

Original Research Article

The “Vertical” Generalization of the Binary Goldbach’s Conjecture as Applied on “Iterative” Primes with (Recursive) Prime Indexes (i-primeths)

Abstract

Aims: This paper proposes the generalization of the binary (strong) Goldbach’s Conjectures (**BGC**), briefly called “the Vertical Binary Goldbach’s Conjecture” (**VBGC**), which is essentially a meta-conjecture, as VBGC states an infinite number of conjectures stronger than BGC, which all apply on “iterative” primes with recursive prime indexes (named “i-primeths” in this article, as derived from the concept of generalized “primeths”, a term first introduced by N. J. A. Sloane and Robert G. Wilson in their “primeth recurrence” concept). VBGC was discovered by the author of this paper in 2007 and perfected (by computational verifications) until 2017 by using the arrays of matrices of Goldbach index-partitions (**GIPs**), which are a useful tool in studying BGC by focusing on prime indexes.

Study design: This article also brings in a synthesized classification of some Goldbach-like conjectures (**GLC**) (including those which are “stronger” than BGC) and a new class of **GLCs** “stronger” than BGC, from which VBGC distinguishes as a very important conjecture of primes (with potential importance in the optimization of the BGC experimental verification and other possible theoretical and practical applications in mathematics and physics) and a very special self-similar property of the primes subset of positive integers.

Keywords: Prime (number), primes with prime indexes aka prime-index primes (**PIPs**), i-primeths (**i-PIPs**), the Binary Goldbach Conjecture (**BGC**), Goldbach index-partition (**GIP**), Goldbach-like conjectures (**GLC**), the Vertical Binary Goldbach Conjecture (**VBGC**) as applied on i-primeths
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Introduction

This paper proposes the generalization of the binary (strong) Goldbach's Conjectures (**BGC**) [1,2,3][4,5,6,7], briefly called "the Vertical Binary Goldbach's Conjecture" (**VBGC**), which is essentially a meta-conjecture, as VBGC states an infinite number of conjectures stronger than BGC, which all apply on "iterative" primes with recursive prime indexes named "i-primeths" in this article, as derived from the concept of generalized "primeths", a term first introduced in 1995 by N. J. A. Sloane and Robert G. Wilson in their "primeth recurrence" concept in their array of integers indexed as A007097 (formerly M0734) [8] in The Online Encyclopedia of Integer Sequences (Oeis.org); the term "primeth" was then used from 1999 by Neil Fernandez in his "The Exploring Primeness Project" [9]). *The "i-primeth" concept is the generalization with iteration order $i \geq 0$ of the known "higher-order prime numbers" (alias "super-prime numbers", "super-prime numbers", "super-primeths", "super-primeths" or "prime-indexed primes[PIPs]") as a subset of (simple or recursive) primes with (also) prime indexes, with P_x^i being the x-th i-primeth, with iteration order $i \geq 0$, as noted in this paper and explained later on.*

VBGC was discovered in 2007 and perfected until 2017 by using the arrays (S_p and $S_{i,p}$) of Matrices (M) of Goldbach index-partitions (**GIPs**) (simple $M_{p,n}$ and recursive $M_{i,p,n}$, with iteration order $i \geq 0$, also related to the concept of "i-primeths"), which are a useful tool in studying BGC/VBGC by focusing on prime indexes (as the function P_n that numbers the primes is bijective).

The author of this article also brings in a **S-M-synthesis** of some Goldbach-like conjectures (**GLC**) (including those which are "stronger" than BGC) and a new class of **GLCs** "stronger" than BGC, from which VBGC (which is essentially a variant of BGC applied on a serial array of subsets of i-primeths with a general iteration order $i \geq 0$) distinguishes as a very important conjecture of primes (with potential importance in the optimization of the BGC experimental verification and other possible useful theoretical and practical applications in mathematics [including cryptography and fractals] and physics [including crystallography and M-Theory]), and a very special self-similar property of the primes subset of N (noted/abbreviated as $\wp$ or $\wp^*$ as explained later on in this paper).

Primes (which are considered natural numbers [positive integers] >1 that each has no positive divisors other than 1 and itself by the latest modern conventional definition, as number 1 is a special case[10,11] which is considered neither prime nor composite, but the unit of all integers) are conjectured (by BGC) to have a sufficiently dense and (sufficiently) uniform distribution in N , so that:

(1) any natural even number $2n$, with $n > 1$ can be splitted in at least one Goldbach partition/pair(GP)corresponding to at least one Goldbach index-partition (**GIP**)[12]

OR

(2) any positive integer $n > 1$ can be expressed as the arithmetic average of at least one pair of primes.

BGC is specifically reformulated by the author of this article in order to emphasize the importance of studying the Primes Distribution (**PD**) [13,14,15,16] defined by a global and local density and uniformity with multiple interesting fractal patterns [17]: BGC is in fact an auto-recursive fractal property of PD in N alias the Goldbach Distribution of Primes (**GDP**) (as the author will try to prove later on in this article), but also a property of $\wp$, a property which is indirectly expressed as BGC, using the subset of even naturals).

Part A. The array S_p of the simple Matrix of Goldbach Index-Partitions $(M_{p,n})$

Definition of $\wp^*$ and $\wp$. We may define the prime subset of N as $\wp^* = \{P_1(=2), P_2(=3), P_3(=5), \dots, P_x, \dots, P_y, \dots, P_\infty\}$, with $x, y \in N^*$ and $0 < x < y$, with $P_x(P_y)$ being the x -th (y -th) primes of $\wp^*$ and P_∞ marking the already proved fact that $\wp^*$ has an infinite number of (natural) elements (Euclid's 2nd theorem [18]). The numbering function of primes (P_n) is a bijection that interconnects $\wp^*$ with N^* so that each element of $\wp^*$ corresponds to only (just) one element of N^* and vice versa: $1 \leftrightarrow P_1(=2)$, $2 \leftrightarrow P_2(=3)$, ..., $x \leftrightarrow P_x$ (the x -th prime), $y \leftrightarrow P_y$ (the y -th prime), ..., $\infty \leftrightarrow P_\infty$. Originally, Goldbach considered that number 1 was the first prime: although still debated until present, today the mainstream considers that number 1 is neither prime or composite, but the unity of all the other integers.^[10,11] However, in respect to the first "ternary" formulation of GC (**TGC**) (which was re-formulated by Euler as the BGC and also demonstrated by the same Euler to be stronger than TGC, as TGC is a consequence of BGC), the author of this article also defines $P_0 = 1$ (the unity of all integers, implicitly the unity of all primes) and $\wp = \{P_0(=1), P_1(=2), P_2(=3), P_3(=5), \dots, P_x, \dots, P_y, \dots, P_\infty\}$, with $x, y \in N$ and $0 \leq x < y$, although only $\wp^* = \wp - \{P_0(=1)\}$ shall be used in this paper (as it is used in the mainstream of modern mathematics).

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The 1st formulation of BGC. For any even integer $\boxed{n > 2}$, it will always exist at least one pair of (other two) integers $x, y \in N^*$ with $x \leq y$ so that $P_x + P_y = n$, with $P_x(P_y)$ being the x -th (y -th) primes of $\wp^*$. **Important observation:** The author considers that analyzing those "homogeneous" triplets of three naturals (n, x, y) (no matter if primes or composites) is more convenient and has more "analytical" potential than analyzing (relatively) "inhomogeneous" triplets of type (n, P_x, P_y) : that's why the author proposes **Goldbach index partitions (GIPs)** as an alternative to the standard **Goldbach partitions (GPs)** proposed by Oliveira e Silva^[12]. The existence of (at least) a triplet (n, x, y) for each even integer $\boxed{n > 2}$ (as BGC claims) may suggest that BGC is profoundly connected to the generic primality (of any P_x and P_y) and, more specifically, argues that *GC is in fact a property of PD in N (and a property of $\wp^*$ as composed of indexed/numbered elements)*. The most important property of Primes and PD and is that $\boxed{P_x \rightarrow x \cdot \ln(x) \Leftrightarrow P_x / x \rightarrow \ln(x), \text{ for } x \rightarrow \infty}$ or $\boxed{P_x \cong x \cdot \ln(x), \text{ for any progressively large } x}$ (which is the alternative [linearithmic] expression of the Prime Number Theorem [19], as if $\wp^*$ is a result of an apparently random quantized linearithmization of $N^* - \{1\}$ so that $\boxed{P_n \rightarrow n \cdot \ln(n)}$). **In conclusion:** For any even integer $\boxed{n > 2}$, at least one GIP exists (**BGC – 1st condensed formulation**)

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The 2nd formulation of BGC using the Matrix of Goldbach index-partitions (M-GIP or M).

[1] Let us consider an infinite string of matrices $S = \{M_1, M_2, M_3, \dots, M_n, \dots, M_\infty\}$, with each generic M_n being composed of lines made by GIPs (X, Y) , such as:

$$M_n = \begin{pmatrix} x_{n,1} & y_{n,1} \\ \vdots & \vdots \\ x_{n,j} & y_{n,j} \\ \vdots & \vdots \\ x_{n,m_n} & y_{n,m_n} \end{pmatrix}, \text{ with } P_{x_{n,j}} + P_{y_{n,j}} = n, \forall j \in [1, m_n]$$

(j is the index of any chosen line of M_n , $j \geq 1$ and $j \leq m_n$)

(m_n is the total maximum number of j -indexed lines of M_n)

($x_{n,i}, y_{n,i} \in N^*$, $x_{n,i} < x_{n,i+1}$ for $m_n \geq 2$, $\forall i \in [1, m_n]$)

[2] Let us also consider the function that counts the lines of any M_n , such as: $l(n) = m_n$. This function (that numbers the lines of a GM) is classically named as $r(n) = l(n) = m_n$ ("r" stands for the number of "rows").[12]

[3] An empty matrix $(M_\emptyset)$ is defined as a matrix with zero rows and/or columns.

Using S , M , $M_\emptyset$ and $r(n)$ as previously defined, BGC has two formulations sub-variants:

1. $M_n \neq M_\emptyset$ (OR S doesn't contain any $M_\emptyset$) for any even integer $n > 2$ or shortly:
 $\forall \text{ even integer } n > 2 \Leftrightarrow M_n \neq M_\emptyset$ (the 2nd formulation of BGC – 1st sub-variant).
2. For any even integer $n > 2$, $r(n) > 0$ or shortly: $\forall \text{ even integer } n > 2 \Leftrightarrow r(n) > 0$ (the 2nd formulation of BGC – 2nd sub-variant).

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The 3rd formulation of BGC using the generalization of $S(S_p)$ and the generalization of $M(M_{p,n})$ for GIPs matrix containing more than 2 columns (as based on GIPs with more than 2 elements).

