

STRUCTURE OF SOME PREGROUPS AND LENGTH FUNCTIONS

Abstract

The concept of Pregroups was introduced in 1971 by Stallings. Subsequent work has been done by many other researchers. Stallings defined a set with binary operation satisfying five axioms, namely, P1, P2, P3, P4, and P5. It has been proved that P3 is a consequence of the other axioms. Stallings has also linked the Pregroups to Free Product of Groups.

This construction is developed to include a new axiom called P6, which enabled to define a length function on the universal group of Pregroups. This led to direct proofs of many other problems.

Keywords: Archimedean elements, Defined product of elements, Length Functions, Pregroup, Universal Group.

1. INTRODUCTION

Stallings [1], in 1971 introduced the concept of a pregroup. Subsequent work is done by Hoare [2], Nesayef [3], Chiswell [4], and many others. Five axioms are originally introduced by Stallings [1], namely P1, P2, P3, P4, and P5. It is proved in [3] that P3 is a consequence of the other axioms. Stallings extended his construction to link Pregroups to free products of groups in [5].

In section one of this paper, we introduce the concept of length function and list all the axioms of length function which are needed in the latter sections. We also introduce the definition and some important properties of pregroups. In section two, we introduce a new axiom called P6 and prove that some axioms are equivalent to the other ones.

We also investigate some basic properties of Pregroups restricting our attention to P6. Finally we show that the universal group of a Pregroup satisfying P6 has a length function given by Lyndon [6] and prove some consequence results.

2. LENGTH FUNCTION

Definition 2.1: A length function $| \cdot |$ on a group G , is a function giving each element x of G , a real number $|x|$, such that for all $x, y, z \in G$, the following axioms are satisfied.

A1' $|e| = 0$, where e is the identity elements of G .

A2 $|x^{-1}| = |x|$

A4 $d(x, y) < d(y, z) \Rightarrow d(x, y) = d(x, z)$, where $d(x, y) = \frac{1}{2} (|x| + |y| - |xy^{-1}|)$

Lyndon showed that A4 is equivalent to: $d(x, y) \geq \min \{ d(y, z), d(x, z) \}$ and to

$$d(y, z), d(x, z) \geq m \Rightarrow d(x, y) \geq m.$$

A1', A2 and A4 imply $|x| \geq d(x, y) = d(y, x) \geq 0$

Assuming, A2 and A4 only, it is easy to show that:

- i. $d(x, y) \geq |e|$, where e is the identity element of G .
- ii. $|x| \geq |e|$
- iii. $d(x, y) \leq |x| - \frac{1}{2} |e|$

A3 State that $d(x, y) \geq 0$, is deductible from A1', A2 and A4' is a weaken version of the axiom:

A1 $|x| = 0$, if and only if $x = 1$ in G .

The following results are introduced by Lyndon [6].

- (1) $d(xy, y) + d(x, y^{-1}) = |y|$
- (2) $d(x, y^{-1}) + d(y, z^{-1}) \leq |y|$ Implies $|x y z| \leq |x| - |y| + |z|$
- (3) $d(x, y^{-1}) + d(y, z^{-1}) \leq |y|$ Implies $d(xy, z^{-1}) = d(y, z^{-1})$
- (4) $d(x, y) + d(x^{-1}, y^{-1}) \geq |x| = |y|$ Implies $|d(xy^{-1})^2| \leq |xy^{-1}|$

It follows from (2) that for any $x, y \in G$, $d(x, y) = |y| - d(x y^{-1}, y^{-1}) \leq |y|$ by A3.

Since $d(x, y) = d(y, x)$, we get $d(x, y) \leq \min\{|x|, |y|\}$

As stated that $d(x, y) + d(x^{-1}, y^{-1}) > |x| = |y| \Rightarrow x = y$

The following definitions are introduced in [7].

Definition 2.2 A non-trivial element g of a group G is **called non-Archimedean** if $|g^2| \leq |g|$

Definition 2.3 Let G be a group with length function an element $x \neq 1$ in G is called Archimedean if $|x| \leq |x^2|$.

The following Axioms and results have added by Lyndon and others

- A0 $x \neq 1 \Rightarrow |x| < |x^2|$
- C0 $d(x, y)$ is always an integer
- C1 $x \neq 1, |x^2| \leq |x|$ implies $|x|$ is odd
- C2 For no x is $|x^2| = |x| + 1$
- C3 if $|x|$ is odd then $|x^2| \geq |x|$
- C1' if $|x|$ is even and $|x| \neq 0$, then $|x^2| > |x|$
- N0 $|x^2| \leq |x|$ implies $x^2 = 1$ is $x = x^{-1}$
- N1* G is general by $\{x \in G : |x| \leq 1\}$

The following two constructions are also added by Lyndon [6], where the set of all Non-Archimedean elements in G is denoted by N :

$$N = \{x \in G : |x^2| \leq |x|\} \dots\dots\dots (1)$$

$$M = \{xy \in G : |xy| + |yx| < 2|x| = 2|y|\} \dots\dots\dots (2)$$

Lyndon showed that $M \subseteq N$. However, the nature of the elements of M and N is investigated in [8].

