

VARIANTS OF COMPATIBLE MAPPINGS IN Menger SPACES

Abstract. In this paper, we introduce the notions of compatible mappings of type (R), type (K) and type (E) in Menger spaces and prove some common fixed point theorems for these mappings. In fact, we call these maps as variants of compatible mappings.

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1. Introduction

The notion of probabilistic metric space as a generalization of metric space was introduced by Menger [6]. In Menger theory, the notion of probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In this note he explained how to replace the numerical distance between two points x and y by a function $F(x, y, t)$ whose value $F(x, y, t)$ at the real number t is interpreted as the probability that the distance between x and y is less than t . In fact the study of such spaces received an impetus with the pioneering work of Schweizer and Sklar [10]. The theory of probabilistic metric space is of paramount importance in Probabilistic Functional Analysis especially due to its extensive applications in random differential as well as random integral equations.

Now, we give preliminaries and basic definitions in Menger space which are useful in this paper.

Definition 1.1[10] A mapping $F: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is called distribution function if it is non decreasing and left continuous with $\inf \{F(t): t \in \mathbb{R}^+\} = 0$ and $\sup \{F(t): t \in \mathbb{R}^+\} = 1$. We will denote the set of all distribution functions by $\mathcal{L}$.

Let $\mathcal{L}$ be the set of all distribution functions whereas H be the set of specific distribution function (Also known as Heaviside function) defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 1.2[6] A probabilistic metric space is a pair (X, F) , where X is a nonempty set and $F: X \times X \rightarrow \mathcal{L}$ is a mapping satisfying the following:

For all $x, y, z \in X$ and $t, s \geq 0$,

(p_1) $F(x, y, t) = 1$ if and only if $x = y$;

(p_2) $F(x, y, 0) = 0$;

(p_3) $F(x, y, t) = F(y, x, t)$;

(p_4) $F(x, y, t) = 1$ and $F(y, z, s) = 1$, then $F(x, z, t + s) = 1$.

Every metric space (X, d) can always be realized as a Probabilistic metric space by $F(x, y, t) = H(t - d(x, y))$, for all $x, y \in X$, where H be the set of specific distribution function (Also known as Heaviside function) defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0 \end{cases}; \quad \text{and } F: X \times X \rightarrow \mathcal{L}.$$

Probabilistic metric space offers a wider framework than that of the metric space and cover even wider statistical situations.

Definition 1.3[10] A mapping $\Delta: [0,1] \times [0,1] \rightarrow [0,1]$ is called a t -norm if for all $a, b, c \in [0,1]$,

(1) $\Delta(a, 1) = a, \Delta(0, 0) = 0$;

(2) $\Delta(a, b) = \Delta(b, a)$;

(3) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a, d \geq b$;

(4) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Example 1.1 The following are the four basic t -norms:

- (i) The minimum t -norm: $\Delta_M(a, b) = \min\{a, b\}$.
- (ii) The product t -norm: $\Delta_P(a, b) = ab$.
- (iii) The Lukasiewicz t -norm: $\Delta_L(a, b) = \max\{a + b - 1, 0\}$.
- (iv) The weakest t -norm, the drastic product:

$$\Delta_D(a, b) = \begin{cases} \min\{a, b\} & \text{if } \max\{a, b\} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We have the following ordering in the above stated norms:

$$\Delta_D < \Delta_L < \Delta_P < \Delta_M.$$

Definition 1.4 [6] A Menger space is a triplet (X, F, Δ) , where (X, F) is a probabilistic metric space and Δ is a t -norm with the following condition:

For all $x, y, z \in X$ and $t, s \geq 0$,

$$(p_5) \quad F(x, z, t + s) \geq \Delta(F(x, y, t), F(y, z, s)).$$

Example 1.2 Let $X = \mathbb{R}$, $\Delta(a, b) = \min(a, b)$, for all a, b in $[0, 1]$ and

$$F(x, y, t) = \begin{cases} H(t), & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases}; \quad \text{where } H(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0. \end{cases}$$

Then (X, F, Δ) is a Menger space.

Definition 1.5 A sequence $\{x_n\}$ in Menger space (X, F, Δ) is said to be:

- (i) Convergent at a point $x \in X$ if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon, \lambda}$ such that $F(x_n, x, \epsilon) > 1 - \lambda$ for all $n \geq N_{\epsilon, \lambda}$.
- (ii) Cauchy sequence in X if for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N_{\epsilon, \lambda}$ such that $F(x_n, x_m, \epsilon) > 1 - \lambda$ for all $n, m \geq N_{\epsilon, \lambda}$.
- (iii) Complete if every Cauchy sequence in X is convergent in X .

In 1996, Jungck [4] introduce the notion of weakly commuting mappings.

Definition 1.6[4] Two self-mapping f and g of a Menger space (X, F, Δ) are said to be weakly commuting if $F(fgx, gfx, t) \geq F(fx, gx, t)$ for each $x \in X$ and for each $t > 0$.

