

Eigenvalue problem with the basis exchange algorithm

Abstract. The eigenvalue problem plays an important role in contemporary methods of exploratory data analysis. For example, the *principal component analysis (PCA)* widely used in data exploration, is based on finding the eigenvalues and eigenvectors of the covariance matrix.

The paper presents a new method of the eigenvalue problem solution which uses the basis exchange algorithms. The basis exchange algorithms, similarly to the linear programming techniques are based on the Gauss-Jordan transformation of the inverted matrices. The proposed approach to the eigenvalue problem can be linked to regularization of squared matrices by single unit vectors. This approach is based on inducing a linear dependence among regularized vectors.

Keywords. Eigenvalue problem, data exploration, principal component analysis, basis exchange algorithms, linear dependency.

1 Introduction

Exploratory data analysis is typically aimed at discovering patterns (regularities) in large, multivariate data sets [1]. Patterns can be discovered by using variety of data mining tools [2]. Observed patterns may allow to extract useful knowledge from a given data set. Principal component analysis (*PCA*) is one of the fundamental, widely used method of data exploration [3]. The origins of the Principal Components Analysis (*PCA*) are linked to the works of Pearson (1901) [4], and Hotelling (1933) [5]. One of the best modern reference to *PCA* is provided by Jolliffe (2002) [6].

The principal component analysis (*PCA*) involves computation of the eigenvalues and eigenvectors of the covariance matrix created on the basis of the given data set [7]. Many computational techniques of the eigenvalue problems solutions and their applications in data exploration have been proposed and developed [8]. Apart from *PCA*, the eigenvalue solutions are used in Correspondence Analysis (*CA*), or Canonical Correlation Analysis (*CCA*). However, new computational methods are still developed which could allow to improve the efficiency of large, multi-variate data exploration.

A new method of the eigenvalue problem solution is proposed in the paper. The presented method uses the matrices inversion with the basis exchange algorithm [9]. A crucial role in the proposed solution of the eigenvalue problem plays induced linear dependency among regularized vectors [10]. The considered approach to the eigenvalue problem can be linked to the regularization techniques of squared matrices by unit vectors [7]. The considered approach should be useful, among others, in colinear biclustering based on the flat pattern extraction [11].

2. Eignvalue problem in principal component analysis

Let us consider the data set $\mathbf{C}$ composed of m feature vectors $\mathbf{x}_j = [x_{j,1}, \dots, x_{j,n}]^T$:

$$\mathbf{C} = \{\mathbf{x}_1, \dots, \mathbf{x}_m\} \quad (1)$$

Feature vectors $\mathbf{x}_j$ can be considered as points in the n -dimensional feature space $F[n]$ ($\mathbf{x}_j \in F[n]$).

We assume that m objects O_j ($j = 1, \dots, m$) are represented in a standardazied manner by the n - dimensional feature vectors $\mathbf{x}_j$. Components $x_{j,i}$ of the feature vector $\mathbf{x}_j$ may represent numerical results of measurements of n different features x_i ($i = 1, \dots, n$) of the j - th object O_j ($x_{j,i} \in \{0,1\}$ or $x_{j,i} \in R^1$).

The data set $\mathbf{C}$ (1) can be characterized by the mean vector $\mathbf{m}$ and the covariance matrix $\mathbf{S}$ [3]:

$$\mathbf{m} = \sum_j \mathbf{x}_j / m \quad (2)$$

and

$$\mathbf{S} = \sum_j (\mathbf{x}_j - \mathbf{m})(\mathbf{x}_j - \mathbf{m})^T / (m - 1) \quad (3)$$

The *PCA* exploratory techniques are aimed at identifying unknown trends in multidimensional data sets. The *PCA* method is expected to reduce the dimensionality of multivariate data while preserving of the variance as much as possible.

The basic idea of the *PCA* method is to describe the variation of a set of multivariate data in terms of a set of new uncorrelated variables, each of which is a particular linear combination of the n features x_i . Such a linear transformation of the feature vectors $\mathbf{x}_j$ (1) is applied that the new variables are uncorrelated and have the greatest variability.