3. PREGROUPS

Definition 3.1 A Pregroup is a set P containing an element called the identity element of P , denoted by 1 , a subset D of $P \times P$ and a mapping $D \rightarrow P$, where $(x, y) \rightarrow xy$, together with a map $i : P \rightarrow P$ where $i(x) = x^{-1}$, satisfying the following axioms:

We say that xy is defined if $(x, y) \in D$, i.e. $xy \in P$.

P1. For all $x \in P$, $1x$ and $x1$ are defined and $1x = x1 = x$.

P2. For all $x \in P$, $x^{-1}x = xx^{-1} = 1$.

P3. For all $x, y \in P$, if xy is defined, then $y^{-1}x$ is defined and $(xy)^{-1} = y^{-1}x^{-1}$.

P4. Suppose that $x, y, z \in P$. If xy and yz are defined, then $x(yz)$ is defined, in which case $x(yz) = (xy)z$.

P5. If $w, x, y, z \in P$, and if wx, xy, yz , are all defined then either $w(xy)$ or $(xy)z$ is defined.

3.1 THE AXIOM P6

In this section we restrict our attention to a special type of pregroups, which satisfy a certain condition. To do this we introduce the following theorems, which are given in [3].

Theorem 3.1 The following two statements are equivalent in P .

P6(1): If (x_1, x_2) is reduced and $x_1a, a^{-1}x_2$ are both defined, then $a \in A_0$

P6(2): If (x_1, x_2) is reduced and $(ax_1)x_2$ is defined for $a \in P$ then $ax_1 \in A_0$

Proof suppose (x_1, x_2) is reduced and let $(ax_1)x_2$ is defined for some $a \in P$.

$x_1(ax_1)^{-1}, (ax_1)x_2$ are both defined so $ax_1 \in A_0$, so P6(1) $\rightarrow$ P6(2).

Conversely, suppose (x_1, x_2) is reduced and $x_1a, a^{-1}x_2$ are both defined for some $a \in P$

Since (x_1, x_2) is reduced, then $(x_1a, a^{-1}x_2)$ is reduced

Since $x_1^{-1}(x_1a)$ is defined and $= a$, and $\{x_1^{-1}(x_1a)\}a^{-1}x_2$ is defined and $= x_2$, then by P6(2) $x_1^{-1}(x_1a) \in A_0$, i.e. $a \in A_0$.

Therefore P6(1) $\Leftrightarrow$ P6(2)

We denote the equivalent statements P6(1) and P6(2) in theorem 3.1, by P6 and the pregroup which satisfies P6, by P^* .

The following construction is introduced in [9].

Definition 3.2 Let P^* be a pregroup satisfying P6. The Universal group, $U(P^*)$ is the set of all equivalence classes of reduced words in P^* .

We define now a length function on $U(P^*)$. Before we achieve this, we introduce the following result, which generalizes the condition P6(2).

Theorem 3.2 Let $a_{n-1}, \dots, a_1$ be any sequence, and $x_1, \dots, x_n$ be reduced, both on P^* . If

$a_{n-1}, \dots, a_1x_1, \dots, x_n$ is defined, then $a_{n-1}, \dots, a_1x_1, \dots, x_n \in A_0, n \geq 2$

Proof The only way in which $a_{n-1}, \dots, a_1x_1, \dots, x_n$ is defined is by $[a_{n-1}, \dots, a_1x_1, \dots, x_n]x_n$ being defined.

Then also by theorem 2.1 either $[(a_{n-1}, \dots, a_1 x_1, \dots, x_{n-2}) x_{n-1}] x_n$ is defined, so by P6(2),
 $(a_{n-1}, \dots, a_1 x_1, \dots, x_{n-2}) x_{n-1} \in A_0$, or $[a_{n-1} (a_{n-2} \dots a_1 x_1 \dots x_{n-1})] x_n$ is defined, where $x_0 = 1$.

Since $(a_{n-2}, \dots, a_1 x_1, \dots, x_{n-1}) x_n$ is not defined by theorem 2.1, then by P6(2)
 $a_{n-1} (a_{n-2} \dots a_1 x_1 \dots x_{n-1}) \in A_0$

Theorem 3.3 Let $U(P^*)$ be the universal group of a pergroup P^* and left $g, h \in (P^*)$. Let

$g = x_1 \dots x_n, h = y_1 \dots y_m, m, n \geq 2$ be in reduced forms. Let $a_i = x_{n-i+1} \dots (x_n y_m^{-1}) \dots y_{m-i+1}^{-1}$ be defined for $1 \leq i \leq s$ for some $s < m, s \leq n$.

