

ON THE STABILITY ANALYSIS OF LINEAR SYSTEMS WITH CONTROL PARAMETERS

ABSTRACT:

In this work, we study the MDGKN system with control parameters. The relationship between the parameters determines the stability or otherwise of the system. The Lyapunov direct method is used to analyse and establish the stability status of the MDGKN system. Stability theorem is formulated by means of the largest and smallest eigenvalues of the combinations of the system matrices. Since the domain is R^n , the conditions are illustrated on a 2x2 and a 3x3 matrix systems.

1.0 INTRODUCTION

Linear systems involving Mass, Damping Force, Gyroscopic Force, Stiffness and Circulatory Force describe the damped gyroscopic system with circulatory effect known as the MDGKN systems. These systems arise in the modeling of mechanical systems with follower forces, in formulation of rotor systems with internal damping and with sliding bearings, in turbines with unsymmetrical steam flow, in articulated pipes, etc. For almost a century it has been well known that circulatory forces can cause instabilities. In this work, we study the stability properties of the MDGKN systems. Consider the non-conservative linear system of the form

$$M\ddot{x} + (\delta D + \mu G)\dot{x} + (K + \nu N)x = f(t) \quad (1)$$

where the dot denotes the time derivative; $x \in R^n$; and

$D = D^* > 0, G = -G^* > 0$ and $N = -N^*$ are real matrices corresponding to dissipative, gyroscopic and circulatory forces whose magnitude are controlled by the parameters δ, μ and ν ,

25 respectively while $f(t)$ describes excitation. The relationship between the control parameters
 26 determines the stability or instability status of system (1). We now examine the following cases:

27 **Case 1:** $\delta \sim \nu \ll \mu$

28 The most interesting in practice is the situation when these forces in the system are small as
 29 compared with the gyroscopic force. The critical gyroscopic parameter μ on the boundary of the
 30 gyroscopic stabilization domain of the non-conservative system is a function of the parameters
 31 corresponding to the dissipative and circulatory forces. Moreover, stability is extremely sensitive
 32 to the choice of a perturbation while the balance of forces leading to the asymptotic stability is
 33 not obvious.

34 **Case 2:** $\mu \ll \delta \sim \nu$

35 This case with a tendency for high perturbation arising from very small gyroscopic effect may be
 36 unstable and will not be considered since the bounds of solutions are not obtained for unstable
 37 systems.

38 In the following we assume the situation of case 1 where the stability is ensured and for
 39 simplicity we omit the parameters and proceed with the analysis. Examples are given to
 40 demonstrate the results obtained.

41 **2.0 METHODOLOGY**

42 If excitation in eqn (1) is negligible *i.e.* $f(t) = 0$, we obtain the following homogeneous linear
 43 system

44
$$M\ddot{x} + (D + G)\dot{x} + (K + N)x = 0 \quad (2)$$

45 The stability or otherwise of system (2) can be determined by eigenvalue method and also by
 46 Routh-Hurwitz method. In this work, we shall use Lyapunov direct method to analyse the
 47 stability or otherwise of the system. The advantage of this method over eigenvalue method is that
 48 the stability status of the system can be determined easily in cases where the eigenvalues cannot
 49 be found easily. Applying the direct method of Lyapunov, we have the following: The system (2)
 50 is equivalent to the system

$$\begin{aligned} 51 \quad & \dot{x}_1 = x_2 \\ 52 \quad & \dot{x}_2 = -M^{-1}(D + G)x_2 - M^{-1}(K + N)x_1 \end{aligned} \quad (3)$$

53 Putting (3) in the equivalent form of a first order system we have the following

$$54 \quad \dot{z} = Az \quad (4)$$

55 where $A = \begin{bmatrix} O & I \\ -M^{-1}(K + N) & -M^{-1}(D + G) \end{bmatrix}$ (I is the identity matrix and O is the zero matrix)

56 We define a function $V(z(t))$ is called a Lyapunov function for system (4) if $V > 0$ and the time
 57 derivative $\dot{V} \leq 0$ for all solutions $z(t)$ of (4). The existence of such a Lyapunov function implies
 58 stability of the system (asymptotic stability if $\dot{V} < 0$). Let

$$59 \quad V = z(t)^* P z(t)$$

60 be the Lyapunov function with a Hermitian matrix $P > 0$. For the solutions of (4) we then have

61 $V = z(t)^* (A^* P + PA) z(t)$, such that condition $\dot{V} \leq 0$ is expressed by the matrix $Q = Q^* \geq 0$ of
 62 the Lyapunov matrix equation.

$$63 \quad A^* P + PA = -Q$$

64 The system (4) (and therefore also system (2) is asymptotically stable, if there exist Hermitian
65 matrices $P>0$ and $Q>0$ which satisfy the Lyapunov matrix equation .