The *PCA* method is based on the solution of the eigenvalue problem with the symmetric covariance matrix $\mathbf{S}$ (3) of the dimension $n \times n$ [3]:

$$\mathbf{S} \mathbf{k}_i = \lambda_i \mathbf{k}_i \quad (4)$$

where $\mathbf{k}_i = [k_{i,1}, \dots, k_{i,n}]^T$ is the i -th eigenvector ($i = 1, \dots, n$) and λ_i is the i -th eigenvalue ($\lambda_i \geq 0$).

The eigenvectors $\mathbf{k}_i$ should have the unit length:

$$(\forall i \in \{1, \dots, n\}) \quad \mathbf{k}_i^T \mathbf{k}_i = 1 \quad (5)$$

Two eigenvectors $\mathbf{k}_i$ and $\mathbf{k}_{i'}$ (4) of the symmetrical matrix $\mathbf{S}$ are perpendicular if their eigenvalues λ_i and $\lambda_{i'}$ are different ($\lambda_i \neq \lambda_{i'}$) [6]:

$$(\forall i \neq i') \quad \text{if } \lambda_i \neq \lambda_{i'}, \text{ then } \mathbf{k}_i^T \mathbf{k}_{i'} = 0 \quad (6)$$

The eigenvectors $\mathbf{k}_i$ are usually set according to decreasing eigenvalues λ_i :

$$\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n \quad (7)$$

where

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$$

The eigenvector $\mathbf{k}_i$ defines the i -th principal component y_i :

$$(\forall i \in \{1, \dots, n\}) \quad y_i = \mathbf{k}_i^T \mathbf{x} = k_{i,1}x_1 + \dots + k_{i,n}x_n \quad (8)$$

In accordance with the above transformation, each feature vector $\mathbf{x}_j$ can be projected on the point $y_{i,j}$ of the line y_i (8). It has been shown that the variance σ_i^2 of the vectors $\mathbf{x}_j$ projected on the line y_i (5) is equal to λ_i^2 [6]:

$$(\forall i \in \{1, \dots, n\}) \quad \sigma_i^2 = \sum_j (y_{i,j} - \mu_i)^2 / (m - 1) = \lambda_i^2 \quad (9)$$

where μ_i is the mean value of the points $y_{i,j} = \mathbf{k}_i^T \mathbf{x}_j$ (5).

On the basis of the equation (7) we can see that the first principal components y_i (8) include the greatest parts of the vectors $\mathbf{x}_j$ variability. The covariance $Cov(y_i, y_{i'})$ of the new variables y_i and $y_{i'}$ (8) is equal to zero [6]:

$$(\forall i \neq i') \quad Cov(y_i, y_{i'}) = \sum_j (y_{i,j} - \mu_i)(y_{i',j} - \mu_{i'}) = 0 \quad (10)$$

It follows from the above equation that the new variables y_i and $y_{i'}$ (8) are uncorrelated.

The eigenvalues λ_i and the eigenvectors $\mathbf{k}_i$ are computed on the basis of the equation (4). The equation (4) can be presented in the below form [8]:

$$(\mathbf{S} - \lambda_i \mathbf{I}) \mathbf{k}_i = \mathbf{0} \quad (11)$$

The determinant $|\mathbf{S} - \lambda_i \mathbf{I}|$ equal to zero is the condition necessary for a non-trivial solution $\mathbf{k}_i \neq \mathbf{0}$ of the equation (11):

$$|\mathbf{S} - \lambda_i \mathbf{I}| = 0 \quad (12)$$

The *characteristic equation* (12) is solved in order to find the matrix's $\mathbf{S}$ eigenvalues λ_i . Based on this, the eigenvectors $\mathbf{k}_i$ (7) can be found.

The Singular Value Decomposition (SVD) is currently the primary method of the eigenvalue problem solution for the Principal Components Analysis [8].