If $a_s y_{m-s}^{-1}$ is defined then $a_i \in A_0$ for all $i \leq s$. Hence by symmetry if $s < n$ and $x_{n-s} a_s$ is defined, then $a_i \in A_0$.

Proof $a_i = x_{n-i+1} a_{i-1} y_{m-i+1}^{-1}$, for $1 \leq i < s$, where $a_0 = 1$. Then by theorem 2.1 either $x_{n-i+1}^{-1} (a_{i-1} y_{m-i+1}^{-1})$ is defined (1)

or $(x_{n-i+1}^{-1} a_{i-1}) y_{m-i+1}^{-1}$ is defined (2)

If (1) holds, then $[x_{n-i+1}^{-1} (a_{i-1} y_{m-i+1}^{-1})] y_{m-i}^{-1}$ is defined

Then $a_{i-1} y_{m-i+1}^{-1}, y_{m-i}^{-1}$ is reduced. So by P6 (2) $[x_{n-i+1} (a_{i-1} y_{m-i+1}^{-1})] = a_i \in A_0$

If (2) holds, then $[(x_{n-i+1} a_{i-1}) y_{m-i+1}^{-1}] y_{m-i}^{-1}$ is defined.

Since $y_{m-i+1}^{-1}, y_{m-i}^{-1}$ is reduced, then by P6 (2), $(x_{n-i+1} a_{i-1}) y_{m-i+1}^{-1} = a_i \in A_0$

Corollary 3.1 In theorem 3.3, $x_{n-s} a_s (a_s y_{m-s}^{-1})$ is defined if and only if $(x_{n-s} a_s) a_s y_{m-s}^{-1}$ is defined.

Proof By theorem 2.3 $a_s \in A_0$ in either case. Then $x_{n-s} a_s$ and $a_s y_{m-s}^{-1}$ are defined. by P4. The result follows.

From Theorems 2.2 and 2.3 and Corollary 2.1, with the same notations, we have shown that:

Corollary 3.2 $gh^{-1} = x_1 \dots x_{n-s} a_s y_{m-s}^{-1} \dots y_{m-s}^{-1}$ is reduced, if and only if $a_s \notin A_0$.

The proof of this Corollary is similar to the main theorem in [10].

3.2 P^* PREGROUPS AND LENGTH FUNCTION

Theorem 3.4 $|\cdot| : U(P^*) \rightarrow \mathbb{R}$ given in definition 3.2 is a length function on $U(P^*)$

Proof $A1', A2$ are clearly satisfied, so we prove $A4$ is also satisfied.

Let $g, h, k \in U(P^*)$. The result is trivial if any one of $|g|, |h|, |k|$ is zero. So let

$g = x_1 \dots x_n, n \geq 1, h = y_1 \dots y_m, m \geq 1$ and $z = z_1 \dots z_\ell, \ell \geq 1$, be reduced where x_1, y_1 and $z_1 \notin A_0$,

i.e. $|g| = n, |h| = m$ and $|k| = \ell$.

Clearly $d(g, h) \geq 0$. Suppose $d(g, h), d(h, k) \geq s$

Case 1: s is an integer

There exists a_s , such that, $gh^{-1} = x_1 \dots x_{n-s} a_s y_{m-s}^{-1} \dots y_1^{-1}$, where

$$a_s = x_{n-s+1} \dots x_n y_{m-s}^{-1} \dots y_{m-s+1}, \text{ and } a_s \in A_0$$

Similarly $hk^{-1} = y_1 \dots y_{m-s} b_s z_{\ell-s}^{-1} \dots z_1^{-1}$ and $b_s \in A_0$ $b_s = y_{m-s+1} \dots y_m z_{\ell-s}^{-1} \dots z_{\ell-s+1}^{-1}$

$$\begin{aligned} gh^{-1} &= x_1 \dots x_n z_{\ell}^{-1} \dots z_1^{-1} = x_1 \dots x_n y_m^{-1} \dots y_{m-s+1}^{-1} y_{m-s+1}^{-1} \dots y_m z_{\ell}^{-1} \dots z_1^{-1} \\ &= x_1 \dots x_{n-s} a_s b_s z_{\ell-s}^{-1} \dots z_1^{-1} \end{aligned}$$

