -2e \cos 1 + 2e \sin 1.$

371 With the exact solution $u(x) = e^x(1-x)\cos x$. We solve this problem with step size $h = \frac{1}{30}$
 372 and compared the errors in those special points given [8]. The maximum absolute errors in
 373 the solutions are tabulated in Table 6. Our results are compared with the results obtained
 374 Reproducing kernel method. The results in this table show the applicability of our method
 375 computationally.

376 **Table 6. Numerical results for Example 4 with $h = \frac{1}{30}$**

x	Our method	Method in [8]
0.125	3.51(-14)	4.74(-10)
0.250	6.09(-13)	5.20(-9)
0.375	5.29(-12)	1.53(-8)
0.500	4.32(-12)	2.45(-8)
0.625	3.04(-12)	2.53(-8)
0.750	2.12(-12)	1.56(-8)

377 **Example 5.** Consider the following nonlinear sixth order boundary value problem

378 $u^{(6)}(x) = u^2(x)e^{-x}, \quad 0 \leq x \leq 1$

379 $u(0) = u''(0) = u^{(4)}(0) = 1, \quad u(1) = u''(1) = u^{(4)}(1) = e.$

380 With the exact solution $u(x) = e^x$. We solve this problem with step size $h = \frac{1}{30}$. The results
 381 are compared with those mentioned in [22, 27]. The maximum absolute errors in the
 382 solutions are tabulated in Table 7. Our results are compared with the results obtained by
 383 sextic spline collocation method and Variational iteration method. The results in this table
 verified that our method is more accurate.

384 **Table 7. Numerical results for Example 5 with $h = \frac{1}{30}$**

x	Our method	Method in [22]	Method in [27]
0.1	2.07(-17)	2.08(-15)	1.23(-4)
0.2	8.33(-16)	8.39(-15)	2.35(-4)
0.3	6.03(-16)	5.94(-15)	3.25(-4)
0.4	5.28(-16)	1.56(-14)	3.85(-4)
0.5	4.06(-16)	2.53(-14)	4.08(-4)
0.6	3.14(-16)	3.18(-14)	3.91(-4)
0.7	2.24(-16)	3.24(-14)	3.36(-4)
0.8	1.35(-16)	2.60(-14)	2.45(-4)
0.9	1.07(-16)	1.38(-14)	1.29(-4)

385 6. CONCLUSION

386 A new super convergent numerical method has been developed to solve the general linear
 387 and nonlinear higher order two-point boundary value problems. The global error estimates
 388 are obtained of order $O(h^{10})$. It has been found that our algorithm gives highly accurate
 389 numerical results and it is efficient in comparison with the other existing methods.
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