3 Iterative matrix inversion

Referring to the equation (11) we introduce the below regularized matrix $\mathbf{S}_\lambda$ with a big value of the parameter λ ($\lambda \in R^1$):

$$\mathbf{S}_\lambda = [\mathbf{s}_1 - \lambda \mathbf{e}_1, \mathbf{s}_2 - \lambda \mathbf{e}_2, \dots, \mathbf{s}_n - \lambda \mathbf{e}_n]^T = [\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_n]^T \quad (13)$$

where $\mathbf{s}_j = [s_{j,1}, \dots, s_{j,n}]^T$ ($\mathbf{s}_j \in R^n$) is the j -th feature vector, $\mathbf{e}_j$ is the j -th unit vector, and $\mathbf{z}_j$ is the regularized vector:

$$(\forall j \in \{1, \dots, n\}) \quad \mathbf{z}_j = \mathbf{s}_j - \lambda \mathbf{e}_j \quad (14)$$

We can assume that the matrix $\mathbf{S}_\lambda$ (13) is created in an iterative manner from the unit matrix $\mathbf{I} = [\mathbf{e}_1, \dots, \mathbf{e}_n]$ [9]. During the first step ($k = 1$) the matrix $\mathbf{S}_\lambda(1) = [\mathbf{z}_1, \mathbf{e}_2, \dots, \mathbf{e}_n]^T$ is created by changing the vector $\mathbf{e}_1$ to $\mathbf{z}_1$. Similarly, in step k , the matrix $\mathbf{S}_\lambda(k)$ is created by changing the vector $\mathbf{e}_k$ to $\mathbf{z}_k$ (14):

$$(\forall k = 1, \dots, n) \quad \mathbf{S}_\lambda(k) = [\mathbf{z}_1, \dots, \mathbf{z}_k, \mathbf{e}_{k+1}, \dots, \mathbf{e}_n]^T \quad (15)$$

We can also assume without limitation that the matrix $\mathbf{S}_\lambda(k)$ is non singular and the inverse matrix $\mathbf{S}_\lambda(k)^{-1}$ can be computed at each step k :

$$\mathbf{S}_\lambda(k)^{-1} = [\mathbf{r}_1(k), \dots, \mathbf{r}_n(k)] \quad (16)$$

The regularized vectors $\mathbf{z}_i$ (14) and the columns $\mathbf{r}_i(k)$ of the inverse matrix $\mathbf{S}_\lambda^{-1}(k)$ (16) fulfil the below equations:

$$\begin{aligned} (\forall i \in \{1, \dots, k\}) \quad \mathbf{z}_i^T \mathbf{r}_i(k) &= \mathbf{r}_i(k)^T \mathbf{z}_i = 1, \text{ and} \\ (\forall i' \in \{1, \dots, k-1\}) \quad \mathbf{z}_i^T \mathbf{r}_{i'}(k) &= \mathbf{r}_i(k)^T \mathbf{z}_{i'} = 0 \end{aligned} \quad (17)$$

If during the next step $k + 1$ the vector $\mathbf{e}_{k+1}$ is replaced by $\mathbf{z}_{k+1}$ (14) then the columns $\mathbf{r}_j(k+1)$ of the inverse matrix $\mathbf{S}_\lambda(k+1)^{-1}$ (16) can be computed on the basis of the columns $\mathbf{r}_j(k)$ (16) in accordance with the Gauss - Jordan transformation [13]:

$$\mathbf{r}_{k+1}(k+1) = (1 / \mathbf{r}_{k+1}(k)^T \mathbf{z}_{k+1}) \mathbf{r}_{k+1}(k) \quad (18)$$

and

$$\begin{aligned} (\forall j \neq (k+1)) \quad \mathbf{r}_j(k+1) &= \mathbf{r}_j(k) - (\mathbf{r}_j(k)^T \mathbf{z}_{k+1}) \mathbf{r}_{k+1}(k+1) = \\ &= \mathbf{r}_j(k) - (\mathbf{r}_j(k)^T \mathbf{z}_{k+1} / \mathbf{r}_{k+1}(k)^T \mathbf{z}_{k+1}) \mathbf{r}_{k+1}(k) \end{aligned} \quad (19)$$

The basis exchange algorithms are based on the Gauss-Jordan transformation (18), (19). The inverse matrices $\mathbf{S}_\lambda(k)^{-1}$ (16) can be computed efficiently in successive steps k by using the basis exchange algorithm even in the case of high dimensional vectors [12].