Since $a_s b_s \in A_0$, then $d(g, k) \geq s$

Case 2: s is not an integer

Let $d(g, k), d(h, k) \geq r - \frac{1}{2} = s, r \geq 1$

Then $gh^{-1} = x_1 \dots x_{n-r} a_r y_{m-s}^{-1} \dots y_1^{-1}$, where $a_r = y_{n-r+1}^{-1} \dots x_n y_m^{-1} \dots y_{m-r+1}^{-1}$

$hk^{-1} = y_1 \dots y_{m-r} b_r z_{\ell-s}^{-1} \dots z_1^{-1}$, where $b_r = y_{m-r+1} \dots y_m$

$z_{\ell}^{-1} \dots z_{\ell-r+1}^{-1}$ and a_r, b_r are not necessarily in A_0 , moreover a_i is defined for all $i \leq r$.

$$\begin{aligned} gh^{-1} &= x_1 \dots x_n z_{\ell}^{-1} \dots z_1^{-1} = x_1 \dots x_n y_{m-r+1}^{-1} y_{m-r+1} \dots y_m z_{\ell}^{-1} \dots z_1^{-1} \\ &= x_1 \dots x_{n-r} a_r b_r z_{\ell-r}^{-1} \dots z_1^{-1} \end{aligned}$$

Let (a_r, b_r) be reduced. By Theorem 2.3, $a_{r-1} \in A_0$.

Therefore $x_{n-r+1} a_{r-1}$ is defined and, $x_{n-r+1} a_{r-1} = ((x_{n-r+1} a_{r-1}) y_{m-r+1}^{-1}) y_{m-r+1} = a_r y_{m-r+1}$.

Similarly $y_{m-r+1}^{-1} b_r$ is defined, Then by 6(1), $y_{m-r+1} \in A_0$, so a contradiction.

Hence $a_r b_r$ is defined, thus $|gk^{-1}| \leq n - r + \ell - r + 1$, i.e. $d(g, k) \geq r - \frac{1}{2} = s$

Therefore A4 is satisfied, and so $|\cdot|$ is a length function.

3.3 APPLICATIONS OF P* PREGROUPS

Let $P = G \cup \{G t_i \mid G\} \cup \{G t_j \mid G\}$ and suppose that the product xy of two elements x and y of P is defined if and only if at least one of $\{x, y, xy\}$ is in G . Thus $xy \in A_0$, provided we exclude the case when

$$G^* = \langle c, t \mid \text{rel } c, tct^{-1} = \alpha(c), t^2 = c' \in C \rangle, \text{ in which case } A_0 = G^*$$

The axioms P1, P2 and P3 are clearly satisfied. For P4, let $x, y, z \in P$, and suppose xy, yz are defined, and $x(yz)$ is also defined, but $(xy)z$ is not defined, then

$$xy \notin G, z \notin G \text{ and } (xy)z \notin G \quad (1)$$

Thus

$$xy \text{ is defined} \Rightarrow \text{either } x \text{ or } y \in G \quad (2)$$

$$yz \text{ is defined} \Rightarrow \text{either } y \text{ or } yz \in G \quad (3)$$

$X(yz)$ is defined $\Rightarrow$ either x or yz or $x(yz) \in G$ (4)

Since $(xy)z \notin G$ and the products $x(yz)$ and $(xy)z$ are equal in G^* , therefore $x(yz) \notin G$

Case 1: If $x \in G$, then $yz \notin G$, so $y \in G$ by (3), and hence $xy \in G$, so a contradiction.

Case 2: If $x \notin G$, then $yz \in G$, so $y \in G$, by (2), and $yz \in G$ by (4). Therefore, $z \in G$ also a contradiction.

Hence $(xy)z$ is defined, i.e. P4 is satisfied.

For P5, let $w, x, y, z \in P$ and let w, x, y, z be defined and suppose neither $w(xy)$ nor $(xy)z$ is defined. Then $w \notin G, xy \notin G$ and $w(xy) \notin G$ since xy is defined. Then either x or $y \in G$.

Case 1: If $x \in G$ then $x(yz)$ is defined, so by P4 $(xy)z$ is also defined and $x(yz) = (xy)z$

Case 2: If $y \in G$ then $(wx)y$ is defined, so by P4 $w(xy)$ is defined hence P5 is satisfied

For P6 suppose (x, y) is reduced, i.e. no one of x, y, xy is in G and suppose that $xa, a^{-1}y$ are both defined. Suppose $a \notin G$ since $x \notin G$, then $xa \in G$, and similarly $a^{-1}y \in G$.

Hence $(xa)(a^{-1}y) \in G$, so $xy \in G$ that is xy is defined, so a contradiction.

Therefore $a \in G$, and P6 is satisfied.

Note The following case are given in [8].

If $J = \emptyset$ and $N \neq \emptyset$, then we get an H.N.N extension, in which case $N \cap P^* = G$ and $M \cap A_0 = M \cap G =$ all elements of the associated subgroups.