[1] Let us consider an infinite set OF infinite strings OF matrix:

- a) $S_2 = \{M_{2,1}, M_{2,2}, M_{2,3}, \dots, M_{2,n}, \dots, M_{2,\infty}\}$ (the generic $M_{2,n}$ of S_2 has 2 columns based on [binary] GIPs with 2 elements);
- b) $S_3 = \{M_{3,1}, M_{3,2}, M_{3,3}, \dots, M_{3,n}, \dots, M_{3,\infty}\}$ (the generic $M_{3,n}$ of S_3 has 3 columns based on [ternary] GIPs with 3 elements);
- c) ...;
- d) $S_p = \{M_{p,1}, M_{p,2}, M_{p,3}, \dots, M_{p,n}, \dots, M_{p,\infty}\}$ (the generic $M_{p,n}$ of S_p has p columns based on [p -nary] GIPs with p elements and natural $p > 3$);
- e) ...;

- f) $S_\infty = \{M_{\infty,1}, M_{\infty,2}, M_{\infty,3}, \dots, M_{\infty,n}, \dots M_{\infty,\infty}\}$ (the generic $M_{\infty,n}$ of S_∞ has potentially infinite (∞) number of columns based on ∞ -nary GIPs with a potentially infinite (∞) number of elements)
- g) With each generic $M_{p,n}$ being composed of $m_{p,n}$ lines and p columns made by p -nary GIPs with p elements, such as:

$$M_{p,n} = \begin{pmatrix} x_{n,1} & \dots & x_{n,k} & \dots & x_{n,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n,j} & \dots & x_{n,k+j} & \dots & x_{n,p+j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n,m_{p,n}} & \dots & x_{n,k+m_{p,n}} & \dots & x_{n,p+m_{p,n}} \end{pmatrix}, \text{ with } P_{x_{n,j}} + \dots + P_{x_{n,j+k}} + \dots + P_{x_{n,p+j}} = n,$$

$\forall j \in [1, m_{p,n}] \text{ and } \forall k \in [1, p],$

(j is the index of any chosen line of $M_{p,n}$, $j \geq 1$ and $j \leq m_{p,n}$

and $m_{p,n}$ is the total maximum number of j -indexed lines of $M_{p,n}$)

(k is the index of any chosen column of $M_{p,n}$, $k \geq 1$ and $k \leq p$

and p is the total number of k -indexed columns of $M_{p,n}$)

($x_{n,k+j} \in N^*$, $x_{n,j} \leq x_{n,j+1}$ for $m_{p,n} \geq 2$, $\forall j \in [1, m_{p,n}]$ and $\forall k \in [1, p]$)

[2] Let us also consider the function that counts the lines of any $M_{p,n}$, such as:

$$r(p,n) = l(p,n) = m_{p,n}.$$

[3] An **empty matrix** ($M_\emptyset$) is defined as a matrix with a 0 number of rows and/or columns.

Using S_p , $M_{p,n}$, $M_\emptyset$ and $r(p,n)$ as previously defined, BGC has two formulations sub-variants:

1. $M_{2,n} \neq M_\emptyset$ (OR S_2 doesn't contain any $M_\emptyset$) for any even integer $n > 2$ or shortly:

$$\forall \text{ even integer } n > 2 \Leftrightarrow M_{2,n} \neq M_\emptyset \text{ (the 3}^{\text{rd}} \text{ formulation of BGC - 1}^{\text{st}} \text{ sub-variant)}.$$

2. For any even integer $n > 2$, $r(2,n) > 0$ or shortly: $\forall \text{ even integer } n > 2 \Leftrightarrow r(2,n) > 0$
(the 3rd formulation of BGC - 2nd sub-variant).

Part B. A synthesis and A/B classification of the main known Goldbach-like conjectures (GLCs) using the $M_{p,n}$ concept

The Goldbach-like conjectures (GLCs) category/class.

GLCs definition. A GLC may be defined as any additional special (observed/conjectured) property of S_p and its elements $M_{p,n}$ other than GC (with $n > 2$), with possibly other inferior limits $a \geq 2$, with $n > a \geq 2$).

GLCs classification. GLCs may be classified in two major classes using a double criterion such as:

1. **Type A GLCs (A-GLCs)** are those GLCs that claim: [1] Not only that all $M_{p,n} \neq M_{\emptyset}$ for a chosen $p > 1$ and for any / any odd / any even integer $n > a \geq 2$ (with a being any finite natural established by that A-GLC and $n > a$) BUT ALSO [2] any other non-trivial(nt) accessory property/properties of all $M_{p,n} (\neq M_{\emptyset})$ of S_p . A specific A-GLC is considered authentic if the other non-trivial accessory property/properties of all $M_{p,n} (\neq M_{\emptyset})$ (claimed by that A-GLC) isn't/aren't a consequence of the 1st claim (of the same A-GLC). Authentic (at least conjectured as such) A-GLCs are (have the potential to be) "stronger" than GC as they claim "more" than GC does.
2. **Type B GLCs (B-GLCs)** are those GLCs that claim: no matter if all $M_{p,n} \neq M_{\emptyset}$ or just some $M_{p,n} \neq M_{\emptyset}$ for a chosen $p > 1$ and for some / some odd / some even integer $n > a \geq 2$ (with a being any finite natural established by that B-GLC and $n > a$), all those $M_{p,n}$ that are yet non- $M_{\emptyset}$ (for $n > a$) have (an)other non-trivial accessory property/properties. A specific B-GLC is considered authentic if the other non-trivial accessory property/properties of all $M_{p,n} (\neq M_{\emptyset})$ (claimed by that B-GLC for $n > a$) isn't/aren't a consequence of the fact that some $M_{p,n} \neq M_{\emptyset}$ for $n > a$. Authentic (at least conjectured as such) B-GLCs are "neutral" to GC (uncertainly "stronger" or "weaker" conjectures) as they claim "more" but also "less" than GC does (although they may be globally weaker and easier to formally prove than GC).

Other variants of the (generic) Goldbach Conjecture (GC) and GLCs include the statements that:

1. "[...] Every [integer] number that is greater than 2 is the sum of three primes" (Goldbach's original conjecture formulated in 1742, sometimes called the "Ternary" Goldbach conjecture (TGC), written in a June 7, 1742 letter to Euler) (which is equivalent to: "every integer > 2 is the sum of at least one triad of primes", *with the specification that number 1 was also considered a prime by the majority of mathematicians contemporary to Goldbach, which is no longer the case now]). This (first) variant of GC (TGC) can be formulated using (ternary) $M_{3,n}$ (based on GIPs with 3 elements) such as:
 - a. **Type A formulation variant as applied to $\wp$ (not just to $\wp^*$):**

$$\boxed{\forall \text{ integer } n > 2 \Leftrightarrow M_{3,n} \neq M_{\emptyset}} \text{ (with } x_{n,j,k} \geq 0 \text{ and } P_{x_{n,j,k}} \in \wp \text{)}$$
 - b. **Type B (neutral) formulation variant:** not supported.
2. "Every even integer $n > 4$ is the sum of 2 odd primes." (Euler's binary reformulation of the original GC, which was initially expressed by Goldbach in a ternary form as previously explained).

Since BGC (as originally reformulated by Euler) contains the obvious triviality that there are infinite many even positive integers of form $2p = p + p$ (with p being any prime), the non-trivial BGC (ntBGC) sub-variant that shall be treated in this paper (alias "BGC" or "ntBGC") is that: **"every even integer $n > 6$ is the sum of at least one pair of distinct odd primes"** [20,21] (which is equivalent to: **"every even integer $m > 3$ is the arithmetic average of at least one pair of distinct odd primes"**). Please note that ntBGC doesn't support the definition of a GLC, as $2p = p + p$ is a trivial property of some even integers implying the complementary relative triviality that: $2c \neq 2p \neq p + p$ (with c being any composite natural number and p being any prime). ntBGC can be formulated using (binary) $M_{2,n}$ (based on GIPs with 2 elements) such as:

- a. **Type A formulation variant:** " $\boxed{\forall \text{ even integer } n > 6}$, $M_{2,n}(M_n) \neq M_{\emptyset}$ AND $M_{2,n}(M_n)$ contains at least one line with both elements (GIPs) $\neq 1$ (as $P_1 = 2$ is the only even prime) AND distinct to each other (as distinct GIPs means distinct primes as based on the bijection of the prime numbering function)"
- b. **Type B (neutral) formulation variant:** " $\boxed{\forall \text{ even integer } n > 6}$, all $M_{2,n}(M_n)$ that are non-empty (as S_p may also contain empty $M_{2,n}(M_n) = M_{\emptyset}$ for some specific [but still unfound] n values) will contain at least one line with both elements (GIPs) $\neq 1$ (as $P_1 = 2$ is the only even prime) AND distinct to each other (as distinct GIPs means distinct primes as based on the bijection of the prime numbering function)"

3. " $\boxed{\forall \text{ odd integer } n > 5}$, n is the sum of 3 (possibly identical) primes." [22] (the [weak] Ternary Goldbach's conjecture/theorem [TGC/TGT] formally proved by Harald Helfgott in 2013 [23,24,25], so that TGC is very probably [but not surely however] a proved theorem (as TGT), and no longer a "conjecture") (which is equivalent to: " $\boxed{\forall \text{ odd integer } n > 5}$, n is the sum of at least one triad of [possibly identical] primes"). TGC can be formulated using (ternary) $M_{3,n}$ (based on GIPs with 3 elements) such as:

- a. **Type A formulation variant:** " $\boxed{\forall \text{ odd integer } n > 5 \Leftrightarrow M_{3,n} \neq M_{\emptyset}}$ "
- b. **Type B (neutral) formulation variant:** not supported.

4. " $\boxed{\forall \text{ integer } n > 17}$, n is the sum of exactly 3 distinct primes." [20] (cited as "Conjecture 3.2" by Pakianathan and Winfree in their article, which is equivalent to: " $\boxed{\forall \text{ integer } n > 17}$, n is the sum of at least one triad of distinct primes") (this is a conjecture stronger than TGC, but weaker than BGC as it is implied by BGC). This stronger version of TGC (sTGC) can also be formulated using (ternary) $M_{3,n}$ (based on GIPs with 3 elements) such as:

- a. **Type A formulation variant:** " $\boxed{\forall \text{ integer } n > 17} \Rightarrow M_{3,n} \neq M_{\emptyset}$ AND $M_{3,n}$ contains at least one line with all 3 elements (GIPs) distinct from each other"
- b. **Type B (neutral) formulation variant:** " $\boxed{\forall \text{ integer } n > 17} \Rightarrow$ those $M_{3,n}$ which are $\neq M_{\emptyset}$ will contain at least one line with all 3 elements (GIPs) distinct from each other"

5. " $\boxed{\forall \text{ odd integer } n > 5}$, n is the sum of a prime and a doubled prime [which is twice of any prime]." (Lemoine's conjecture [LC] [26,27] which was erroneously attributed by MathWorld to Levy H. who pondered it in 1963 [27,28,29]). LC is stronger than TGC, but weaker than BGC. LC also has an extension formulated by Kiltinen J. and Young P. (alias the "refined Lemoine

conjecture" [30]), which is stronger than LC, but weaker than BGC and won't be discussed in this article (as this paper mainly focuses on those GLCs stronger than BGC). LC can be formulated using (ternary, not binary) $M_{3,n}$ (based on GIPs with 3 elements) such as:

- a. **Type A formulation variant:** " $\boxed{\forall \text{ odd integer } n > 5} \Rightarrow M_{3,n} \neq M_{\emptyset} \text{ AND } M_{3,n} \text{ contains at least one line with at least 2 identical elements (GIPs)}$ "
- b. **Type B (neutral) formulation variant:** " $\boxed{\forall \text{ odd integer } n > 5} \Rightarrow \text{those } M_{3,n} \text{ which are } \neq M_{\emptyset} \text{ will contain at least one line with at least 2 identical elements (GIPs)}$ "

6. There are also a few original conjectures on partitions of integers as summations of primes published by Smarandache F. [31] that won't be discussed in this article, as these conjectures depart from VBGC (as VBGC presentation is the main purpose of this article).