For a big values of the parameter λ the determinant $|\mathbf{S}_\lambda|$ of the matrix $\mathbf{S}_\lambda$ (13) is different from zero [8]:

$$(\exists \lambda' \in \mathbb{R}^1) \quad |\mathbf{S}_{\lambda'}| \neq 0 \quad (20)$$

If the above condition $|\mathbf{S}_{\lambda'}| \neq 0$ is fulfilled then the n regularized vectors $\mathbf{z}_j$ (14) are linearly independent. Therefore the inverse matrix $\mathbf{S}_{\lambda'}^{-1}$ (13) exists and can be computed by using the basis exchange algorithm [9]. The condition $|\mathbf{S}_{\lambda'}| \neq 0$ (20) causes the trivial solutions $\mathbf{k}_i = \mathbf{0}$ of the eigen equation (11). In order to avoid unconstructive solutions $\mathbf{k}_i = \mathbf{0}$ of the equation (11) the procedure based on inducing linear dependence between the regularized vectors $\mathbf{z}_j$ (14) is proposed.

4 Induced linear dependency among regularized vectors

The sequence of the matrices $\mathbf{S}_\lambda(k)$ (15) can be created in successive steps k ($k = 1, \dots, n$) by changing the vectors $\mathbf{e}_k$ to $\mathbf{z}_k$ (14). Let us assume that the non singular matrix $\mathbf{S}_\lambda(n-1)$ (15) has been created in an iterative manner from the unit matrix $\mathbf{I} = [\mathbf{e}_1, \dots, \mathbf{e}_n]$ during the first $n-1$ steps k :

$$\mathbf{S}_\lambda(n-1) = [\mathbf{z}_1, \dots, \mathbf{z}_{n-1}, \mathbf{e}_n]^T \quad (21)$$

where (14) $(\forall j \in \{1, \dots, n\}) \mathbf{z}_j = \mathbf{s}_j - \lambda \mathbf{e}_j$.

The inverse matrix $\mathbf{S}_\lambda(n-1)^{-1}$ (21) can be represented as:

$$\mathbf{S}_\lambda(n-1)^{-1} = [\mathbf{r}_1(n-1), \dots, \mathbf{r}_n(n-1)] \quad (22)$$

The only one unit vector remaining in the matrix $\mathbf{S}_\lambda(n-1)$ (21) is $\mathbf{e}_n$. Changing the vector $\mathbf{e}_n$ to $\mathbf{z}_n$ should cause the singularity of the matrix $\mathbf{S}_\lambda(n)$ (13). In accordance with the Gauss - Jordan transformation (18) the replacement of the vector $\mathbf{e}_n$ by $\mathbf{z}_n$ can be described as:

$$\mathbf{r}_n(n) = (1 / \mathbf{r}_n(n-1)^T \mathbf{z}_n) \mathbf{r}_n(n-1) \quad (23)$$

The transformation (23) cannot be used if the division by zero occurs:

$$\mathbf{r}_n(n-1)^T \mathbf{z}_n = 0 \quad (24)$$

The condition (24) can also be interpreted geometrically as the move in the parameter space along the parallel hyperplane $h_n = \{\mathbf{w}: \mathbf{z}_n^T \mathbf{w} = 1\}$ [12].

The matrix $\mathbf{S}_\lambda(n) = [\mathbf{s}_1 - \lambda \mathbf{e}_1, \mathbf{s}_2 - \lambda \mathbf{e}_2, \dots, \mathbf{s}_n - \lambda \mathbf{e}_n]^T$ (13) obtained from the non singular matrix $\mathbf{S}_\lambda(n-1)$ (21) becomes singular if the last vector $\mathbf{s}_n - \lambda \mathbf{e}_n$ is a linear combination of the first $n-1$ vectors $\mathbf{s}_j - \lambda \mathbf{e}_j$ ($j = 1, \dots, n-1$) [12]:

$$\mathbf{s}_n - \lambda \mathbf{e}_n = \alpha_{n,1} (\mathbf{s}_1 - \lambda \mathbf{e}_1) + \dots + \alpha_{n,n-1} (\mathbf{s}_{n-1} - \lambda \mathbf{e}_{n-1}) \quad (25)$$

where $\alpha_{n,i} \in \mathbb{R}^1$.