If $J = \emptyset$, then we get a qusai- H.N.N. extension, (excluding $|J| = 1, I = \emptyset$), we have .

$$N \cap P^* = G \cup \{g(t_j C_j)\}g^{-1} : g \in G, j \in J \text{ and } M \cap A_0 = \{G_j, A_i, B_i\}.$$

Theorem 3.5 If $\cup (P^*)$ is the universal group of a pergroup P^* then N consists of conjugates of elements of $P^* \cap N$. i.e. $g \in N \Leftrightarrow g = x a x^{-1}, x \in \cup (P^*)$ and, $a \in P^* \cap N$. Moreover if $|g|$ is even then $a \in A_0$ and $|x| = \frac{|g|}{2}$ and $|x| = \frac{|g|-1}{2}$, if $|g|$ is odd.

Proof Suppose $g \in N$, and let $g = x_1 \dots x_n$ be reduced, then $|g| = n$ or $g = 0$ if $n = 1$ and $x_1 \in A_0$.

The result is trivial if $n = 0$ or 1 , so let $n > 1$.

Case 1: If n is even, then put $n = 2s, s \geq 1, g^2 = x_1 \dots x_s x_{s+1} \dots x_n x_1 \dots x_s x_{s+1} \dots x_n$

Since $|g^2| \leq |g|$, then $g^2 = x_1 \dots x_s x_{s+1} \dots x_n$ where $a_s = x_{s+1} \dots x_n x_1 \dots x_s \in A_0$

By theorem 2.3, and since $|x_s a_s x_{s+1}| \leq 2$, then

$$g = x_1 \dots x_s x_{s+1} \dots x_n x_1 \dots x_s (x_1 \dots x_s)^{-1} = (x_1 \dots x_s) a_s (x_1 \dots x_s)^{-1}$$

= a conjugate of an element of A_0 , Moreover, $a_s^2 \in A_0$, i.e. $|a_s^2| = 0 = |a_s|$, so $a_s \in N$.

Case 2: If n is odd, then let $n = 2r + 1, r \geq 1$.

$$g^2 = x_1 \cdot \dots \cdot x_r x_{r+1} \cdot \dots \cdot x_n x_1 \cdot \dots \cdot x_r x_{r+1} x_{r+2} \cdot \dots \cdot x_n$$

Since $|g^2| \leq |g|$, then

$$g^2 = x_1 \cdot \dots \cdot x_r (x_{r+1} a_r x_{r+1}) x_{r+2} \cdot \dots \cdot x_n, \text{ where } a_r x_{r+2} \cdot \dots \cdot x_n x_1 \cdot \dots \cdot x_r \text{ and } |x_{r+1} a_r x_{r+1}| \leq 1.$$

By Theorem 2.2, $a_r \in A_0$.

$$g = x_1 \cdot \dots \cdot x_r x_{r+1} x_{r+2} \cdot \dots \cdot x_n x_1 \cdot \dots \cdot x_r (x_1 \cdot \dots \cdot x_r)^{-1}$$

$$g = (x_1 \cdot \dots \cdot x_r)(x_{r+1} a_r)(x_1 \cdot \dots \cdot x_n)^{-1}, x_{r+1} a_r = b \in P \setminus A_0.$$

Since $|x_{r+1} a_r x_{r+1}| \leq 1$ and since $a_r \in A_0$, then $b^2 = x_{r+1} a_r x_{r+1} a_r$ and $|b^2| \leq 1$, i.e. $|b^2| \leq |b|$.

Hence $b \in P \cap N$.

Conversely, suppose, $g = (x_1 \cdot \dots \cdot x_s) a (x_1 \cdot \dots \cdot x_s)^{-1}$, where $a \in P \cap N$

If $x_s a x_s^{-1}$ is defined, then put $x_s a x_s^{-1} = a_1$, so $a \in A_0$, by theorem 2.3.

If $|a_1| = 0$, then $|a_1^2| = 0$, so $a_1 \in P \cap N$.

Let $|a_1| = 1$. $a_1^2 = x_s a^2 x_s^{-1}$, where $a^2 \in A_0$.

Suppose $(x_s a^2 x_s^{-1})$ is reduced, i.e. $|a_1^2| = 2$. Apply P6(1) on $(x_s a^2 x_s^{-1})$.

Since $(x_s a^2)(a^{-1} x_s^{-1})$ and $(x_s a^2 x_s^{-1})$ are both defined, then $x_s a \in A_0$, so a contradiction.

Thus $x_s a^2 x_s^{-1}$ is defined, i.e. $|a_1^2| \leq |a_1|$, so $a_1 \in P \cap N$.