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There are also a number of (relative recently discovered) GLCs stronger than BGC (and implicitly stronger than TGC), that can also be synthesized using $M_{p,n}$ concept: **these stronger GLCs (as VBGC also is) are tools that can inspire new strategies of finding a formal proof for BGC, as I shall try to demonstrate next.** Additionally, there are some arguments that Twin Prime Conjecture (TPC) [32] may be also (indirectly) related to BGC as part of a more extended and profound conjecture^[17] [33,34, 35], so that any new clue for BGC formal proof may also help in TPC (formal) demonstration. Moreover, TPC may be weaker (and possibly easier to proof) than BGC (at least regarding the efforts towards the final formal proof) as the superior limit of the primes gap was recently "pushed" to be ≤ 246 [36], but the Chen's Theorem I (that "every sufficiently large even number can be written as the sum of either 2 primes, OR a prime and a semiprime [the product of just 2 primes]" [37,38,39]) has not been improved since a long time (at least by the set of proofs that are accepted in the present by the mainstream) except Cai's new proved theorem published in 2002 ("There exists a natural number N such that every even integer n larger than N is a sum of a prime $\leq n^{0.95}$ and a semi-prime" [40,41], a theorem which is a similar but a weaker statement than LC that hasn't a formal proof yet).

1. **The Goldbach-Knizek conjecture [GKC] [42]** (which is stronger than BGC): " $\boxed{\forall \text{ even integer } n > 4}$, there is at least one prime number p [so that] $\sqrt{n} < p \leq n/2$ and $q = n - p$ is also prime [with $n = p + q$ implicitly]". GKC can also be reformulated as: "every even integer $n > 4$ is the sum of at least one pair of primes with at least one element in the semi-open interval $(\sqrt{n}, n/2]$ ". GKC can be also formulated using (binary) $M_{2,n}$ (based on GIPs with 2 elements) such as:

- a. **Type A formulation variant:** " $\boxed{\forall \text{ even integer } n > 4} \Rightarrow M_{2,n}(M_n) \neq M_{\emptyset} \text{ AND } M_{2,n}(M_n) \text{ contains at least one line with at least one element } x, \text{ so that } P_x \in (\sqrt{n}, n/2]$."
- b. **Type B (neutral) formulation variant:** " $\boxed{\forall \text{ even integer } n > 4} \Rightarrow \text{those } M_{2,n}(M_n) \text{ which are } \neq M_{\emptyset} \text{ will contain at least one line with at least one element } x, \text{ so that } P_x \in (\sqrt{n}, n/2]$."

2. **The Goldbach-Knizek-Rivera conjecture [GKRC] [43]** (which is obviously stronger than BGC, but also stronger than GKC for $n \geq 64$): " $\boxed{\forall \text{ even integer } n > 4}$, there is at least one prime number p [so that] $\sqrt{n} < p < 4\sqrt{n}$ and $q = n - p$ is also prime [with $n = p + q$ implicitly]". GKRC can also be reformulated as: " $\boxed{\forall \text{ even integer } n > 4}$, n is the sum of at least one pair of

primes with one element in the double-open interval $(\sqrt{n}, 4\sqrt{n})$ ". GKRC can be formulated using (binary) $M_{2,n}$ (based on GIPs with 2 elements) such as:

- a. **Type A formulation variant:** " $\boxed{\forall \text{ even integer } n > 4} \Rightarrow M_{2,n}(M_n) \neq M_\emptyset$ AND $M_{2,n}(M_n)$ contains at least one line with one element x , so that $P_x \in (\sqrt{n}, 4\sqrt{n})$."
- b. **Type B (neutral) formulation variant:** " $\boxed{\forall \text{ even integer } n > 4} \Rightarrow$ those $M_{2,n}(M_n)$ which are $\neq M_\emptyset$ will contain at least one line with one element x , so that $P_x \in (\sqrt{n}, 4\sqrt{n})$."

3. Any other GLC that establishes an additional inferior limit $a > 0$ for $r(2, n)$ so that $r(2, n) \geq a > 0$ (like Woon's GLC [44]) can also be considered stronger than BGC, as BGC only suggests $r(2, n) > 0$ for any even integer $n > 6$ (which implies a greater average number of GIPs per each n than the more selective Woon's GLC does).

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There is also a remarkable set of original conjectures (many of them stronger than BGC and/or TPC) originally proposed by Sun Zhi-Wei ^[1,2] [45,46], a set from which I shall cite ^[3] (by rephrasing) some of those conjectures that have an important element in common with the first special case of VBGC: the recursive P_{P_x} function in which P_x is the x -th prime and P_{P_x} is the P_x -th prime (which is denoted in the next cited conjectures as P_q which is the q -th prime, with q being also a prime number).

1. **Conjecture 3.1 (Unification of GC and TPC, 29 Jan. 2014).** For any integer $n > 2$ there is at least one triad of primes $\left[(1 < q < 2n - 1), (2n - q), (P_{q+2} + 2) \right]$ (Sun's Conjecture 3.1 [SC3.1 or U-GC-TPC], which is obviously stronger than BGC and was tested up to $n = 2 \times 10^8$)
2. **Conjecture 3.2 (Super TPC [SPTC], 5 Feb. 2014).** For any integer $n > 2$ there is at least one triad $\left[(0 < k < n), (P_k + 2 = \text{prime}), (P_{P_{n-k}} + 2 = \text{prime}) \right]$ (Sun's Conjecture 3.2 [SC3.2 or SPTC], which is obviously stronger than TPC and was tested up to $n = 10^9$) ^[4,5]
3. **Conjecture 3.3 (28 Jan. 2014).** For any integer $n > 2$ there is at least one pentad $\left[(0 < k < n - 1), (6k - 1 = \text{prime}), (6k + 1 = \text{prime}), (P_{n-k} = \text{prime}), (P_{n-k} + 2 = \text{prime}) \right]$ (Sun's Conjecture 3.3 [SC3.3], which is obviously stronger than TPC as it implies TPC; SC3.3 was tested up to $n = 2 \times 10^7$)

[1] Wikipedia page about Sun Zhi-Wei: https://en.wikipedia.org/wiki/Sun_Zhiwei

[2] The personal page of Sun Zhi-Wei: <http://maths.nju.edu.cn/~zwsun/>

[3] See also Sun's Z-W. personal web page on which all conjectures are presented in detail (URL)

[4] See also the first announcement of this conjecture made by Sun Z-W. himself on 6 Feb 2014) (URL)

[5] See also the sequence A218829 on OEIS.org proposed by Sun Z-W. (URL1, URL2)

- 4. Conjecture 3.7-i (1 Dec. 2013).** There are infinite many positive even integers $n > 3$ which are associated with a hexad of primes $\left[(n+1), (n-1), (P_n + n), (P_n - n), (nP_n + 1), (nP_n - 1) \right]$ (Sun's Conjecture 3.7-1 [SC3.7-i]), which is obviously stronger than TPC as it implies TPC; $n = 22110$ is the first/smallest value of n predicted by SC3.7-1)
- 5. Conjecture 3.12-i (5 Dec. 2013).** All positive integers $n > 7$ have at least one associated pair $\left[(k < n-1), (2^k + P_{n-k} = \text{prime}) \right]$ (Sun's Conjecture 3.12-i [SC3.12-i])
- 6. Conjecture 3.12-ii (6 Dec. 2013).** All positive integers $n > 3$ have at least one associated pair $\left[(k < n-1), (k! + P_{n-k} = \text{prime}) \right]$ (Sun's Conjecture 3.12-ii [SC3.12-ii])
- 7. Remark 3.19 (which is an implication of the Conjecture 3.19 not cited in this article).** There is an infinite number of triads of primes $\left[(q > 1), (r = P_q - q + 1), (P_r - r + 1) \right]$ (Sun's Remark on Sun's Conjecture 3.19 [SRC3.19])
- 8. Conjecture 3.21-i (6 Mar. 2014).** For any integer $n > 5$ there will always exist at least one triad $\left[(0 < k < n), (2k + 1 = \text{prime}), (P_{k \cdot n} + k \cdot n = \text{prime}) \right]$ (Sun's Conjecture 3.21-i [SC3.21-i])
- 9. Conjecture 3.23-i (1 Feb. 2014).** For any integer $n > 13$ there is at least one triad of primes $\left[(1 < q < n), (q + 2), (P_{n-q} + q + 1) \right]$ (the Sun's Conjecture 3.23-i [SC3.23-i])

Part C. The ‘i-primeths’ (${}^i\wp^*$) definition

The definition of the generalized “i-primeths” concept ${}^i\wp^*$. This paper chooses to use the term “primeth(s)” because this is the shortest and also the most suggestive of all the alternatives ^[6] used until now (as the “th” suffix includes, by abbreviation, the idea of “index of primes”). “Primeths” were originally defined as a subset of primes with (also) prime indexes (with the numbering of the elements of $\wp^*$ starting from $P_1 = 2$). As primes are in fact those positive integers with a prime index (the “prime index” being non-tautological defined as a positive integer >1 that has only 2 distinct divisors: 1 and itself), all the standard primes may be considered primeths with iteration order $i=0$ (or shortly: 0-primeths) NOT with $i=1$ (as Fernandez first considered) (as the $i=0$ marks the genesis of $\wp^*$ from the ordinary $N \supset \wp^*$ and cannot be considered an iteration on $\wp^*$). This new alternative definition (and notation) of i-primeths (${}^iP$ containing ${}^iP_x$ elements with $i \geq 0$ and $x \in N^*$) has three advantages, with an accent strictly on the number (i) of P-on-P iterations and NOT on the general standard definition (and notation) of iterated functions like

$$P^1(x) \equiv P^{o1}(x) = P(x) \text{ and } P^i(x) \equiv P^{oi}(x) = P\left(\frac{P..P(x)}{(i-1)\text{ nested functions } P}\right):$$

1. the iteration order i is also the number of (“vertical”) iterations for producing the i-primeths from the 0-primeths (${}^0\wp^* = \wp^*$) (as in the original primeths definition, the standard primes were considered 1-primeths not 0-primeths, as if they were produced from N using 1 vertical iteration, but N doesn’t contain just primes, as $\wp^* \neq N$);
 - a. these iterations numbered by order i are easy to follow when implemented in different algorithms using a programming language on a computer;
2. the concept of primes can be generalized as “i-primeths” ${}^i\wp^*$, with ${}^i\wp^*$ also including $\wp^*$ as the special case of 0-primeths (${}^0\wp^* = \wp^*$) $\subset {}^i\wp^*$;
3. this definition clearly separates $\wp^*$ from the ordinary N using 0 (not 1) as a starting order (i) for $\wp^*$ (${}^0\wp^*$) and considering N as a ${}^{(-1)}\wp^*$ (a “bulky” ${}^{(-1)}\wp^*$ “contaminated” with composite positive integers that can be considered “(-1)-primeths” convertible to 0-primeths by different sieves of primes.
 - a. ${}^0\wp^*$ inevitably “contains” N^* by its indexes, in the sense that ${}^0\wp^*$ contains all the generic ${}^0P_x$ elements with indexes $x \in N^*$ (an index x that “scrolls” all N^*). The same prime may be part of more than one i-primeths subset ${}^i\wp^*$, as x is not necessarily a prime.
 - b. This slightly different definition of the i-primeths (${}^i\wp^*$ containing generic ${}^iP_x$ elements with $i \geq 0$ and $x \in N^*$, as explained previously) is NOT a new “anomaly” and it was also used by Smarandache F. as cited by Murthy A.[47] and also by Seleacu V. and Bălăcenoiu I. [48]

[6] Alternative terms for “primeths”: “higher-order prime numbers”, “superprime numbers”, “super-prime numbers”, “super-primes”, “superprimes” or “prime-indexed primes[PIPs]” ([URL-OEIS page](#))

The elements of the generalized set of i-primeths ${}^i\mathcal{P}^*$:

${}^0\mathcal{P}^* = \mathcal{P}^* = \{ {}^0P_1 (= P_1 = 2), {}^0P_2 (= P_2 = 3), {}^0P_3 (= P_3 = 5), \dots, {}^0P_x (= P_x), \dots \}$ (alias 0-primeths)

${}^1\mathcal{P}^* = \{ {}^1P_1 (= P_{p_1} = P_2 = 3), {}^1P_2 (= P_{p_2} = P_3 = 5), \dots, {}^1P_x (= P_{p_x}), \dots \}$ (alias 1-primeths [7])

${}^2\mathcal{P}^* = \{ {}^2P_1 (P_{p_{p_1}} = P_3 = 5), {}^2P_2 (P_{p_{p_2}} = P_5 = 11), \dots, {}^2P_x (= P_{p_{p_x}}), \dots \},$

...