Definition 1: The linear dependency (25) of the regularized vector $\mathbf{z}_n = \mathbf{s}_n - \lambda \mathbf{e}_n$ (14) on the remaining $n-1$ vectors $\mathbf{z}_j = \mathbf{s}_j - \lambda \mathbf{e}_j$ ($j = 1, \dots, n-1$) is *induced* by an adequate choice of the parameter λ .

The vector $\mathbf{r}_n(n-1)$ (22) fulfils the equations (17) of the inverse matrix $\mathbf{S}_\lambda(n-1)^{-1}$:

$$(\forall i \in \{1, \dots, n-1\}) \quad \mathbf{r}_n(n-1)^T (\mathbf{s}_i - \lambda \mathbf{e}_i) = 0 \quad (26)$$

Therefore the equation (24) may be obtained by multiplying both sides of the equation (25) by the vector $\mathbf{r}_n(n-1)$ (22).

Remark 1: The equation (24) results from the linear dependence (25) of the regularized vector $\mathbf{s}_n - \lambda_n \mathbf{e}_n$ on the remaining $n-1$ vectors $\mathbf{s}_j - \lambda_n \mathbf{e}_j$ ($j \neq n$).

The equation (24) allows to formulate the below condition for the parameter λ :

$$\mathbf{r}_n(n-1)^T (\mathbf{s}_n - \lambda_n \mathbf{e}_n) = 0 \quad (27)$$

thus

$$\lambda_n = \mathbf{r}_n(n-1)^T \mathbf{s}_n / \mathbf{r}_n(n-1)^T \mathbf{e}_n = \mathbf{r}_n(n-1)^T \mathbf{s}_n / r_{n,n} \quad (28)$$

where $\mathbf{r}_n(n-1) = [r_{n,1}, \dots, r_{n,n}]^T$ ($\mathbf{r}_n \in R^n$).

The condition (27) can be interpreted geometrically by using the below hyperplane $H(\mathbf{r}_n(n-1), 0)$ defined in the feature space $F[n]$ ($\mathbf{x} \in F[n]$) by the vector $\mathbf{r}_n(n-1)$ (22) obtained from the inverse matrix $\mathbf{S}_\lambda^{-1}(n-1)$ (16) [12]:

$$H(\mathbf{r}_n(n-1), 0) = \{\mathbf{x} : \mathbf{r}_n(n-1)^T \mathbf{x} = 0\} \quad (29)$$

Remark 2: The condition (27) means that the vector $(\mathbf{s}_n - \lambda_n \mathbf{e}_n)$ is located on the hyperplane $H(\mathbf{r}_n(n-1), 0)$ (29).

The hyperplane $H(\mathbf{z}_1, \dots, \mathbf{z}_{n-1})$ can be defined in the feature space $F[n]$ ($\mathbf{x} \in F[n]$) also in a different manner by using $n-1$ regularized vectors $\mathbf{z}_j = \mathbf{s}_j - \lambda_n \mathbf{e}_j$ (14) ($j = 1, \dots, n-1$):

$$H(\mathbf{z}_1, \dots, \mathbf{z}_{n-1}) = \{\mathbf{x} : \mathbf{x} = \alpha_1 (\mathbf{s}_1 - \lambda_n \mathbf{e}_1) + \dots + \alpha_{n-1} (\mathbf{s}_{n-1} - \lambda_n \mathbf{e}_{n-1})\} \quad (30)$$

where $\alpha_i \in R^1$ and the parameter λ_n has been determined by the condition (28).