66 2.1 DERIVATION OF P AND Q

67 To derive suitable positive definite Hermitian matrices P and Q from the Lyapunov function of
68 the dynamical system (2), we start with the energy equation which is a first integral of the
69 equations of motion. By multiplying (2) from the left with $\dot{x}^*(t)$ and adding the complex
70 transpose of this equation we get

$$71 \quad \dot{x}^* M \dot{x} + x^* K x + 2 \int_0^t x^* D \dot{x} dt + \int_0^t (\dot{x}^* N x - x^* N \dot{x}) dt = 2E_0 \quad (5)$$

72 where $E_0 = 1/2 (\dot{x}^*(0) M \dot{x}(0) + x^*(0) K x(0))$ is the initial mechanical energy of the system. It
73 is obvious that we cannot use the energy $V = \dot{x}^* M \dot{x} + x^* K x > 0$ as a Lyapunov function since
74 the sign of $\dot{V}$ is indefinite due to the circulatory forces described by $N \neq 0$. The idea is now to
75 add terms to (5) to obtain a function V for which $\dot{V}$ is negative definite. For this purpose, we
76 construct a new first integral of the system (2) by now multiplying from left with $\dot{x}(t)$ and adding
77 the complex transpose of this new equation. This leads to

$$78 \quad (\dot{x}^* M \dot{x} + x^* M \dot{x} + x^* D \dot{x}) + \int_0^t (2x^* K \dot{x} - 2\dot{x}^* M \dot{x} + (x^* G \dot{x} - \dot{x}^* G x)) dt = c \quad (6)$$

79 where c is an integration constant. To find a Lyapunov function we introduce a proper positive
80 constant γ , which has to be determined. Multiplying equation (6) by $\gamma/2$ and adding (5) and (6)
81 we get after rearranging terms the following

$$x^* \left(K + \frac{\gamma}{2} D \right) x + \dot{x}^* M \dot{x} + \frac{\gamma}{2} (\dot{x}^* M x + x^* M \dot{x}) = 2E_0 + \frac{\gamma}{2} c$$

$$- \int_0^t (\gamma x^* K x + \frac{\gamma}{2} (x^* G \dot{x} - \dot{x}^* G x) + (\dot{x}^* N x - x^* N \dot{x}) + \dot{x}^* (2D - \gamma M) \dot{x}) dt \quad (7)$$

putting (7) in the quadratic form, $V = z^* P z$

$$\text{where } z^* = \begin{bmatrix} x^* \\ \dot{x}^* \end{bmatrix} \text{ and } z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

we have that

$$V = \begin{bmatrix} x^* \\ \dot{x}^* \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \quad V = \begin{bmatrix} x^* P_{11} x & x^* P_{12} \dot{x} \\ \dot{x}^* P_{21} x & \dot{x}^* P_{22} \dot{x} \end{bmatrix}$$

But eqn (7) is in the form

$$V = P + \int_0^t Q(s) ds$$

Where using (7) we have

$$P = \begin{bmatrix} K & \frac{\gamma}{2} M \\ \frac{\gamma}{2} M & M \end{bmatrix}, \quad Q = \begin{bmatrix} \gamma K & \frac{\gamma}{2} G \\ \frac{\gamma}{2} G^* & 2G - \gamma M \end{bmatrix} \quad (8)$$

where γ is a real number.

3.0 STABILITY ANALYSIS

Considering the MDGKN system with control parameters, we proceed with the analysis as

follows: We assume that $M = M^* > 0$, $D = D^* > 0$ and $K = K^* > 0$. Since P and Q must be

positive definite, we are interested in finding the condition for the existence of a real number γ that guarantees the positive definiteness of P and Q . Schur's lemma provides this condition.

Schur's Lemma

A matrix $R \begin{bmatrix} R_1 & R_2 \\ R_2^* & R_3 \end{bmatrix}$ with Hermitian submatrices R_1 and R_3 is positive definite

if and only if R_1 and $R_3 - R_2^* R_1^{-1} R_2$ are positive definite.