If $x_{s-1} a_1 x_{s-1}^{-1}$ is also defined, then we apply the same argument and so on until we have

$$g = (x_1 \cdot \dots \cdot x_r) b (x_1 \cdot \dots \cdot x_r)^{-1}, \text{ where } b \in P \cap N, \text{ and } x_r b x_r^{-1} \text{ is not defined.}$$

If $b \in A_0$, then $|g| = 2r, b^2 \in A_0$ and $|g^2| = |(x_1 \dots x_s) b^2 (x_1 \dots x_s)^{-1}| \leq 2r$, so $g \in N$.

If $b \notin A_0$, and $x_r b$ and $b x_r^{-1}$ are not defined then $|g| = 2r + 1$

Since $|b^2| \leq 1$, then $|g^2| = |(x_1 \dots x_s) b^2 (x_1 \dots x_s)^{-1}| \leq 2r + 1 = |g|$, so $g \in N$.

Finally, if $b \notin A_0$ and either $x_r b$ or $b x_r^{-1}$ is defined, then $|g| = 2r$.

Since $b \in N$ then $|b^2| \leq |b|$, so b^2 is defined.

Consider $g^2 = (x_1 \dots x_s) b^2 (x_1 \dots x_s)^{-1}$, and suppose x_r, b^2, x_r^{-1} is reduced

Either (i) $b x_r^{-1}$ is defined, then apply P6 (2) on (b^2, x_r^{-1}) .

Since $(b^{-1}, b^2) x_r^{-1} = b x_r^{-1}$ is defined, then $b^{-1} \in A_0$, so $b \in A_0$ and so a contradiction.

Or (ii) $x_r b$ is defined, and this is similar to (i)

Therefore x_r, b^2, x_r^{-1} is not reduced, i.e. $|x_r b^2 x_r^{-1}| \leq 2$

Therefore, $|g^2| \leq 2r = |g|$, i.e. $g \in N$.

Theorem 3.6 The equivalent elements of N in $U(P^*)$ are the same conjugates of P^* i.e. $|g| = |h|$, then $g \sim h \Leftrightarrow g = xax^{-1}$, $h = xbx^{-1}$ and $a \sim b$, where $a, b \in P^*$, $x \in U(P^*)$

Proof Suppose $g \sim h$ in N , then $|gh^{-1}| \leq |g| = |h|$

Let $g = x_1 \dots x_n$, $h = y_1 \dots y_n$ be reduced

Case 1: The result is trivial if $|g| = |h| = 0, 1$

Case 2: If n is even, then put $n = 2s$, $s \geq 1$

Then, $g = (x_1 \dots x_s) a_s (x_1 \dots x_s)^{-1}$, $a_s \in A_0$ and $h = (y_1 \dots y_s) b_s (y_1 \dots y_s)^{-1}$, $b_s \in A_0$

$$gh^{-1} = (x_1 \dots x_{s-1}) (x_s a_s) (x_1 \dots x_s)^{-1} (y_1 \dots y_s) (b_s^{-1} y_s^{-1}) (y_1 \dots y_{s-1})^{-1}$$

Since $|gh^{-1}| \leq n = 2s$, then by theorem 2.3, $(x_1 \dots x_s)^{-1} (y_s \dots y_s) = a_s \in A_0$, and

$$y_1 \dots y_s = (x_1 \dots x_s) a$$

Thus $h = (x_1 \dots x_s) ab_s a^{-1} (x_1 \dots x_s)^{-1}$, where $ab_s a^{-1} = b \in A_0$

Hence g, h are the same conjugates, moreover, $|a_s b^{-1}| = 0$, for $a_s, b^{-1} \in A_0$

Case 3: If n is odd and $n = 2r + 1$ where $r \geq 1$, then we have:

$$g = (x_1 \dots x_r) (x_{r+1} a_r) (x_1 \dots x_r)^{-1}, a_r \in A_0, x_{r+1} a_r \in P^* \setminus A_0$$

$$\text{Also } h = (y_1 \dots y_r) (y_{r+1} b_r) (y_1 \dots y_r)^{-1}, \text{ where } b_r \in A_0, y_{r+1} b_r \in P^* \setminus A_0$$

$$\text{Since } |gh^{-1}| \leq 2r + 1 \text{ then } |(x_{r+1} a_r) (x_1 \dots x_r)^{-1} (y_1 \dots y_r) (y_{r+1} b_r)^{-1}| \leq 1$$

Since $(x_{r+1} a_r) (x_1 \dots x_r)^{-1}$ and $(y_1 \dots y_r) (y_{r+1} b_r)^{-1}$ are reduced it follows that

$$(x_1 \dots x_r)^{-1} (y_1 \dots y_r) \text{ is defined and by theorem 2.3 we have, } (x_1 \dots x_r)^{-1} (y_1 \dots y_r) = b \in A_0$$

$$\text{Thus } (y_1 \dots y_r) = (x_1 \dots x_r) b$$

$$\text{So } h = (x_1 \dots x_r) b (y_{r+1} b_r) b^{-1} (x_1 \dots x_r)^{-1} \text{ where } b (y_{r+1} b_r) b^{-1} \in P^* \setminus A_0$$

Hence g and h are conjugates of elements of $P^* \setminus A_0$ determined by the same element of (P^*) .