${}^i\mathcal{P}^* = \left\{ {}^iP_1 = P \underbrace{\hspace{1.5cm}}_{\substack{P \dots p_1 \\ i \text{ iterations of } P}}, {}^iP_2 = P \underbrace{\hspace{1.5cm}}_{\substack{P \dots p_2 \\ i \text{ iterations of } P}}, \dots, {}^iP_x = P \underbrace{\hspace{1.5cm}}_{\substack{P \dots p_x \\ i \text{ iterations of } P}}, \dots \right\}, \text{ with}$

$x \in \mathbb{N}^* - \{1, 2\}$

[7] Primes subset (3, 5, 11, 17, 31, 41, 59, 67, 83, 109, 127, 157, ...), also known as sequence **A006450** in OEIS ([URL-OIES page](#))

Part D. The Meta-conjecture VBGC - The extension and generalization of BGC as applied on i-primeths ($^i\wp^*$)

Meta-conjecture VBGC – main co-statements:

$$1. \text{ Alternatively defining i-primeths as: } {}^0P_x = P\left(\frac{x}{[0 \text{ iterations of } P \text{ on } P]}\right), {}^1P_x = P\left(\frac{P(x)}{[1 \text{ iteration of } P \text{ on } P]}\right),$$

$${}^2P_x = P\left(\frac{P(P(x))}{[2 \text{ iterations of } P \text{ on } P]}\right) \dots {}^iP_x = P\left(\frac{P(P(\dots P(x)))}{[i(i \geq 0) \text{ iterations}]} \right), \text{ with } P(x) \text{ being the } x\text{-th prime in the}$$

set of standard primes (usually denoted as $P(x)$ or P_x and equivalent to ${}^0P_x$ alias “0-primeths”)

and the generic ${}^iP_x$ being named the generic set of i-primeths (with “i” being the “iterative”/recursive order of that i-primeth which measures the number of P-on-P iterations associated with that specific i-primeth subset).

2. The inductive variant of (the meta-conjecture) VBGC (iVBGC) proposed in this paper states that:

“All even positive integers $2m \geq 2 \cdot f_x(a, b)$ AND (also) $2m \geq 2 \cdot f_{x_2}(a, b)$, can be written

as the sum of at least one pair of DISTINCT odd i-primeths ${}^aP_x > {}^bP_y$, with the positive

integers pair (a, b) , with $a \geq b \geq 0$ defining the (recursive) orders of each of those i-primeths

pair AND the pair of distinct positive integers (x, y) , with $x > y > 1$ defining the indexes of each of those i-primeths pair, with

$$f_x(a, b) = \begin{cases} 2^{(a+1)(b+1)(a+b+2)} & \text{for } (a = b = 0) \\ 2^{[(a+1)(b+1)(a+b+3)/a] - a} & \text{for } (a = b) \text{ AND } (a > 0) \\ 2^{(a+1)(b+1)(a+b+2) - (a+b-2)} & \text{for } (a \neq b) \text{ AND } [(a > 0) \text{ OR } (b > 0)] \end{cases}$$

AND

$$f_{x_2}(a, b) = \begin{cases} 2^{(a+1)(b+1)(a+b+2)} & \text{for } (a = b = 0) \\ 2^{[(a+1)(b+1)(a+b+3)/a] - 2a} & \text{for } (a = b) \text{ AND } (a > 0) \\ 2^{(a+1)(b+1)(a+b+2) - (a+b-2)} & \text{for } (a \neq b) \text{ AND } [(a > 0) \text{ OR } (b > 0)] \end{cases}."$$

a. A secondary inductive (form of) (the meta-conjecture) VBGC (siVBGC[a,0]) proposed in

this paper states that: “All even positive integers $2m \geq 2 \cdot \text{int}[f_y(a)]$, with

$f_y(a) = e^{4a}$, can be written as the sum of at least one pair of DISTINCT odd i-

primeths ${}^aP_x > {}^0P_y$, with the positive integers pair $(a, 0)$, with $a > 0$ defining

the (recursive) orders of the i-primeths pair $({}^aP_x, {}^0P_y)$ AND the distinct positive

- integers pair (x, y) , with $x > y > 1$ defining the indexes of each of those i-primeths.”.
3. **The analytical variant of (the meta-conjecture) VBGC (aVBGC)** proposed in this paper (from which the previous inductive VBGC was derived) states that: “For any pair of finite positive integers (a, b) , with $a \geq b \geq 0$ defining the (recursive) orders of an a-primeth (^aP) and a b-primeth respectively (^bP) , there will always exist a single finite positive integer $(n_{a,b} = n_{b,a}) \geq 3$ so that, for any positive integer $m > n_{a,b}$ it will always exist at least one pair of finite *distinct* positive integers (x, y) , with $x > y > 1$ (indexes of distinct odd i-primeths) so that:
- $^aP_x + ^bP_y = 2m$ AND $^aP_x > ^bP_y$ AND the function
- $f(a, b) = f(b, a) = (n_{a,b} = n_{b,a}) \geq 3$ has a finite positive integer value for any combination of finite positive integers (a, b) , without any catastrophic-like infinities for any (a, b) pair of finites positive integers.”
- a. Important note. I have chosen the additional conditions $(a \geq b \geq 0) \wedge (x > y > 1) \Leftrightarrow ^aP_x > ^bP_y$ so that to lower the nof. lines per each Goldbach Matrix (GM) and to simplify the algorithm of searching $(^aP_x, ^bP_y)$ pairs, as the set ^aP is much less dense that the set ^bP for $a > b$ AND the sieve using ^aP (which searches an ^aP starting from $2m$ to 3) finds a $(^aP_x, ^bP_y)$ pair much more quicker than a sieve using ^bP (if $a > b$).
- b. $f(0, 0) = (n_{0,0}) = 3$
- c. $f(1, 0) = f(0, 1) = (n_{1,0} = n_{0,1}) = 3$; $f(1, 0) < f(2, 0) = 2564$;
- d. $f(2, 0) = f(0, 2) = (n_{2,0} = n_{0,2}) = 2\ 564$; $f(2, 0) < f(1, 1)$; the conjectured sequence of all even integers that cannot be expressed as the sum of two distinct 2-primeth and 0-primeth $^2P_x > ^0P_y$ was also submitted to OEIS, reviewed and approved as A282251^[8]
- e. $f(1, 1) = (n_{1,1}) = 40\ 306$; $f(1, 1) > f(2, 0) = 2564$, as also predicted by $fx(1, 1) > fx(2, 0)$;
- f. $f(3, 0) = f(0, 3) = (n_{3,0} = n_{0,3}) = 125\ 771$; $f(3, 0) > f(2, 0) = 2564$, as also predicted by $fx(3, 0) > fx(2, 0)$; $f(3, 0) < f(2, 1) = 1\ 765\ 126$, as also

[8] Official page at: <https://oeis.org/A282251>; Complete review: <https://oeis.org/draft/A282251>; Review history: <https://oeis.org/history?seq=A282251>

predicted by $\hat{f}(3,0) < \hat{f}(2,1)$; $f(3,0) > f(1,1) = 40\ 306$, as also predicted by $\hat{f}(3,0) > \hat{f}(1,1)$

g. $\mathbf{f}(2,1) = f(1,2) = (n_{2,1} = n_{1,2}) = \mathbf{1\ 765\ 126}$; $f(2,1) > f(3,0) = 125\ 771$, as also predicted by $\hat{f}(2,1) > \hat{f}(3,0)$; $f(2,1) < f(2,2) = 161\ 352\ 166$, as also predicted by $\hat{f}(2,1) < \hat{f}(2,2)$;

h. $\mathbf{f}(4,0) = f(0,4) = (n_{4,0} = n_{0,4}) = \mathbf{6\ 204\ 163}$; $f(4,0) > f(3,0) = 125\ 771$, as also predicted by $\hat{f}(4,0) > \hat{f}(3,0)$; $f(4,0) < f(2,2) = 161\ 352\ 166$, which is also predicted by $\hat{f}(4,0) < \hat{f}(2,2)$;

i. $\mathbf{f}(3,1) = f(1,3) = (n_{3,1} = n_{1,3}) = \mathbf{32\ 050\ 472}$;
 $f(3,1) > f(2,1) = 1\ 765\ 126$, as also predicted by $\hat{f}(3,1) > \hat{f}(2,1)$;
 $f(3,1) > f(4,0) = 6\ 204\ 163$ as also predicted by $\hat{f}(3,1) > \hat{f}(4,0)$;
 $\hat{f}(3,1) > \hat{f}(2,2)$ erroneously predicts that $f(3,1)$;
 $\hat{f}(3,1) \cong 7.04 \times 10^{13}$ overestimates the computed $f(3,1)$;

j. $\mathbf{f}(2,2) = (n_{2,2}) = \mathbf{161\ 352\ 166}$; $f(2,2) > f(2,1) = 1\ 765\ 126$, as also predicted by $\hat{f}(2,2) > \hat{f}(2,1)$; $f(2,2) > f(4,0) = 6\ 204\ 163$, as also predicted by $\hat{f}(2,2) > \hat{f}(4,0)$; $f(2,2) < f(5,0) = 260\ 535\ 479$, which is also predicted by $\hat{f}(2,2) < \hat{f}(5,0)$;

k. $\mathbf{f}(5,0) = f(0,5) = (n_{5,0} = n_{0,5}) = \mathbf{260\ 535\ 479}$; $f(5,0) > f(4,0) = 6\ 204\ 163$, as also predicted by $\hat{f}(5,0) > \hat{f}(4,0)$; however, $\hat{f}(5,0) \cong 5.5 \times 10^{11}$ overestimates $f(5,0)$ over $2m = 10^{10}$, as also in the case of $\hat{f}(3,1) \cong 7.04 \times 10^{13}$ overestimating $f(3,1)$;

l. $\mathbf{f}(4,1) = f(1,4) = (n_{4,1} = n_{1,4}) = ?$ (computing in progress); $\mathbf{f}(4,1)$ is expected to be smaller than $f(3,2)$ according to the prediction $\hat{f}(4,1) < \hat{f}(3,2)$;
obviously $\mathbf{f}(4,1)$ is expected to be larger than $f(3,1)$ as also according to the prediction $\hat{f}(4,1) > \hat{f}(3,1)$; $\mathbf{f}(4,1)$ is ALSO expected to be larger than $f(3,3)$ as according to the prediction $\hat{f}(4,1) > \hat{f}(3,3)$; however, $\hat{f}(4,1) \cong 1.5 \times 10^{20}$ surely overestimates $f(4,1)$;

m. $\mathbf{f}(3,2) = f(2,3) = (n_{3,2} = n_{2,3}) = ?$ (computing in progress);
 $\hat{f}(3,2) \cong 2.4 \times 10^{24}$ surely overestimates $f(3,2)$;

n. $\mathbf{f}(3,3) = (n_{3,3}) = ?$ (computing in progress); $\hat{f}(3,3) \cong 3.5 \times 10^{13}$ surely overestimates $f(3,3)$ over $2m = 10^{10}$;

- o. ...[working progress on other higher indexes function values]
- p. The 2D matrix/array of the finite values $f(a, b)$ is a conjectured meta-sequence of integers and was also proposed to OEIS, BUT rejected in the meantime, with the main argument that OEIS doesn't accept conjectured meta-sequences (the sequence of f values was considered "too ambitious") and that it hadn't an "appropriate form", although OEIS doesn't mention this (main) exclusion-criterion (applied to VBGC $f[a, b]$ meta-sequence) explicitly in their publishing policy^[9].