Applying the lemma to Q given by (91), we get that $Q > 0$ if and only if there exists $\gamma > 0$ such that

$$2D - \gamma M - \left(-N + \frac{\gamma}{2}G\right)^* (\gamma K)^{-1} \left(-N + \frac{\gamma}{2}G\right) > 0$$

Rearranging terms we get the following conditions

$$-\gamma^2 \left(M + \frac{1}{4}G^* K^{-1}G\right) + \gamma \left(2D + \frac{1}{2}(G^* K^{-1}N + N^* K^{-1}G)\right) - N^* K^{-1}N > 0 \quad (9)$$

We define all $z \in C^n$, then (9) is equivalent to the inequality

$$-\gamma^2 z^* \left(M + \frac{1}{4}G^* K^{-1}G\right) z + \gamma z^* \left(2D + \frac{1}{2}(G^* K^{-1}N + N^* K^{-1}G)\right) z - z^* N^* K^{-1}N z > 0 \quad (10)$$

Taking $z^* z = I$, the coefficients of the quadratic polynomial in γ are Rayleigh quotients for Hermitian matrices. These Rayleigh quotients are limited by the smallest eigenvalue λ_{min} and the largest eigenvalue λ_{max} of the respective matrices. The Rayleigh quotients for the matrices $M, D, G^* K^{-1}G, M + \frac{1}{4}G^* K^{-1}G$ and $N^* K^{-1}N$ are all positive since M, D , and K^{-1} are assumed to be positive definite. Introducing the scalars a, b and c defined by

$$\left. \begin{aligned} 109 \quad & a = \lambda_{\max} \left(M + \frac{1}{4} G^* K^{-1} G \right) > 0 \\ & b = \lambda_{\min} \left(2D + \frac{1}{2} (G^* K^{-1} N + N^* K^{-1} G) \right) \\ & c = \lambda_{\max} (N^* K^{-1} N) > 0 \end{aligned} \right\} \quad (11)$$

110 Inequality (10) is now satisfied if there exists $\gamma > 0$ with

$$111 \quad -\gamma^2 a + \gamma b - c > 0 \quad (12)$$

112 There are solutions if and only if

$$113 \quad b^2 - 4ac > 0 \quad \text{and} \quad b > 0 \quad (13)$$

114 In this case γ can be chosen as any number in the interval

$$115 \quad \frac{b - \sqrt{b^2 - 4ac}}{2a} < \gamma < \frac{b + \sqrt{b^2 - 4ac}}{2a} \quad (14)$$

116 and then matrix Q will be positive definite.

117 Next, if $Q > 0$ then $2D - \gamma M > 0$. This implies $\frac{\gamma}{2} D - \frac{\gamma^2}{4} M > 0$.

118 Also, $P > 0$ and applying the Schur's lemma on P we have

$$M - \frac{\gamma}{2} M \left(K + \frac{\gamma}{2} D \right)^{-1} \frac{\gamma}{2} M > 0$$

119 Multiplying through by $\left(K + \frac{\gamma}{2} D \right)$ we have

$$M \left(K + \frac{\gamma}{2} D \right) - \frac{\gamma}{2} M \left(K + \frac{\gamma}{2} D \right)^{-1} \left(K + \frac{\gamma}{2} D \right) \frac{\gamma}{2} M > 0$$

$$M \left(K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right) > 0$$

$$\Rightarrow K + \frac{\gamma}{2}D - \frac{\gamma^2}{4}M > 0$$

120 We now formulate the following theorem that provides the condition for the stability of system
121 (1) and then asymptotic stability of system (2).

122 **Stability theorem**

Assume a, b and c defined by (11). If $b^2 - 4ac > 0$ and $b > 0$,

then system (1) is asymptotically stable.