In 1982, Sessa [11] weakened the concept of commutativity to weakly commuting mappings. Afterwards, Jungck [3] enlarged the concept of weakly commuting mappings to compatible mappings. In 1991, Mishra [7] introduced the notion of compatible mappings in the setting of probabilistic metric space.

Definition 1.7[7] Let (X, F, Δ) be a Menger space such that the t -norm Δ is continuous and f, g be mappings from X into itself. Then f and g are said to be compatible if $\lim_{n \rightarrow \infty} F(fgx_n, gfx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = u$, for some $u \in X$ and for all $t > 0$.

Definition 1.8 Let f and g be self-mapping on Menger space (X, F, Δ) . Then f and g are said to be non-compatible if either $\lim_{n \rightarrow \infty} F(fgx_n, gfx_n, t)$ is non-existent or $\lim_{n \rightarrow \infty} F(fgx_n, gfx_n, t) \neq 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = u$, for some $u \in X$ and for all $t > 0$.

Further, Singh and Jain [12] proved some fixed point theorems for weakly compatible maps in the setting of Menger space.

Definition 1.9[12] Two maps f and g are said to be weakly compatible if they commute at their coincidence points.

In 1999, Pant [8] introduced a new continuity condition in Menger space, known as reciprocal continuity as follows:

Definition 1.10[8] Let f and g be self-mapping of a Menger space (X, F, Δ) . Then f and g are said to be reciprocally continuous if $\lim_{n \rightarrow \infty} fgx_n = fz$, $\lim_{n \rightarrow \infty} ggz_n = gz$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z$ for some $z \in X$.

Remark 2.1[8] If f and g are both continuous, then they are obviously reciprocally continuous, but the converse is not true. Moreover, common fixed point theorems for compatible pair of self mappings satisfying contractive conditions, continuity of one of the mappings implies their reciprocal continuity, but not conversely.

In 2004, Rohan et al. [9] introduced the concept of compatible mappings of type (R) in a metric space as follows:

Definition 1.11[9] Let f and g be mappings from metric space (X, d) into itself. Then f and g are said to be compatible of type (R) if

$$\lim_{n \rightarrow \infty} d(fgx_n, gfx_n) = 0 \text{ and } \lim_{n \rightarrow \infty} d(ffx_n, ggx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$, for some t in X .

In 2007, Singh and Singh et al. [13] introduced the concept of compatible mappings of type (E) in a metric space as follows:

Definition 1.12[13] Let f and g be mappings from metric space (X, d) into itself. Then f and g are said to be compatible of type (E) if $\lim_{n \rightarrow \infty} ffx_n = \lim_{n \rightarrow \infty} fgx_n = gt$ and $\lim_{n \rightarrow \infty} ggx_n = \lim_{n \rightarrow \infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$ for some t in X .

In 2014, Jha et al. [1] introduced the concept of compatible mappings of type (K) in a metric space as follows:

Definition 1.13[1] Let f and g be mappings from metric space (X, d) into itself. Then f and g are said to be compatible of type (K) if

$$\lim_{n \rightarrow \infty} d(ffx_n, gt) = 0 \text{ and } \lim_{n \rightarrow \infty} d(ggx_n, ft) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = t$, for some t in X .

2. Properties of variants of compatible mappings.

Now we introduce the notions of variants of compatible mappings in the setting of a Menger space as follows:

Definition 2.1 Let f and g be self- mapping on Menger space (X, F, Δ) . Then f and g are called

(1) *Compatible of type (R)* if $\lim_{n \rightarrow \infty} F(fgx_n, gfx_n, t) = 1$, and

$\lim_{n \rightarrow \infty} F(ffx_n, ggx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = u$, for some $u \in X$ and for all $t > 0$.

(2) *Compatible of type (K)* if $\lim_{n \rightarrow \infty} F(ffx_n, gx, t) = 1$, and $\lim_{n \rightarrow \infty} F(ggx_n, fx, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = x$, for some $x \in X$ and for all $t > 0$.

(3) *Compatible of type (E)* if $\lim_{n \rightarrow \infty} ffx_n = \lim_{n \rightarrow \infty} fgx_n = gt$ and $\lim_{n \rightarrow \infty} ggx_n = \lim_{n \rightarrow \infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = x$ for some x in X .

Now we give some properties related to compatible mappings of type (R) and type (E).

Proposition 2.1 Let f and g be compatible mappings of type (R) of a Menger space (X, F, Δ) into itself. If $fx = gx$ for some $x \in X$, then $fgx = ffx = ggx = gfx$.

Proposition 2.2 Let f and g be compatible mappings of type (R) of a Menger space (X, F, Δ) into itself. Suppose that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = x$ for some x in X . Then

- (a) $\lim_{n \rightarrow \infty} gfx_n = fx$ if f is continuous at x .
- (b) $\lim_{n \rightarrow \infty} fgx_n = gx$ if g is continuous at x .
- (c) $fgx = gfx$ and $fx = gx$ if f and g are continuous at x .