4. Interestingly, $f(a, b)$ applied on $a \in [0, 5]$ and $b \in [0, 5]$ has its values in the set

$$\mathbf{F} = \left\{ (3), (2564), (40\ 306), (125\ 771), (1\ 765\ 126), \right. \\ \left. (6\ 204\ 163), (32\ 050\ 472), (161\ 352\ 166), (260\ 535\ 479) \right\} \text{ which has an}$$

exponential (relatively) compact pattern such as:

$$\mathbf{E}_F = \left\{ (1.1), (7.8), (10.6), (11.7), (14.4), (15.6), (17.3), (18.9), (19.4) \right\}, \text{ with a}$$

relatively constant geometric progression (of about 1.2 ± 0.15) between its last 7 elements so that

$$\left[\left(\frac{19.4}{18.9} \right) \cong \left(\frac{18.9}{17.3} \right) \cong \left(\frac{17.3}{15.6} \right) \cong \left(\frac{15.6}{14.4} \right) \cong \right. \\ \left. \left(\frac{14.4}{11.7} \right) \cong \left(\frac{11.7}{10.6} \right) \cong \left(\frac{10.6}{7.8} \right) \right] \cong (1.2 \pm 0.15)$$

. The single exception of this rule is the gap between the exponents $\cong 1.1$ and $\cong 7.8$. See the next figures.

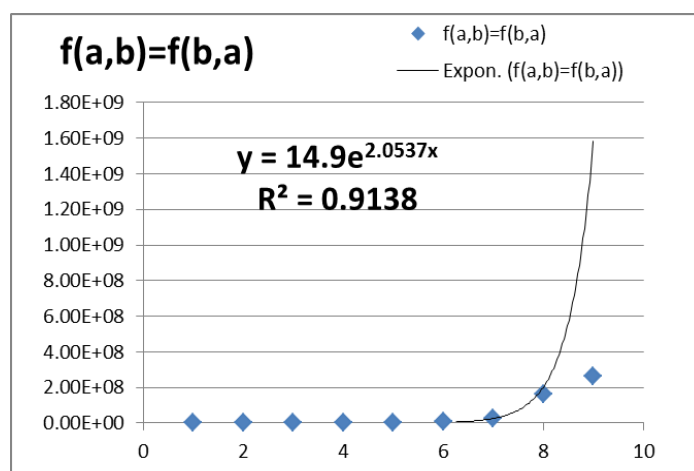


Figure D-1. The exponential pattern-1 of the $f(a, b)$ values for $a \in [0, 5]$ and $b \in [0, 5]$

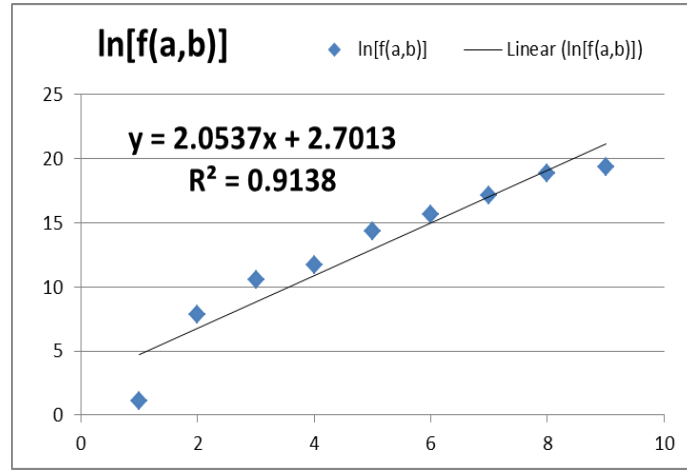


Figure D-2. The exponential pattern-2 of the $f(a,b)$ values for $a \in [0,5]$ and $b \in [0,5]$

a.
$$\mathbf{F} = \left\{ \begin{array}{l} (3), (2\ 564), (40\ 306), (125\ 771), (1\ 765\ 126), \\ (6\ 204\ 163), (32\ 050\ 472), (161\ 352\ 166), (260\ 535\ 479) \end{array} \right\}$$
 has ALSO

a correspondent matrix in which a is a column index and b is a line index

$$M_{f(a,b)} = \begin{bmatrix} (n_{0,0}) & (n_{1,0} = n_{0,1}) & (n_{2,0} = n_{0,2}) & (n_{3,0} = n_{0,3}) & (n_{4,0} = n_{0,4}) & (n_{5,0} = n_{0,5}) \\ (n_{0,1} = n_{1,0}) & (n_{1,1}) & (n_{2,1} = n_{1,2}) & (n_{3,1} = n_{1,3}) & (n_{4,1} = n_{1,4}) & (n_{5,1} = n_{1,5}) \\ (n_{0,2} = n_{2,0}) & (n_{1,2} = n_{2,1}) & (n_{2,2}) & (n_{3,2} = n_{2,3}) & (n_{4,2} = n_{2,4}) & (n_{5,2} = n_{2,5}) \\ (n_{0,3} = n_{3,0}) & (n_{1,3} = n_{3,1}) & (n_{2,3} = n_{3,2}) & (n_{3,3}) & (n_{4,3} = n_{3,4}) & (n_{5,3} = n_{3,5}) \\ (n_{0,4} = n_{4,0}) & (n_{1,4} = n_{4,1}) & (n_{2,4} = n_{4,2}) & (n_{3,4} = n_{4,3}) & (n_{4,4}) & (n_{5,4} = n_{4,5}) \\ (n_{0,5} = n_{5,0}) & (n_{1,5} = n_{5,1}) & (n_{2,5} = n_{5,2}) & (n_{3,5} = n_{5,3}) & (n_{4,5} = n_{5,4}) & (n_{5,5}) \end{bmatrix}$$

$$= \begin{bmatrix} (3) & (3) & (2564) & (125\ 771) & (6\ 204\ 163) & (260\ 535\ 479) \\ (3) & (40\ 306) & (1\ 765\ 126) & (32\ 050\ 472) & (?) & (?) \\ (2564) & (1\ 765\ 126) & (161\ 352\ 166) & (?) & (?) & (?) \\ (125\ 771) & (32\ 050\ 472) & (?) & (?) & (?) & (?) \\ (6\ 204\ 163) & (?) & (?) & (?) & (?) & (?) \\ (260\ 535\ 479) & (?) & (?) & (?) & (?) & (?) \end{bmatrix}$$

and a matrix of exponents in which a is also a column index and b is also a line index

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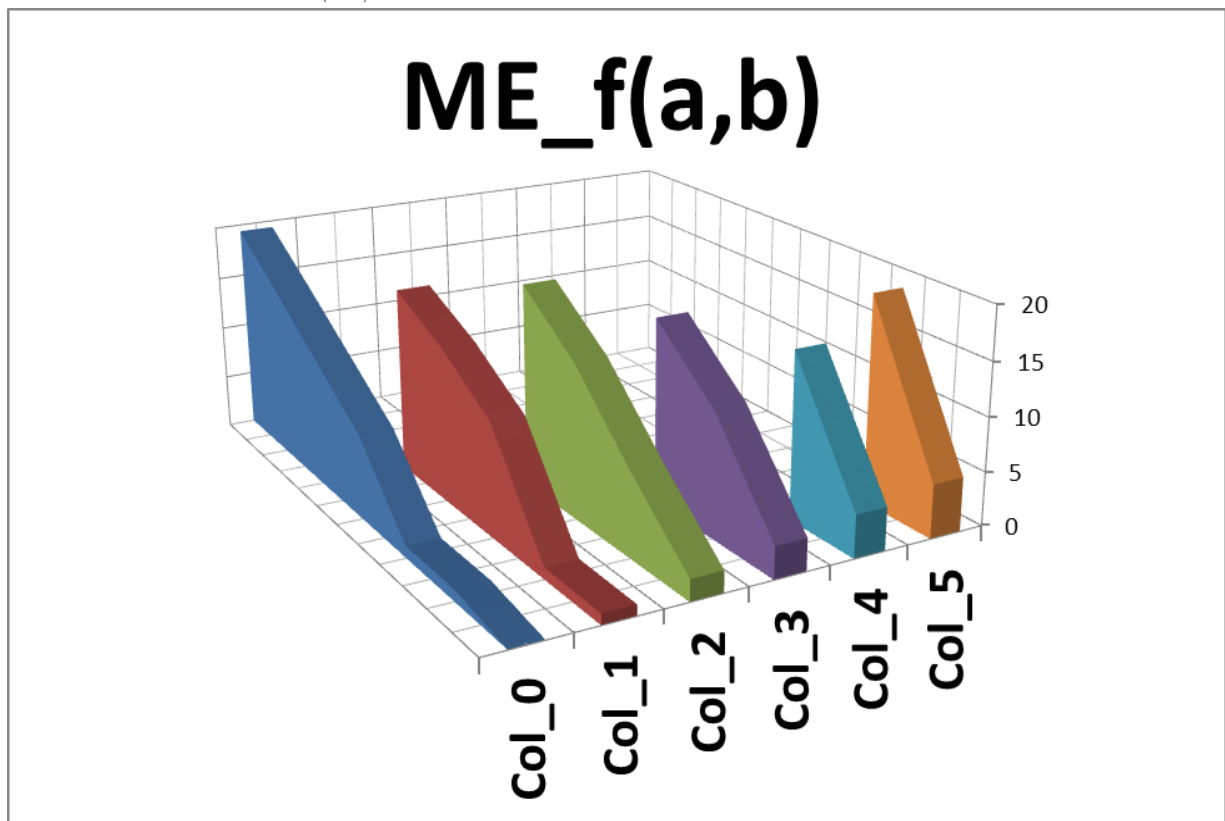
$$ME_{f(a,b)} = LN \begin{bmatrix} (n_{0,0}) & (n_{1,0} = n_{0,1}) & (n_{2,0} = n_{0,2}) & (n_{3,0} = n_{0,3}) & (n_{4,0} = n_{0,4}) & (n_{5,0} = n_{0,5}) \\ (n_{0,1} = n_{1,0}) & (n_{1,1}) & (n_{2,1} = n_{1,2}) & (n_{3,1} = n_{1,3}) & (n_{4,1} = n_{1,4}) & (n_{5,1} = n_{1,5}) \\ (n_{0,2} = n_{2,0}) & (n_{1,2} = n_{2,1}) & (n_{2,2}) & (n_{3,2} = n_{2,3}) & (n_{4,2} = n_{2,4}) & (n_{5,2} = n_{2,5}) \\ (n_{0,3} = n_{3,0}) & (n_{1,3} = n_{3,1}) & (n_{2,3} = n_{3,2}) & (n_{3,3}) & (n_{4,3} = n_{3,4}) & (n_{5,3} = n_{3,5}) \\ (n_{0,4} = n_{4,0}) & (n_{1,4} = n_{4,1}) & (n_{2,4} = n_{4,2}) & (n_{3,4} = n_{4,3}) & (n_{4,4}) & (n_{5,4} = n_{4,5}) \\ (n_{0,5} = n_{5,0}) & (n_{1,5} = n_{5,1}) & (n_{2,5} = n_{5,2}) & (n_{3,5} = n_{5,3}) & (n_{4,5} = n_{5,4}) & (n_{5,5}) \end{bmatrix},$$