123 To apply the theorem, it will be beneficial and less cumbersome to estimate a and b as follows

$$a \leq \lambda_{\max}(M) + \frac{1}{4}\lambda_{\max}(G^*K^{-1}G)$$

$$124 \quad b \geq 2\lambda_{\min}(D) + \frac{1}{2}\lambda_{\min}(G^*K^{-1}N + N^*K^{-1}G) \quad (15)$$

125 Thus,

$$126 \quad \lambda_{\max}(G^*K^{-1}G) \leq g_{\max}^2/k_{\min} ,$$

$$127 \quad \lambda_{\max}(N^*K^{-1}N) \leq n_{\max}^2/k_{\min} \quad (16)$$

$$128 \quad \lambda_{\min}(G^*K^{-1}N + N^*K^{-1}G) \geq -2g_{\max}n_{\max}/k_{\min} ,$$

129 where $g_{\max} = |\lambda(G)|_{\max}$, $n_{\max} = |\lambda(N)|_{\max}$ are the maximum of the absolute values of the
130 eigenvalues of G and N, respectively, and $k_{\min}$ is the smallest eigenvalue of $K > 0$. Additionally
131 we use $m_{\max}$ for $\lambda_{\max}(M)$ and $d_{\min}$ for $\lambda_{\min}(D)$. Applying (15) and (16) conditions (96) for
132 the existence of $\gamma > 0$ become

$$133 \quad \left(2d_{min} - \frac{g_{max}n_{max}}{k_{min}}\right)^2 - \frac{4\left(m_{max} + \frac{1}{4}\frac{g_{max}^2}{k_{min}}\right)n_{max}^2}{k_{min}} > 0, \quad (17)$$

$$2d_{min} - \frac{g_{max}n_{max}}{k_{min}} > 0$$

134 Obviously, both inequalities in (17) are satisfied if

$$135 \quad d_{min}^2 k_{min} - d_{min} g_{max} n_{max} - m_{max} n_{max}^2 > 0 \quad (18)$$

136 (18) is a more restrictive condition than (13) and contains the smallest and largest eigenvalues of
 137 the system matrices. It is therefore a simple sufficient condition for asymptotic stability of
 138 system (2). Choosing appropriate $\gamma > 0$ by adding the two limits in (14) we have $\gamma = \frac{b}{a}$

139 And using the estimates of a and b we have the following

$$140 \quad \gamma = (d_{min} k_{min} - \frac{1}{2} g_{max} n_{max}) / (m_{max} k_{min} + \frac{1}{4} g_{max}^2) \quad (19)$$

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142 4.0 APPLICATIONS

143 Example 1

144 To illustrate the formulas for response bounds for the inhomogeneous system let us consider the
 145 2x2 system described by

$$146 \quad \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \left(\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \right) \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \left(\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = f(t) \quad (20)$$

147 We obtain the constants a, b and c defined in (11) as follows:

$$a = \lambda_{max} \left(M + \frac{1}{4} G^* K^{-1} G \right) = 7$$

$$b = \lambda_{min} \left(2D + \frac{1}{2} (G^* K^{-1} N + N^* K^{-1} G) \right) = \frac{32}{5}$$

$$c = \lambda_{max} (N^* K^{-1} N) = 1$$

148 Applying the values on the stability condition we have that

$$b^2 - 4ac = 12.96 > 0$$

149 The system is therefore stable according to the stability theorem.

150 Example 4

151 Consider the 3x3 system

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \left(\begin{bmatrix} 8 & -2 & 2 \\ -2 & 8 & -2 \\ 2 & -2 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix} \right) \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} +$$

$$\begin{bmatrix} 4 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (21)$$

153 We compute the constants defined in (11) as follows:

$$a = \lambda_{max} \left(M + \frac{1}{4} G^* K^{-1} G \right) = \frac{37 + \sqrt{809}}{8}$$

$$b = \lambda_{min} \left(2D + \frac{1}{2} (G^* K^{-1} N + N^* K^{-1} G) \right) = 13$$

$$c = \lambda_{max} (N^* K^{-1} N) = 3$$

155 Since $b > 0$ and $b^2 - 4ac = 70.8 > 0$, the system is stable according to the stability
156 theorem.

157 5.0 CONCLUSION

158 The gyroscopic systems (damped and undamped) are generally stable systems but the addition
159 of circulatory forces can destroy stability. For the MDGKN system with control parameters, the
160 relationship between the control parameters determines the stability or otherwise of the system.
161 Bounds of the solution can be obtained if the system is stable. Stability theorem is formulated by
162 means of the largest and smallest eigenvalues of the combinations of the system matrices. Since

the domain is R^n , the conditions were illustrated on a 2x2 and a 3x3 matrix systems. With the conditions, the stability status of systems can be determined without explicit computation of eigenvalues.

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