Proposition 2.3 Let f and g be compatible mappings of type (E) of a Menger space (X, F, Δ) into itself. Let one of f and g be continuous. Suppose that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = x$ for some $x \in X$. Then

- (a) $fx = gx$ and $\lim_{n \rightarrow \infty} ffx_n = \lim_{n \rightarrow \infty} ggx_n = \lim_{n \rightarrow \infty} fgx_n = \lim_{n \rightarrow \infty} gfx_n$.
- (b) If there exists $u \in X$ such that $fu = gu = x$, we have $fgu = gfu$.

LEMMA 2.1[12] Let $\{x_n\}$ be a sequence in a Menger space (X, F, Δ) with continuous t -norm Δ and $\Delta(t, t) \geq t$. If there exists a constant $k \in (0, 1)$ such that

$$F(x_n, x_{n+1}, kt) \geq F(x_{n-1}, x_n, t)$$

for all $t > 0$ and $n = 1, 2, 3, \dots$ then $\{x_n\}$ is a Cauchy sequence in X .

LEMMA 2.2[12] Let (X, F, Δ) be a Menger space. If there exists $k \in (0, 1)$ such that

$$F(x, y, kt) \geq F(x, y, t) \text{ for all } x, y \in X \text{ and } t > 0, \text{ then } x = y.$$

3. MAIN RESULTS

Rohan et.al [9], Singh and Singh [13] and Jha [1] proved the common fixed point theorem in a complete metric space

Theorem 3.1 Let P, R, S and T be mappings of a complete Menger space (X, F, Δ) into itself satisfying the conditions:

$$(3.1) \quad T(X) \subset P(X), \quad S(X) \subset R(X);$$

$$(3.2) \quad F(Sz, Tw, kt) \geq \min \left\{ F(Pz, Rw, t), F(Pz, Sz, t), F(Rw, Tw, t), F(Sz, Rw, \alpha t), F(Pz, Tw, (2 - \alpha)t) \right\};$$

hold for all z, w in X , where $\alpha \in (0, 2)$, $t > 0$,

$$(3.3) \quad \text{one of the mappings } P, R, S \text{ and } T \text{ is continuous.}$$

Assume that the pairs P, S and R, T are compatible of type (R) . Then P, R, S and T have a unique common fixed point in X .

Proof Since $S(X) \subset R(X)$. Now consider a point $z_0 \in X$, there exists $z_1 \in X$ such that $Sz_0 = Rz_1 = w_0$, for this point z_1 there exists $z_2 \in X$ such that $Tz_1 = Pz_2 = w_1$. Continuing in this way, we can define a sequence $\{w_n\}$ in X such that

$$w_{2n} = Sz_{2n} = Rz_{2n+1}; \quad w_{2n+1} = Tz_{2n+1} = Pz_{2n+2};$$

Now we prove that $\{w_n\}$ is Cauchy sequence in X .

Putting $z = z_{2n}$, $w = z_{2n+1}$, $\alpha = 1 - \beta$ with $\beta \in (0, 1)$ in inequality (3.2), we have

$$\begin{aligned} F(Sz_{2n}, Tz_{2n+1}, kt) &\geq \min \left\{ \begin{array}{l} F(Pz_{2n}, Rz_{2n+1}, t), F(Pz_{2n}, Sz_{2n}, t), \\ F(Rz_{2n+1}, Tz_{2n+1}, t), F(Sz_{2n}, Rz_{2n+1}, (1 - \beta)t), \\ F(Pz_{2n}, Tz_{2n+1}, (1 + \beta)t) \end{array} \right\} \\ F(w_{2n}, w_{2n+1}, kt) &\geq \min \left\{ \begin{array}{l} F(w_{2n-1}, w_{2n}, t), F(w_{2n-1}, w_{2n}, t), \\ F(w_{2n}, w_{2n+1}, t), F(w_{2n}, w_{2n}, (1 - \beta)t), \\ F(w_{2n-1}, w_{2n+1}, (1 + \beta)t) \end{array} \right\} \\ F(w_{2n}, w_{2n+1}, kt) &\geq \min \left\{ \begin{array}{l} F(w_{2n-1}, w_{2n}, t), F(w_{2n-1}, w_{2n}, t), \\ F(w_{2n}, w_{2n+1}, t), 1, \\ F(w_{2n-1}, w_{2n+1}, (1 + \beta)t) \end{array} \right\} \\ &\geq \min \left\{ \begin{array}{l} F(w_{2n-1}, w_{2n}, t), F(w_{2n}, w_{2n+1}, t), \\ F(w_{2n-1}, w_{2n+1}, (1 + \beta)t) \end{array} \right\} \\ &\geq \min \left\{ \begin{array}{l} F(w_{2n-1}, w_{2n}, t), F(w_{2n}, w_{2n+1}, t), \\ F(w_{2n-1}, w_{2n}, t), F(w_{2n}, w_{2n+1}, (\beta t)) \end{array} \right\} \\ &\geq \min \left\{ \begin{array}{l} F(w_{2n-1}, w_{2n}, t), F(w_{2n}, w_{2n+1}, t), \\ F(w_{2n}, w_{2n+1}, (\beta t)) \end{array} \right\} \end{aligned}$$