Moreover $y_{r+1} a_r \sim b (y_{r+1} b_r) b^{-1}$, since $|x_{r+1} a_r b (y_{r+1} b_r) b^{-1}| \leq 1$ and $|b| = 0$

Conversely, suppose $g = (x_1 \dots x_r) a (x_1 \dots x_r)^{-1}$, $a \in P^* \cap N$, $h = (x_1 \dots x_r) b (x_1 \dots x_r)^{-1}$, $a \in P^* \cap N$, where $a \sim b$.

Similar arguments show that: $x_r a x_r^{-1}$ is defined where $r = \frac{|g|}{2}$ if $|g|$ is even, and $r = \frac{|g|-1}{2}$ if g is odd.

Since $a \sim b$, either $a, b \in A_0$ or $a, b \notin A_0$. So we consider the possible cases.

Case 1: If $a, b \in A_0$ then $ab^{-1} \in A_0$ and $|g| = |h| = 2r$

$$|gh^{-1}| = |(x_1 \dots x_r) b (x_1 \dots x_r)^{-1}| \leq 2r, \text{ so } g \sim h$$

Case 2: If $a, b \notin A_0$ and both x_r, b, x_r^{-1} are reduced, then $|g| = |h| = 2r + 1$ and $|a| = |b| = 1$

Since $|ab^{-1}| \leq |a| = |b| = 1$ by assumption, then $|gh^{-1}| = |(x_1 \dots x_r)ab^{-1}(x_1 \dots x_r)^{-1}| \leq 2r + 1$, so $g \sim h$

Case 3: If $a, b \notin A_0$ and either $x_r a$ is defined (1)

Or ax_r^{-1} is defined (2)

i.e. $|g| = 2r$ and either $x_r b$ is defined (3)

Or bx_r^{-1} is defined (4)

i.e. $|g| = 2r$

Suppose ax_r^{-1} is defined, and (x_r, a) is reduced, since $\in N$, then $|a^2| \leq 1$ and then $x_r a^{-1}$ and a^2 are both defined. So applying P6(1) to $(x_r a^{-1}, aa)$, then $a \in A_0$ so contradiction

Hence $x_r a$ and $a x_r^{-1}$ are both defined. Similarly $x_r b$ and $b x_r^{-1}$ are both defined

So $|gh^{-1}| = |(x_1 \dots x_r a)(x_1 \dots x_r b)^{-1}| \leq 2r$

Therefore $g \sim h$

Theorem 3.7 If $U(P^*)$ is the universal group of a pregroup $P^* \neq A_0$ then the elements of M are conjugates of elements of length zero in $U(P^*)$, i.e. $h \in M \Rightarrow gh = xax^{-1}$, where $x \in U(P^*)$ and $a \in A_0$

Proof Let $g = x_1 \dots x_n, h = y_1 \dots y_n$ be reduced, and suppose $|gh| + |hg| < 2|h|$, then $|g|, |h| \geq 1$

case 1: If $n = 1$, then $g = x_1 \notin A_0$ and $h = h_1 \notin A_0$, since $|x_1 y_1| + |y_1 x_1| < 2|x_1| = 2|y_1| = 2$,

Then at least one of $|x_1 y_1|$ or $|y_1 x_1|$ is zero

Suppose $|y_1 x_1| = 0$, i.e. $y_1 x_1 \in A_0$, then $gh = x_1 y_1 x_1 x_1^{-1} = x_1 (y_1 x_1) x_1^{-1} = \text{conjugate of } y_1 x_1 \in A_0$

Similarly, if $|x_1 y_1| = 0$, then hg is a conjugate of $x_1 y_1$

Case 2: If $n \geq 2$, then let s be the maximum such that

$$gh = x_1 \dots x_{n-s} a_s y_{s+1} \dots y_n \quad (1)$$

$s \leq n$. Then either (1) is reduced in which case

$$|gh| = 2n - 2s + 1 \quad (2)$$

Or $a_s \in A_0$ and $x_{n-s} a_s y_{s+1}$ is not defined in which case

$$|gh| = 2n - 2s, \text{ where } a_s = x_{n-s+1} \dots x_n y_1 \dots y_s \quad (3)$$