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$$ME_{f(a,b)} \cong \begin{bmatrix} (1.1) & (1.1) & (7.85) & (11.74) & (15.64) & (19.38) \\ (1.1) & (10.6) & (14.38) & (17.28) & (?) & (?) \\ (7.85) & (14.38) & (18.9) & (?) & (?) & (?) \\ (11.74) & (17.28) & (?) & (?) & (?) & (?) \\ (15.64) & (?) & (?) & (?) & (?) & (?) \\ (19.38) & (?) & (?) & (?) & (?) & (?) \end{bmatrix}.$$

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b. $ME_{f(a,b)}$ can be graphed as a 2D surface in a 3D space: **see the next figure.**



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Figure D-3. The 3D graph of $ME_{f(a,b)}$ cell values for $a \in [0,5]$ and $b \in [0,5]$

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c. The previous matrices generate half-dome-like graphs, apparently with no closed “depression” regions, as all elements tend to become greater when: moving on the lines from left to right, moving on the columns from up to down, moving on the diagonals, from sides to the center. The exponents from each column of $ME_{f(a,b)}$ tend to grow

almost linearly from up to down (but also on diagonals, from left to center-right and vice versa): see the next figure.

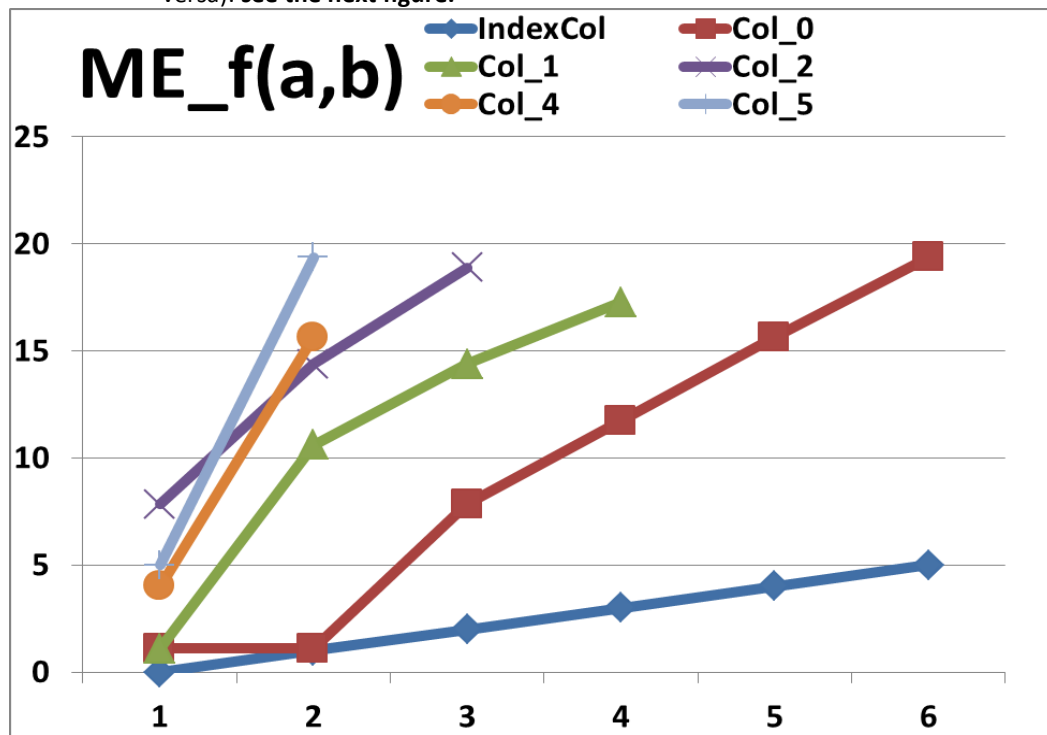


Figure D-4. The almost linear growth of $ME_{f(a,b)}$ cell values on lines, columns and diagonals for $a \in [0,5]$ and $b \in [0,5]$

d. Furthermore, the symmetrical function $fx(a,b) = fx(b,a)$ (proposed in the inductive variant of VBGC) generates positive integers that are relatively close BUT strictly larger than the values of $f(a,b)$ for $a \in [0,5]$ and $b \in [0,5]$, with also a half-dome-like graph.

e. The function $fx(a,b)$ has its values in the matrix

$$M_{fx(a,b)} \equiv \begin{bmatrix} (4) & (128) & (4\ 096) & (5.2 \times 10^5) & (2.7 \times 10^8) & (5.5 \times 10^{11}) \\ (128) & (5.2 \times 10^5) & (5.4 \times 10^8) & (7 \times 10^{13}) & (...) & (...) \\ (4\ 096) & (5.4 \times 10^8) & (7.6 \times 10^8) & (...) & (...) & (...) \\ (5.2 \times 10^5) & (7 \times 10^{13}) & (...) & (...) & (...) & (...) \\ (2.7 \times 10^8) & (...) & (...) & (...) & (...) & (...) \\ (5.5 \times 10^{11}) & (...) & (...) & (...) & (...) & (...) \end{bmatrix} \text{ in}$$

which each element is strictly larger than its correspondent element from $M_{f(a,b)}$

$$M_{f(a,b)} = \begin{bmatrix} (3) & (3) & (2564) & (125\ 771) & (6\ 204\ 163) & (260\ 535\ 479) \\ (3) & (40\ 306) & (1\ 765\ 126) & (32\ 050\ 272) & (?) & (?) \\ (2564) & (1\ 765\ 126) & (161\ 352\ 166) & (?) & (?) & (?) \\ (125\ 771) & (32\ 050\ 272) & (?) & (?) & (?) & (?) \\ (6\ 204\ 163) & (?) & (?) & (?) & (?) & (?) \\ (260\ 535\ 479) & (?) & (?) & (?) & (?) & (?) \end{bmatrix}$$

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f. Additionally, the function

$$fx_2(a,b) = \begin{cases} 2^{(a+1)(b+1)(a+b+2)} & \text{for } (a=b=0) \\ 2^{[(a+1)(b+1)(a+b+3)/a]-2a} & \text{for } (a=b) \text{ AND } (a>0) \\ 2^{(a+1)(b+1)(a+b+2)-(a+b-2)} & \text{for } (a \neq b) \text{ AND } [(a>0) \text{ OR } (b>0)] \end{cases}$$

has strictly

562 larger but even more closer values to the values of $M_{f(a,b)}$, BUT predicts distorted inequalities between

563 some terms (when compared to the inequalities between some elements of $M_{f(a,b)}$).

$$M_{fx_2(a,b)} \cong \begin{bmatrix} (4) & (128) & (4\ 096) & (5.2 \times 10^5) & (2.7 \times 10^8) & (5.5 \times 10^{11}) \\ (128) & (2.6 \times 10^5) & (5.4 \times 10^8) & (7 \times 10^{13}) & (...) & (...) \\ (4\ 096) & (5.4 \times 10^8) & (1.8 \times 10^8) & (...) & (...) & (...) \\ (5.2 \times 10^5) & (7 \times 10^{13}) & (...) & (...) & (...) & (...) \\ (2.7 \times 10^8) & (...) & (...) & (...) & (...) & (...) \\ (5.5 \times 10^{11}) & (...) & (...) & (...) & (...) & (...) \end{bmatrix}$$

565 g. The first line of $ME_{f(a,b)}$ (which is identical to its first column), has sufficiently many

566 terms to create a function that reasonably approximates the elements on this first

567 line/column, such as: $fey(a) = 4a = [0\ 4\ 8\ 12\ 16\ 20]$, with

$$fy(a) = e^{fey(a)} = e^{4a} \cong [1\ 54.6\ 2981\ 162754.8\ 8.8 \times 10^6\ 485 \times 10^6]$$

569 , which is very close (with larger values) to the first of $ME_{f(a,b)}$

570 $[(1.1)\ (1.1)\ (7.85)\ (11.74)\ (15.64)\ (19.38)]$ and the first line of

571 $M_{f(a,b)} [(3)\ (3)\ (2564)\ (125\ 771)\ (6\ 204\ 163)\ (260\ 535\ 479)]$

572 respectively.

573 i. $fy(6)$ predicts a value for $f(6,0) \cong fy(6) \cong 2.65 \times 10^{10}$ which is
574 beyond the verification capabilities of our current software: this hypothesis was
575 also verified with our software and confirmed that $f(6,0)$ is larger than the
576 limit $2m = 10^{10}$. The exception of VBGC(6,0) smaller-and-closest to
577 $2m = 10^{10}$ is **9 997 202 434 = 2 × 4 998 601 217**

ii. $fy(7)$ predicts a value for $f(7,0) \cong fy(7) \cong 1.45 \times 10^{12}$ which is far beyond the verification capabilities of our current software.

iii. On the second line/column of $ME_{f(a,b)}$

$[(1.1) (10.6) (14.38) (?) (?) (?)]$, the elements may also grow in a

arithmetical progression with an (exponential) step $\boxed{s \cong 4}$ (starting from

$Ef(1,1) \cong 10.6$ to $Ef(2,1) \cong 14.38 \cong 10.6 + s$), with the exception of a

first gap between $Ef(0,1) \cong 1.1$ and $Ef(1,1) \cong 10.6$, which is

correspondent to the gap between $Ef(1,0) \cong 1.1$ and $Ef(2,0) \cong 7.85$. As

observed, the step $\boxed{s \cong 4}$ is conserved on all lines, columns and secondary

diagonals, so that the main diagonal probably has a step of $\boxed{2s \cong 8}$.