As Δ is continuous, letting $\beta \rightarrow 1$ we get

$$F(w_{2n}, w_{2n+1}, kt) \geq \min \left\{ \begin{array}{l} F(w_{2n-1}, w_{2n}, t), F(w_{2n}, w_{2n+1}, t), \\ F(w_{2n}, w_{2n+1}, t) \end{array} \right\}$$

$$= \min\{F(w_{2n-1}, w_{2n}, t), F(w_{2n}, w_{2n+1}, t)\},$$

$$\text{Hence } F(w_{2n}, w_{2n+1}, kt) \geq \min\{F(w_{2n-1}, w_{2n}, t), F(w_{2n}, w_{2n+1}, t)\}$$

Similarly

$$F(w_{2n+1}, w_{2n+2}, kt) \geq \min\{F(w_{2n}, w_{2n+1}, t), F(w_{2n+1}, w_{2n+2}, t)\} \text{ Therefore, for all } n \text{ even or odd we have}$$

$$F(w_n, w_{n+1}, kt) \geq \min\{F(w_{n-1}, w_n, t), F(w_n, w_{n+1}, t)\}.$$

Consequently,

$$F(w_n, w_{n+1}, t) \geq \min\{F(w_{n-1}, w_n, \frac{t}{k}), F(w_n, w_{n+1}, \frac{t}{k})\}.$$

By repeated application of above inequality, we get

$$F(w_n, w_{n+1}, t) \geq \min\{F(w_{n-1}, w_n, \frac{t}{k}), F(w_n, w_{n+1}, \frac{t}{k^m})\}.$$

Since $F(w_n, w_{n+1}, \frac{t}{k^m}) \rightarrow 1$ as $m \rightarrow \infty$, it follows that

$$F(w_n, w_{n+1}, kt) \geq F(w_{n-1}, w_n, t) \text{ for all } n \in N \text{ for all } t > 0$$

Therefore, by Lemma 2.1, $\{w_n\}$ is a Cauchy sequence in X and hence it converges to some point $u \in X$. Consequently, the subsequence $\{Sz_{2n}\}$, $\{Rz_{2n+1}\}$, $\{Tz_{2n+1}\}$ and $\{Pz_{2n}\}$ of $\{w_n\}$ also converges to u .

Now, suppose that P is continuous. Since P and S are compatible of type (R), by Proposition 2.2, PPz_{2n} and SPz_{2n} converges to Pu as $n \rightarrow \infty$.

We claim that $u = Pu$. Putting $z = Pz_{2n}$ and $w = z_{2n+1}$, $\alpha = 1$ in inequality (3.2) we have

$$F(SPz_{2n}, Tz_{2n+1}, kt) \geq \min \left\{ \begin{array}{l} F(PPz_{2n}, Rz_{2n+1}, t), F(PPz_{2n}, SPz_{2n}, t), \\ F(Rz_{2n+1}, Tz_{2n+1}, t), F(SPz_{2n}, Rz_{2n+1}, t), \\ F(PPz_{2n}, Tz_{2n+1}, t) \end{array} \right\}$$

Letting $n \rightarrow \infty$ we have

$$F(Pu, u, kt) \geq \min \left\{ \begin{array}{l} F(Pu, u, t), F(Pu, Pu, t), \\ F(u, u, t), F(Pu, u, t), F(Pu, u, t) \end{array} \right\}$$

$$F(Pu, u, kt) \geq F(Pu, u, t). \text{ By Lemma 2.2, we } Au = u.$$

Next we claim that $u = Su$. Putting $z = u$ and $w = z_{2n+1}$, $\alpha = 1$ in inequality (3.2), we have

$$F(Su, Tz_{2n+1}, kt) \geq \min \left\{ \begin{array}{l} F(Pu, Rz_{2n+1}, t), F(Pu, Su, t), F(Rz_{2n+1}, Tz_{2n+1}, t), \\ F(Su, Rz_{2n+1}, t), F(Pu, Tz_{2n+1}, t) \end{array} \right\}$$

Letting $n \rightarrow \infty$ we have

$$F(Su, u, kt) \geq \min \left\{ \begin{array}{l} F(u, u, t), F(u, Su, t), \\ F(u, u, t), F(u, u, t), F(u, u, t) \end{array} \right\}$$

we get $F(Su, u, kt) \geq F(Su, u, t)$. By Lemma 2.2, we $Su = u$.

Since $S(X) \subset R(X)$ and hence exists a point v in X such that $u = Su = Rv$.