Similarly, let r be the maximum such that

$$gh = y_1 \dots y_{n-r} b_r x_{r+1} \dots x_n \quad (4)$$

$$\text{Then either (4) is reduced, so } |hg| = 2n - 2r + 1 \quad (5)$$

Or $b_r \in A_0$ and $y_{n-r} b_r x_{r+1}$ is not defined, so $|hg| = 2n - 2r$ (6)

Where $b_r = y_{m-r+1} y_m x_1 \dots x_r$

If (2) and (5) hold, then $2n - 2s + 1 + 2n - 2r + 1 < 2n$

$$2n - 2s - 2r + 2 < 0 \Rightarrow r + s > n - 1$$

$$r > n - s + 1 \text{ and } s > n - r + 1$$

If other cases hold, then it is clear that $r > n - s$ and $s > n - r$

Subcase1: If (2) and (5) hold, then :

$$hg = (y_s \dots y_n)^{-1} y_s \dots y_n x_1 \dots x_{n-s+1} x_{n-s+2} \dots x_n y_1 \dots y_{s-1}$$

$$y_s \dots y_n = (y_s \dots y_n)^{-1} b_{n-s+1} \dots a_{s-1} (y_s \dots y_n)$$

Since $n - s + 1 < r$, then $b_{n-s+1} \in A_0$, and since $a_{s-1} \in A_0$, then $b_{n-s+1} a_{s-1} \in A_0$. So gh is a conjugate of an element of A_0 .

Subcase 2: If (3) and (5) or (6) hold, then

$$gh = (y_{s+1} \dots y_n)^{-1} y_{s+1} \dots y_n x_1 \dots x_{n-s} x_{n-s+1} \dots x_n$$

$$y_1 \dots y_s y_{s-1} \dots y_n = (y_{s+1} \dots y_n)^{-1} b_{n-s} \dots a_s (y_{s+1} \dots y_n)$$

Since $n - s < r$ then $b_{n-s} \in A_0$, and since $a_s \in A_0$, then $b_{n-s} a_s \in A_0$, then gh = conjugate of an element of $b_{n-s} a_s \in A_0$

Subcase 3: If (2) and (6) hold, then

$$gh = (y_{n-r+1} \dots y_n)^{-1} y_{n-r+1} \dots y_n x_1 \dots x_r x_{r+1} \dots x_n y_1 \dots y_{n-r} y_{n-r+1} \dots y_n$$

$$= (y_{n-r+1} \dots y_n)^{-1} b_r a_{n-r} (y_{n-r+1} \dots y_n)$$

Since $b_r a_{n-r} \in A_0$ for $s > n - r$, then gh = conjugate of an element of $b_r a_{n-r} \in A_0$

Therefore, the elements of M are conjugate of zero length elements of $U(P^*)$

Since $n - s + 1 < r$ then $b_{n-s+1} \in A_0$, and since $a_{s-1} \in A_0$, then $b_{n-s+1} a_{s-1} \in A_0$, so gh is a conjugate of an element of A_0 .

Subcase 4 If (3) and (5) or (6) hold, then

$$gh = (y_{s+1} \dots y_n)^{-1} y_{s+1} \dots y_n x_1 \dots x_{n-s} x_{n-s+1} \dots x_n y_1 \dots y_s y_{s+1} \dots y_n$$

$$= (y_{s+1} \dots y_n)^{-1} b_{n-s} \dots a_s (y_{s+1} \dots y_n)$$

Since $n - s < r$ then $b_{n-s} \in A_0$, and since $a_s \in A_0$ then $b_{n-s} a_s \in A_0$.

Therefore, gh = conjugate of an element of $b_{n-s} a_s \in A_0$

Subcase 5: If (2) and (6) hold, then

$$gh = (y_{n-r+1} \dots y_n)^{-1} y_{n-r+1} \dots y_n x_1 \dots x_r x_{r+1} \dots x_n$$

$$y_1 \dots y_{n-r} y_{n-r+1} \dots y_n = (y_{n-r+1} \dots y_n)^{-1} b_r a_{n-r} (y_{n-r+1} \dots y_n)$$

Since $b_r, a_{n-r} \in A_0$ for $s > n - r$ then, $g h$ = conjugate of an element of $b_r a_{n-r} \in A_0$

Therefore, the elements of M are conjugate of zero length elements of $U(P^*)$.

4. CONCLUSION

This paper shows that a special type of pregroups which satisfy an additional condition namely P6 can be occupied with Length Function defined by Lyndon [6]. Therefore it will have all the combinatorial group properties which are open for investigation. This paper particularly proved the following:

- (1) The elements of N consists of conjugates of elements of $P^* \cap N$.
- (2) The equivalent elements of N in $U(P^*)$ are the same conjugates of P^*
- (3) The elements of M are conjugate of elements of length zero in $U(P^*)$.

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