1. The 5th unknown element $Ef(4,1) = (?)$ from the 2nd line may have

a value of $Ef(4,1) \stackrel{?}{\cong} [ef(2,1) + 2s = 14.38 + 8 \cong 22.38]$ as

predicted by the same step $\boxed{s \cong 4}$. An $Ef(4,1) \stackrel{?}{\cong} 22.38$

corresponds to a hypothetical

$$f(4,1) \stackrel{?}{=} [e^{Ef(4,1)} \cong e^{22.38} \cong 5\,242\,162\,809 \cong 5.2 \times 10^9]$$

which is ALSO under the limit $2m = 10^{10}$ and may also be (relatively)

verified with our software. However, as $f(4,1) \stackrel{?}{\cong} 5.2 \times 10^9$ is

probably very close to the limit $2m = 10^{10}$, the conjecture VBGC[4,1] may not be testified by a "sufficiently" large gap)

2. Other values of $f(a,b)$ which are predicted to be under the

limit $2m = 10^{10}$ are: $[f(1,4) = f(4,1)] \stackrel{?}{\cong} 5.2 \times 10^9$,

$[f(3,2) = f(2,3)] \stackrel{?}{\cong} (5.2 \text{ to } 8.8) \times 10^9$: all the predicted values

are very close to the limit $2m = 10^{10}$, so that the conjectures VBGC[a,b] (corresponding to those predicted values) may not be testified by a "sufficiently" large gap. **See the next table.**

605

Table T-1. The verified values of $f(a, b)$, with $a \geq b \geq 0$ (written as exact positive integers: the shaded cells of the table) and the estimated maximum values of $f(a, b)$ using the step $s \cong 4$ "rule" (written in exponential format)

$f(a, b)$	0	1	2	3	4	5	6	7
0	3	3	2,564	125,771	6,204,163	260,535,479	1.4E+10	7.8E+11
1	3	40,306	1,765,126	32,050,472	5.2E+09	2.9E+11	1.6E+13	8.5E+14
2	2,564	1,765,126	161,352,166	5.2E+09	2.9E+11	1.6E+13	8.5E+14	4.7E+16
3	125,771	32,050,472	8.8E+09	2.9E+11	1.6E+13	8.5E+14	4.7E+16	2.5E+18
4	6,204,163	5.2E+09	4.8E+11	1.6E+13	8.5E+14	4.7E+16	2.5E+18	1.4E+20
5	260,535,479	2.9E+11	2.6E+13	8.5E+14	4.7E+16	2.5E+18	1.4E+20	7.6E+21
6	1.4E+10	1.6E+13	1.4E+15	4.7E+16	2.5E+18	1.4E+20	7.6E+21	4.1E+23
7	7.8E+11	8.5E+14	7.8E+16	2.5E+18	1.4E+20	7.6E+21	4.1E+23	2.3E+25

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h. $f_y(a) = e^{fey(a)} = e^{4a}$ predicts so accurately the first line of $M_{f(a,b)}$, so that this

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paper also proposes a **secondary inductive (form of) VBGC (siVBGC[a,0])** which states that:

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"Any/every even positive integer $2m \geq 2 \cdot \text{int}[f_y(a)]$, with $f_y(a) = e^{4a}$, can

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be written as the sum of at least one pair of DISTINCT odd i-primeths ${}^aP_x > {}^0P_y$,

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with the positive integers pair $(a, 0)$, with $a > 0$ defining the (recursive) orders of

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the i-primeths pair $({}^aP_x, {}^0P_y)$ AND the distinct positive integers pair

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(x, y) , with $x > y > 1$ defining the indexes of each of those i-primeths."

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i. The set of conjectures siVBGC(a,0) can be used to verify much more rapidly (cost/time-efficiently) ntBGC, by searching using *only* the subsets ${}^aP$, starting

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from ${}^aP_x$ which is closest to $2m \geq 2 \cdot \text{int}(e^{4a})$ down to ${}^aP_2$ and testing

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the primality of $(2m - {}^aP_x)$

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i. Interestingly, the differences between consecutive elements on any line or column of

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$$ME_{f(a,b)} \cong \begin{bmatrix} (1.1) & (1.1) & (7.85) & (11.74) & (15.64) & (19.38) \\ (1.1) & (10.6) & (14.38) & (17.28) & (?) & (?) \\ (7.85) & (14.38) & (18.9) & (?) & (?) & (?) \\ (11.74) & (17.28) & (?) & (?) & (?) & (?) \\ (15.64) & (?) & (?) & (?) & (?) & (?) \\ (19.38) & (?) & (?) & (?) & (?) & (?) \end{bmatrix}$$

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have a 1st or a 2nd value that is slightly above $s \cong 4$, with all the other values (the 2nd

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/3rd, the 4th etc) being smaller or approximately equal to $s \cong 4$: see the next tables

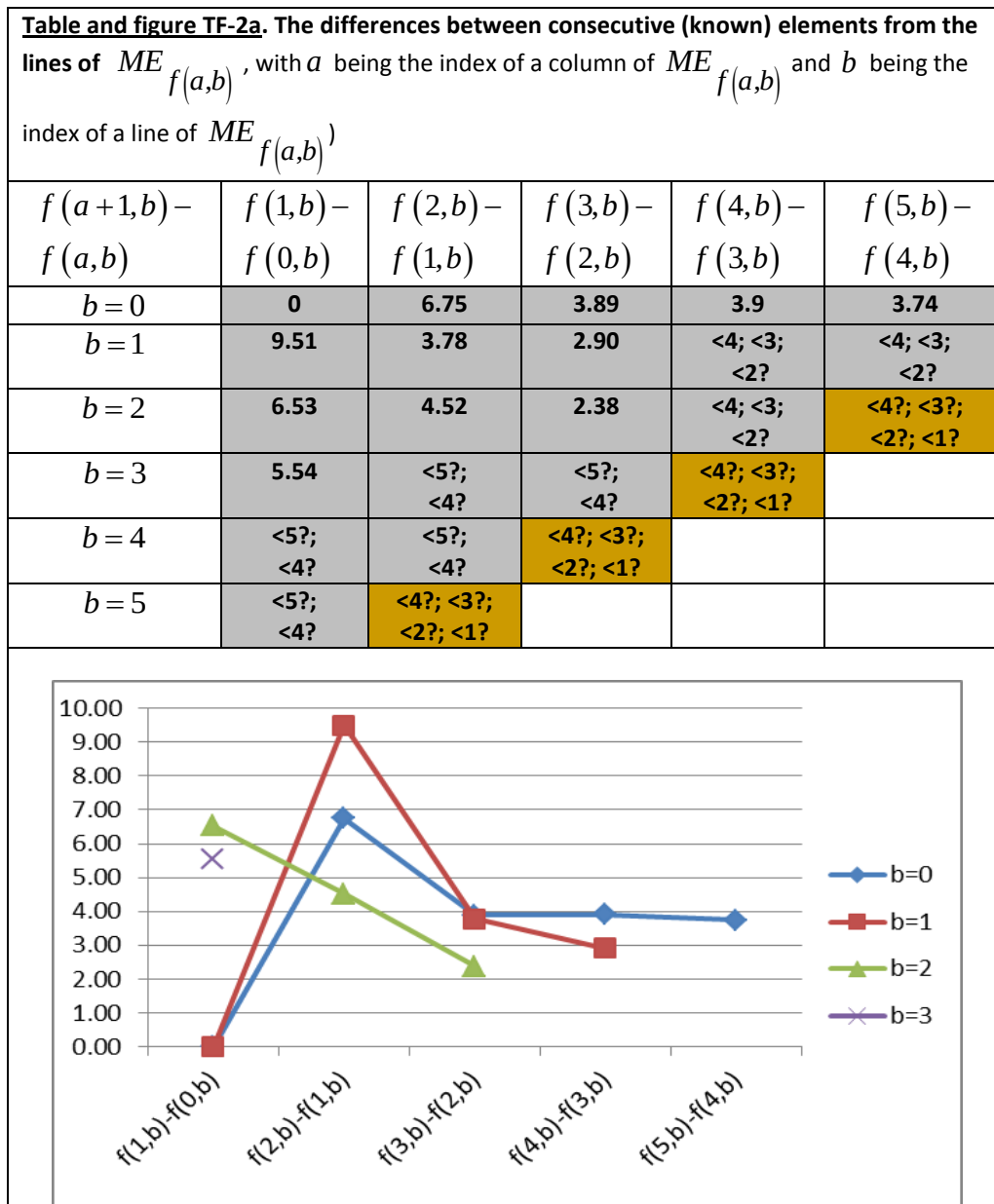
623

and graphs.

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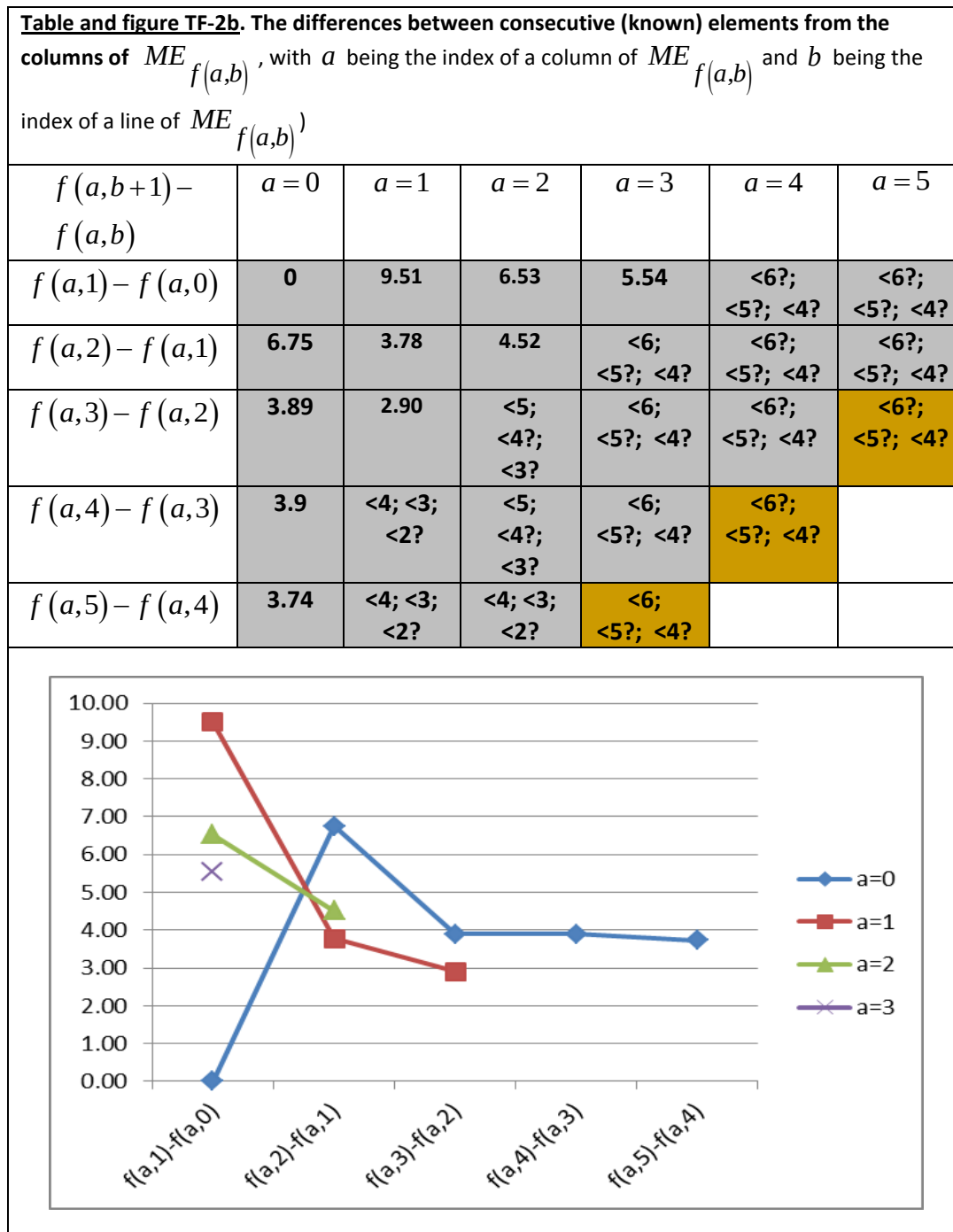
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Conclusions on VBGC