We claim that $u = Tv$. Putting $z = u$ and $w = v$, $\alpha = 1$ in inequality (3.2) we have

$$F(u, Tv, kt) = F(Su, Tv, kt) \geq \min \left\{ \begin{array}{l} F(Pu, Rv, t), F(Pu, Su, t), F(Rv, Tv, t) \\ F(Su, Rv, t), F(Pu, Tv, t) \end{array} \right\}$$

$$F(u, Tv, kt) \geq \min \{ F(u, u, t), F(u, u, t), F(u, Tv, t), F(u, u, t), F(u, Tv, t) \}.$$

we get $F(u, Tv, kt) \geq F(u, Tv, t)$. By Lemma 2.2, we get $u = Tv$.

Since R and T are compatible of type (R) and $Rv = Tv = u$, by Proposition 2.1, $RTv = TRv$ and hence $Ru = RTv = TRv = Tu$. Also, we have

$$\begin{aligned} F(u, Ru, kt) = F(Su, Tu, kt) &\geq \min \left\{ \begin{array}{l} F(Pu, Ru, t), F(Pu, Su, t), F(Ru, Tu, t) \\ F(Su, Ru, t), F(Pu, Tu, t) \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} F(u, Ru, t), F(u, u, t), F(Ru, Ru, t) \\ F(u, Ru, t), F(u, Ru, t) \end{array} \right\} \end{aligned}$$

we get $F(u, Ru, kt) \geq F(u, Ru, t)$. By Lemma 2.2, we get $u = Ru$. Hence $u = Ru = Tu = Pu = Sz$. Therefore, u is a common fixed point of P, S, R and T .

Similarly, we can complete the proof when R is continuous.

Next, suppose that S is continuous. Since P and S are compatible of type (R), by Proposition 2.2, SS_{2n} and Sz_{2n} converges to Su as $n \rightarrow \infty$.

We claim that $u = Su$. Putting $z = Sz_{2n}$ and $w = z_{2n+1}$, $\alpha = 1$ in inequality (3.2), we have

$$F(SS_{2n}, Tz_{2n+1}, kt) \geq \min \left\{ \begin{array}{l} F(PSz_{2n}, Rz_{2n+1}, t), F(PSz_{2n}, SSz_{2n}, t), \\ F(Rz_{2n+1}, Tz_{2n+1}, t), F(SSz_{2n}, Rz_{2n+1}, t), \\ F(PSz_{2n}, Tz_{2n+1}, t) \end{array} \right\}$$

Letting $n \rightarrow \infty$ we have

$$F(Su, u, kt) \geq \min \left\{ \begin{array}{l} F(Su, u, t), F(Su, Su, t), F(u, u, t), \\ F(Su, u, t), F(Su, u, t) \end{array} \right\}$$

we get $F(Su, u, kt) \geq F(Su, u, t)$. By Lemma 2.2, we $Su = u$.

Since $S(X) \subset R(X)$ and hence exists a point $p \in X$ such that $u = Su = Rp$.

We claim that $u = Tp$. Putting $z = Sz_{2n}$ and $w = p$, $\alpha = 1$ in inequality (3.2), we have

$$F(SSz_{2n}, Tp, kt) \geq \min \left\{ \begin{array}{l} F(PSz_{2n}, Rp, t), F(PSz_{2n}, Sz_{2n}, t), F(Rp, Tp, t) \\ F(SSz_{2n}, Rp, t), F(PSz_{2n}, Tp, t) \end{array} \right\}$$

Letting $n \rightarrow \infty$ we have

$$F(u, Tp, kt) \geq \min \left\{ \begin{array}{l} F(u, u, t), F(u, u, t), F(u, Tp, t), \\ F(u, u, t), F(u, Tp, t) \end{array} \right\}$$

we get $F(u, Tp, kt) \geq F(u, Tp, t)$. By Lemma 2.2, we get $u = Tp$. Since R and T are compatible of type (R) and $Rp = Tp = u$, by Proposition 2.2, $RTp = TRp$ and hence $Ru = RTp = TRp = Tu$.

We claim that $u = Tu$. Putting $z = z_{2n}$ and $w = u$, $\alpha = 1$ in inequality (3.2), we have

$$F(Sz_{2n}, Tu, kt) \geq \min \left\{ \begin{array}{l} F(Pz_{2n}, Ru, t), F(Pz_{2n}, Sz_{2n}, t), F(Ru, Tu, t), \\ F(Sz_{2n}, Ru, t), F(Pz_{2n}, Tu, t) \end{array} \right\}$$

Letting $n \rightarrow \infty$ we have

$$F(u, Tu, kt) \geq \min \left\{ \begin{array}{l} F(u, Tu, t), F(u, u, t), F(Tu, Tu, t), \\ F(u, Tu, t), F(u, Tu, t) \end{array} \right\}$$

we get $F(u, Tu, kt) \geq F(u, Tu, t)$. By Lemma 2.2, we $Tu = u$.

Since $T(X) \subset P(X)$ and hence exists a point $x \in X$ such that $u = Su = Ax$.

We claim that $u = Sx$. Putting $z = x$ and $w = u$, $\alpha = 1$ in inequality (3.2), we have

$$\begin{aligned} F(Sx, u, kt) = F(Sx, Tu, kt) &\geq \min \left\{ \begin{array}{l} F(Px, Ru, t), F(Px, Sx, t), F(Ru, Tu, t), \\ F(Sx, Rx, t), F(Px, Tu, t) \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} F(u, u, t), F(u, Sx, t), F(Tu, Tu, t), \\ F(Sx, u, t), F(u, u, t) \end{array} \right\} \end{aligned}$$

we get $F(Sx, u, kt) \geq F(u, Sx, t)$. By Lemma 2.2, we get $u = Sx$.

Since P and S are compatible of type (R) and $Sx = Px = u$, by Proposition 2.2, $PSx = SPx$ and hence $Pu = PSx = SPx = Su$. Hence $u = Ru = Tu = Pu = Su$. Therefore, u is a common fixed point of P, S, R and T .

Similarly, we can complete the proof when T is continuous.

Uniqueness If possible let u and v be two fixed point of the mappings P, R, S and T .

Finally, we claim that $u = v$. Putting $z = u$ and $w = v$, $\alpha = 1$ in inequality (3.2), we have

$$\begin{aligned} F(Su, Tv, kt) = F(u, v, kt) &\geq \min \left\{ \begin{array}{l} F(Pu, Rv, t), F(Pu, Su, t), F(Rv, Tv, t), \\ F(Su, Rv, t), F(Pu, Tv, t) \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} F(u, v, t), F(u, u, t), F(v, v, t), \\ F(u, v, t), F(u, v, t) \end{array} \right\} \end{aligned}$$

we get $F(u, v, kt) \geq F(u, v, t)$. By Lemma 2.2, we get $u = v$. Hence P, S, R and T have a unique common fixed point.

Next we prove the following theorem for compatible mappings of type (K).

Theorem 3.2 Let P, S, R and T be mappings of a complete Menger space (X, F, Δ) into itself satisfying the following conditions (3.1), (3.2). Suppose that the pairs P, S and R, T are reciprocally continuous. Assume that the pairs P, S and R, T are compatible of type (K). Then P, S, R and T have a unique common fixed point in X .

Proof Now from the proof of Theorem 3.1, we can easily prove that $\{w_n\}$ is Cauchy sequence in X and hence it converges to some point $u \in X$. Consequently, the subsequence $\{Sz_{2n}\}$, $\{Rz_{2n+1}\}$, $\{Tz_{2n+1}\}$ and $\{Pz_{2n}\}$ of $\{w_n\}$ also converges to u .

Since the pairs P, S and R, T are compatible of type (K), we have $PPz_{2n} \rightarrow Su, SSz_{2n} \rightarrow Pu$ and $RRz_{2n} \rightarrow Tu, TTz_{2n+1} \rightarrow Ru$ as $n \rightarrow \infty$.

We claim that $Ru = Pu$. Putting $z = Sz_{2n}$ and $w = Tz_{2n+1}$, $\alpha = 1$ in inequality (3.1), we have

$$F(SSz_{2n}, TTz_{2n+1}, kt) \geq \min \left\{ \begin{array}{l} F(PSz_{2n}, RTz_{2n+1}, t), F(PSz_{2n}, SSz_{2n}, t), \\ F(RTz_{2n+1}, TTz_{2n+1}, t), F(SSz_{2n}, RTz_{2n+1}, t), \\ F(RSz_{2n}, Tz_{2n+1}, t) \end{array} \right\}$$

Letting $n \rightarrow \infty$ and using reciprocal continuity of the pairs P, S and R, T we have

$$F(Pu, Ru, kt) \geq \min \left\{ \begin{array}{l} F(Au, Ru, t), F(Pu, Pu, t), F(Ru, Ru, t), \\ F(Pu, Ru, t), F(Pu, Ru, t) \end{array} \right\}$$

we get $F(Pu, Ru, kt) \geq F(Pu, Ru, t)$. By Lemma 2.2, we $Pu = Ru$.

Next we claim that $Ru = Su$. Putting $z = u$ and $w = Tz_{2n+1}$, $\alpha = 1$ in inequality (3.1), we have

$$F(Su, TTz_{2n+1}, kt) \geq \min \left\{ \begin{array}{l} F(Pu, RTz_{2n+1}, t), F(Pu, Su, t), \\ F(RTz_{2n+1}, TTz_{2n+1}, t), F(Su, RTz_{2n+1}, t), \\ F(Pu, TTz_{2n+1}, t) \end{array} \right\}$$

Letting $n \rightarrow \infty$ and using reciprocal continuity of the pairs P, S and R, T we have

$$F(Su, Ru, kt) \geq \min \left\{ \begin{array}{l} F(Ru, Ru, t), F(Ru, Su, t), F(Ru, Ru, t), \\ F(Su, Ru, t), F(Ru, Ru, t) \end{array} \right\}$$

we get $F(Su, Ru, kt) \geq F(Su, Ru, t)$. By Lemma 2.2, we $Su = Ru$.