1. Essentially, VBGC(a,b) is an extension and generalization of BGC as applied on the generalized concept of all subsets of super-primes of any iteration order i, generically named "i-primeths" in this paper.
 - a. VBGC has an inductive variant and an analytical variant, which both apply to any i-primeth subset.
 - b. Obviously, VBGC contains the attribute "vertical" in its name motivated by the fact that VBGC is a "vertical" (recursive) generalization of the ntBGC on the infinite superset of i-primeths.
2. VBGC can be considered a "meta-conjecture", as it essentially (and alternatively) states/predicts an infinite number of BGC-like conjectures ("stronger" than BGC) which are generically indexed as VBGC(a,b), each associated with a pair $(a,b) \in N$ and a finite positive integer $n_{(a,b)} \in N$
 - a. **VBGC(0,0)** is equivalent to the non-trivial variant of BGC (ntBGC), as defined in the Part B of this article.
 - b. **VBGC(1,0)** is a Goldbach-like Conjecture (**GLC**) stronger and more elegant than ntBGC, as it acts on a limit $2f(1,0) = 6$ identical to ntBGC inferior limit (which is $2f(0,0) = 6$) BUT the associated $G_{1,0}(m)$ (which counts the number of pairs of possible GIPs for any even integer $m > 3$) has significantly smaller values than the function $G_{0,0}(m)$ of ntBGC [which is VBGC(0,0)]
 - c. **VBGC(2,0)** is obviously stronger than VBGC(1,0) with $G_{2,0}(m)$ having smaller non-zero values than $G_{1,1}(m)$ for $m \in (f(2,0), \infty)$
 - d. **VBGC(1,1)** (anticipated by my discovery of VBGC(1,0) from 2007 and officially registered in 2012 at OSIM) is obviously stronger than VBGC(1,0) and is equivalent to Bayless-Klyve-Oliveira e Silva Goldbach-like Conjecture (BKOS-GLC) published in Oct. 2013 [49] alias "Conjecture 9.1" (rephrased) (tested by these authors up to $2m = 10^9$): "all even integers $2m > [2 \cdot 40306 (= 2f(1,1))]$ can be expressed as the sum of at least one pair of prime-indexed primes [PIPs] (1-primeths ^1P_x and ^1P_y)". This article of Bayless. Klyve and Oliveira (2012, 2013) was based on a previous article by Barnett and Broughan (published in 2009) [50], but BKOS-GLC was an additional result to this 2009 article. Our new software retested and reconfirmed VBGC(1,1) up to $2m = 10^{10}$ and also helped verifying VBGC(a,b) for much more (a,b) positive integer pairs [10].
3. VBGC is "much stronger" and general than BGC and proposes a much more rapid and efficient (at-least-one Goldbach index partition [GIP])-sieve than the Goldbach-Knjzek-Rivera conjecture (**GKRC**).
 - a. The Goldbach Matrices (**GMs**) (containing all possible GIPs) generated by VBGC has a smaller nof. lines than the GMs of GIPs generated by GKRC. VBGC is a useful optimized sieve to push forward the limit $4 \cdot 10^{18}$ to which BGC was verified to hold [51,52].
 - b. All VBGC(a>0,b≥0) (including siVBGC) can be used to produce more rapid algorithms for the experimental verification of ntBGC for very large positive integers. A first experiment would be to re-test BGC up to that limit $2m = 4 \times 10^{18}$ alternatively using siVBGC and to compare the global times of computing. When verifying ntBGC for a very large number N , one can use the aVBGC(a,b) or siVBGC(a,b) with a minimal positive value for the difference

[10] The code-source (written by Mr. George Anescu in Microsoft Studio 2015 - Visual C++ language/environment using parallel processing) that was used to test BKOS-GLC and VBGC up to $n=10^{10}$ (on a laptop PC with an Intel^R CoreTM processor i7-3630 QM CPU at 2.4 GHz with 4 processors (8 hyper-threads), can be found at this [URL](#) (the old variant can be found at this [URL-old](#))

$\lceil N - f(a, b) \rceil$. For example, in the case of VBGC(1,0), the average number of attempts to find the first pair (x,y) for each integer m, in the interval [3,2m] tends asymptotically to $\ln(\sqrt{n}) = \ln(n)/2$ when searching just the 1-primeths subset in descending manner, starting from the largest 1-primeth ${}^1P_x \leq 2m - 1$ and verifying if $(2m - {}^1P_x)$ is a 0-primeth)

4. When $a \rightarrow \infty, b \rightarrow \infty$ and $m \rightarrow \infty$, $G_{a,b}(f(a, b) + 1) \rightarrow 1$ and the “comets” of VBGC(a,b) tend to narrow progressively for each pair of positive integers (a_2, b_2) , with $a_2 > a_1$ and $b_2 > b_1$.

5. VBGC is a potential important (unified) conjecture of primes and a very special self-similar property of the primes as the rarefied set ${}^i\wp^*$ is self-similar to the more dense set ${}^{(i-1)}\wp^*$ in respect to the ntBGC. In other words, each of the i-primeths sets behaves as a “summary of” the 0-primeths set in respect to the ntBGC: this is a (quasi)fractal-like BGC-related behavior of the infinite number of the i-primeths sets.

a. Essentially, VBGC conjectures that ntBGC is a common property of all i-primeths sets (for any positive integer order i), differing just by the inferior limit $f(a, b) = n_{(a,b)}$ of each

VBGC(a,b). The set of values of $f(a, b)$ is a set of critical density thresholds/points of each i-primeths set in respect to the superset of VBGC(a,b) conjectures. All Goldbach comets associated with all known VBGC(a,b) are self-similar to each other and to the entire superset of comets.

b. Batchko R.G. has also reported other quasi-fractal/quasi-self-similar structure in the distribution of the prime-indexed primes [53]: Batchko also used a similar general definition for primes with (recursive) prime indexes (PIPs), briefly named in my article as “i-primeths”.
c. Carlo Cattani and Armando Ciancio also reported a quasi-fractal distribution of primes (including i-primeths) similar to a Cantor set (Cantor dust) by mapping primes and i-primeths into a binary image which visualizes the distribution of i-primeths [54]. VBGC may be an intrinsic property of all sets of i-primeths that can also explain OR be explained by this Cantor dust-like distribution of these i-primeths sets.

d. Obviously, all sets ${}^{(i>0)}\wp^*$ are subsets of ${}^0\wp^* = \wp^*$ and come in an infinite number: this family of subsets is governed/defined by the Prime Number Theorem. There is a potential infinite number of rules/criterions/theorems to extract an infinite number of subsets from $\wp^*$ (grouped in a family of subsets defined by that specific rule/criterion/theorem: like the Dirichlet's theorem on arithmetic progressions for example). It would be an interesting research subfield of BGC to test what are those families (of subsets of primes) that respect ntBGC and generate functions with finite values similar to $f(a, b) = n_{a,b}$. This potential future research subfield may also help in optimizing the algorithms used in the present for ntBGC verification on large numbers.

e. It is an interesting fact per se that all ${}^{(i>0)}\wp^*$ subsets have very low densities (when compared to ${}^0\wp^* = \wp^*$ and N^*), but these low densities are sufficiently... large to allow the existence of a function $f(a, b)$ with finite values for any pair of finites (a, b) .

f. A real challenge in the future (concerning VBGC) is to calculate the values of the function $f(a, b) = n_{a,b}$ and test/verify VBGC(a,b) for large positive integers pairs (a,b), including the pairs (a, b) with relatively large $(a - b)$ differences.

Potential applications of VBGC:

1. As the (weak) Ternary Goldbach Conjecture (**TGC**) is considered a consequence of BGC, VBGC can be used as a model to also formulate a Vertical (generalization) of the Ternary Goldbach Conjecture (**VTGC**) as an analogous consequence of VBGC, with a corresponding (potential infinite) meta-sequence of conjectures VTGC(a,b,c) with an associated function $f(a,b,c) = n_{a,b,c}$.
2. VBGC can be used to optimize the algorithms of finding very large new primes (i-primeths) smaller but closest possible to a chosen (very large) even number $q = 2m$:
 - a. **Step 1.** One may choose a set ${}^a\mathcal{P}^*$ and a conjecture VBGC(a,b) with positive integer order b chosen so that the known $n_{(a,b)}$ to be smaller but closest possible to $q = 2m$.
 - b. **Step 2.** One may then test only the primality of the differences $d_x = q - ({}^aP_x)$ (starting from $x = 1$ to larger positive integer indexes, in ascendant order) which have the potential to be i-primeths of type ${}^bP_y$.
3. VBGC can offer a rule of symmetric/asymmetric decomposition of Euclidean/non-Euclidean finite/infinite spaces with a finite (positive integer) number of dimensions $d = 2n$ into pairs of spaces, both with a (positive) i-primeth number of dimensions. According to VBGC, an finite regular Euclidian/non-Euclidean $2n$ -space with volume V_{2n} (with $n > 2$) can always be decomposed to permit symmetry/asymmetry such as:

$$V_{2n} = k \cdot \left(V_{({}^aP_x)} \times V_{({}^bP_y)} \right) = k \cdot \left(r^{({}^aP_x)} \times r^{({}^bP_y)} \right), \text{ with } k = \text{space volume specific constant}$$

 - a. In this way, VBGC can also be used in M-Theory to simulate decompositions of $2N$ -branes (with finite [positive] integer number of dimensions $d = 2n$) into pair of ${}^aP_x$ -brane and ${}^bP_y$ -brane, both branes with a (positive) i-primeth number of dimensions.
4. This type of vertical generalization (generating a meta-conjecture) may be the start of a new research sub-field in which other conjectures may be hypothesized to also have vertical generalizations applied on i-primeths. For example, a hypothetical vertical Polignac's conjecture (a "minus" version of BGC: "for any positive even number n , there are infinitely many prime gaps of size n " or "there are infinitely many cases of two consecutive prime numbers with difference n ") may speed up the searching algorithms to find very large primes (smaller but closest to a chosen positive integer superior limit m).

Acknowledgements

The computing time for determining and verifying $f(2,1) = 1\,765\,126$ and $f(2,2) = 161\,352\,166$ was about 30 hours in total. The computing time for determining and verifying $f(3,0) = 125\,771$, $f(4,0) = 6\,204\,163$ and $f(5,0) = 260\,535\,479$ was also about 30 hours for each value. The computing time for determining and verifying $f(3,1) = 32\,050\,472$ was a couple of days: no exceptions found between $2 \cdot f(3,1)$ and $2m = 10^{10}$ so that $f(3,1)$ may be a veritable last exception of VBGC[3,1].

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