We claim that $Su = Tu$. Putting $z = u$ and $w = u$, $\alpha = 1$ in inequality (3.1), we have

$$\begin{aligned} F(Su, Tu, kt) &\geq \min \left\{ \begin{array}{l} F(Pu, Ru, t), F(Pu, Su, t), F(Ru, Tu, t), \\ F(Su, Ru, t), F(Pu, Tu, t) \end{array} \right\} \\ F(Su, Tu, kt) &\geq \min \left\{ \begin{array}{l} F(Ru, Ru, t), F(Pu, Pu, t), F(Su, Tu, t), \\ F(Su, Su, t), F(Su, Tu, t) \end{array} \right\} \end{aligned}$$

we get $F(Su, Tu, kt) \geq F(Su, Tu, t)$. By Lemma 2.2, we get $Su = Tu$.

We claim that $u = Tu$. Putting $z = z_{2n}$ and $w = u$, $\alpha = 1$ in inequality (3.2), we have

$$F(Sz_{2n}, Tu, kt) \geq \min \left\{ \begin{array}{l} F(Pz_{2n}, Ru, t), F(Pz_{2n}, Sz_{2n}, t), \\ F(Ru, Tu, t), F(Sz_{2n}, Ru, t), F(Rz_{2n}, Tu, t) \end{array} \right\}$$

Letting $n \rightarrow \infty$ we have

$$F(u, Tu, kt) \geq \min \{ F(u, Tu, t), F(u, u, t), F(u, Tu, t), F(u, Tu, t), F(u, Tu, t) \}.$$

we get $F(u, Tu, kt) \geq F(u, Tu, t)$. By Lemma 2.2, we get $u = Tu$. Hence $u = Bu = Tu = Au = Su$. Therefore, u is a common fixed point of P, S, R and T .

Uniqueness If possible let u and v be two fixed point of the mappings P, R, S and T .

Finally, we claim that $u = v$. Putting $z = u$ and $w = v$, $\alpha = 1$ in inequality (3.2), we have

$$\begin{aligned} F(Su, Tv, kt) &= F(u, v, kt) \geq \min \left\{ \begin{array}{l} F(Pu, Rv, t), F(Pu, Su, t), F(Rv, Tv, t), \\ F(Su, Rv, t), F(Pu, Tv, t) \end{array} \right\} \\ &= \min \{ F(u, v, t), F(u, u, t), F(v, v, t), F(u, v, t), F(u, v, t) \}. \end{aligned}$$

we get $F(u, v, kt) \geq F(u, v, t)$. By Lemma 2.2, we get $u = v$. Hence P, S, R and T have a unique common fixed point.

Next we prove the following theorem for compatible mappings of type (E).

Theorem 2.15 Let P, R, S and T be mappings of a complete Menger space into itself satisfying the following conditions (3.1), (3.2). Suppose that one of P and S is continuous and one of R and T is continuous. Assume that the pairs P, S and R, T are compatible of type (E). Then P, R, S and T have a unique common fixed point in X .

Proof Now from the proof of Theorem 3.1, we can easily prove that $\{w_n\}$ is Cauchy sequence in X and hence it converges to some point $u \in X$. Consequently, the subsequence $\{Sz_{2n}\}, \{Rz_{2n+1}\}, \{Tz_{2n+1}\}$ and $\{Pz_{2n}\}$ of $\{w_n\}$ also converges to u .

Now, suppose that one of the mappings P and S is continuous. Since P and S are compatible of type (E), by Proposition 2.3, $Pu = Su$. Since $S(X) \subset R(X)$ and hence exists a point $v \in X$ such that $Su = Rv$.

We claim that $Su = Tv$. Putting $z = u$ and $w = v$, $\alpha = 1$ in inequality (3.2), we have

$$F(Su, Tv, kt) \geq \min \left\{ \begin{array}{l} F(Pu, Rv, t), F(Pu, Su, t), F(Rv, Tv, t) \\ F(Su, Rv, t), F(Pu, Tv, t) \end{array} \right\}$$

$$= \min \left\{ \begin{array}{l} F(Pu, Su, t), F(Su, Su, t), F(Su, Tv, t) \\ F(Su, Rv, t), F(Su, Tv, t) \end{array} \right\}$$

we get $F(Su, Tv, kt) \geq F(Su, Tv, t)$. By Lemma 2.2, we get $Su = Tv$. Thus we have $Pu = Su = Tv = Rv$.

We claim that $Su = u$. Putting $z = u$ and $w = z_{2n+1}$, $\alpha = 1$ in inequality (3.2), we have

$$F(Su, Tz_{2n+1}, kt) \geq \min \left\{ \begin{array}{l} F(Pu, Rz_{2n+1}, t), F(Pu, Su, t), F(Rz_{2n+1}, Tz_{2n+1}, t) \\ F(Su, Rz_{2n+1}, t), F(Pu, Tz_{2n+1}, t) \end{array} \right\}$$

$$= \min \left\{ \begin{array}{l} F(Su, u, t), F(u, u, t), F(u, u, t) \\ F(Su, u, t), F(Su, u, t) \end{array} \right\}$$

we get $F(Su, u, kt) \geq F(Su, u, t)$. By Lemma 2.2, we get $u = Su$. Hence $u = Ru = Tu = Pu = Su$. Therefore, u is a common fixed point of P, S, R and T .

Again, suppose R and T are compatible of type (E) and one of the mappings R and T is continuous. Then we get $Rv = Tv = u$. By Proposition 2.3, we have $RRv = RTv = TRv = TTv$.

Hence $Ru = Tu$.

We claim that $u = Tu$. Putting $z = z_{2n}$ and $w = u$, $\alpha = 1$ in inequality (3.2), we have

$$F(Sz_{2n}, Tu, kt) \geq \min \left\{ \begin{array}{l} F(Pz_{2n}, Ru, t), F(Pz_{2n}, Sz_{2n}, t), F(Ru, Tu, t) \\ F(Sz_{2n}, Ru, t), F(Pz_{2n}, Tu, t) \end{array} \right\}$$

Letting $n \rightarrow \infty$ we have

$$F(Su, u, kt) \geq \min \left\{ \begin{array}{l} F(Su, u, t), F(Su, Su, t), F(u, u, t) \\ F(Su, u, t), F(Su, u, t) \end{array} \right\}$$

we get $F(Su, u, kt) \geq F(Su, u, t)$. By Lemma 2.2, we $Su = u$.

Since $S(X) \subset R(X)$ and hence exists a point $p \in X$ such that $u = Su = Rp$.

We claim that $u = Tp$. Putting $z = Sz_{2n}$ and $w = p$, $\alpha = 1$ in inequality (3.2), we have

$$F(SSz_{2n}, Tp, kt) \geq \min \left\{ \begin{array}{l} F(PSz_{2n}, Rp, t), F(PSz_{2n}, Sz_{2n}, t), F(Rp, Tp, t), \\ F(SSz_{2n}, Rp, t), F(Pu, Tp, t) \end{array} \right\}$$

Letting $n \rightarrow \infty$ we have

$$F(u, Tp, kt) \geq \min \left\{ \begin{array}{l} F(u, u, t), F(u, u, t), F(u, Tp, t), \\ F(u, u, t), F(u, Tp, t) \end{array} \right\}$$

we get $F(u, Tp, kt) \geq F(u, Tp, t)$. By Lemma 2.1, we get $u = Tp$.

Since R and T are compatible of type (R) and $Rp = Tp = u$, by Proposition 2.3, $RTp = TRp$ and hence $Ru = RTp = TRp = Tu$.

We claim that $u = Tu$. Putting $z = z_{2n}$ and $w = u$, $\alpha = 1$ in inequality (3.2), we have

$$F(Sz_{2n}, Tu, kt) \geq \min \left\{ \begin{array}{l} F(Pz_{2n}, Ru, t), F(Pz_{2n}, Sz_{2n}, t), F(Ru, Tu, t) \\ , F(Sz_{2n}, Ru, t), F(Pz_{2n}, Tu, t) \end{array} \right\}$$

Letting $n \rightarrow \infty$ we have

$$F(u, Tu, kt) \geq \min \left\{ \begin{array}{l} F(u, Tu, t), F(u, u, t), F(Tu, Tu, t), \\ F(u, Tu, t), F(u, Tu, t) \end{array} \right\}$$

we get $F(u, Tu, kt) \geq F(u, Tu, t)$. By Lemma 2.2, we $Tu = u$.

Hence $u = Ru = Tu$ Therefore, u is a common fixed point of P, S, R and T .

Similarly, we can complete the proof when T is continuous.

Uniqueness If possible let u and v be two fixed point of the mappings P, R, S and T .

Finally, we claim that $u = v$. Putting $z = u$ and $w = v$, $\alpha = 1$ in inequality (3.2), we have

$$\begin{aligned} F(Su, Tv, kt) = F(u, v, kt) &\geq \min \left\{ \begin{array}{l} F(Pu, Rv, t), F(Pu, Su, t), F(Rv, Tv, t), \\ F(Su, Rv, t), F(Pu, Tv, t) \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} F(u, v, t), F(u, u, t), F(v, v, t), \\ F(u, v, t), F(u, v, t) \end{array} \right\} \end{aligned}$$

we get $F(u, v, kt) \geq F(u, v, t)$. By Lemma 2.2, we get $u = v$. Hence P, S, R and T have a unique common fixed point.

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