Definition 2: The hyperplane $H(\mathbf{z}_1, \dots, \mathbf{z}_{n-1})$ (30) is *consistent* with the hyperplane $H(\mathbf{r}_n(n-1), 0)$ (29) if the regularized vector $\mathbf{z}_n = \mathbf{s}_n - \lambda_n \mathbf{e}_n$ (14) is located on $H(\mathbf{z}_1, \dots, \mathbf{z}_{n-1})$ (30):

$$\mathbf{s}_n - \lambda_n \mathbf{e}_n = \alpha_1 (\mathbf{s}_1 - \lambda_n \mathbf{e}_1) + \dots + \alpha_{n-1} (\mathbf{s}_{n-1} - \lambda_n \mathbf{e}_{n-1}) \quad (31)$$

where the parameter λ_n is defined by the equation (28).

5 Iterative matrix inversion aimed at eigenvectors

The eigenvalues λ_i and the eigenvectors $\mathbf{k}_i$ should satisfy the equation (11). This matrix equation can be represented in the feature space $F[n]$ ($\mathbf{x} \in F[n]$) as the below set of n linear equations for each eigenvalue λ_i and eigenvector $\mathbf{k}_i$ ($i = 1, \dots, n$):

$$(\forall j \in \{1, \dots, n\}) \quad (\mathbf{s}_j - \lambda_i \mathbf{e}_j)^T \mathbf{k}_i = \mathbf{k}_i^T (\mathbf{s}_j - \lambda_i \mathbf{e}_j) = 0 \quad (32)$$

where $\mathbf{S} = [\mathbf{s}_1, \dots, \mathbf{s}_n]^T$ (11) and $\mathbf{e}_j$ is the j -th unit vector.

The described earlier calculation scheme have been based on the matrix $\mathbf{S}_{\lambda}(n-1)$ (21). A similar scheme can be based on the below regularized matrices $\mathbf{S}_{\lambda}^i(n-1)$:

$$\mathbf{S}_{\lambda}^i(n-1) = [\mathbf{z}_1, \dots, \mathbf{z}_{i-1}, \mathbf{e}_i, \mathbf{z}_{i+1}, \dots, \mathbf{z}_n]^T \quad (33)$$

where $(\forall j \in \{1, \dots, n\}) \quad \mathbf{z}_j = \mathbf{s}_j - \lambda_i' \mathbf{e}_j$ (14), and λ_i' is a current estimate of the i -th eigenvalue λ_i (for example $\lambda_i' = \lambda_0$).

The matrix $(\mathbf{S}_{\lambda}^i(n-1))^{-1}$ inverse to (33) can be represented similarly to (22):

$$(\mathbf{S}_{\lambda}^i(n-1))^{-1} = [\mathbf{r}_1^i(n-1), \dots, \mathbf{r}_i^i(n-1), \dots, \mathbf{r}_n^i(n-1)] \quad (34)$$

The vector $\mathbf{r}_i^i(n-1)$ (34) fulfils the equations (26) of the inverse matrix $S_\lambda(n-1)^{-1}$:

$$(\forall j \in \{1, \dots, n: j \neq i\}) \quad \mathbf{r}_i^i(n-1)^T(\mathbf{s}_j - \lambda \mathbf{e}_j) = 0 \quad (35)$$

By comparing the equations (35) with (32) we can note that only one equation (35) is misising for equalizing the eigenvector $\mathbf{k}_i$ (32) with the vector $\mathbf{r}_i^i(n-1) / \|\mathbf{r}_i^i(n-1)\|$. As a result, the vector $\mathbf{r}_i^i(n-1) / \|\mathbf{r}_i^i(n-1)\|$ can be treated as an approximation of the i -th eigenvector $\mathbf{k}_i$ of the matrix $\mathbf{S}$ (11).

The prognosed value λ_i^p of the i -th eigenvalue λ_i can be computed as (28) but with using the i -th column $\mathbf{r}_i^i(n-1)$ of the matrix $(S_\lambda^i(n-1))^{-1}$ (34):

$$\lambda_i^p = \mathbf{r}_i^i(n-1)^T \mathbf{s}_i / \mathbf{r}_i^i(n-1)^T \mathbf{e}_i = \mathbf{r}_i^i(n-1)^T \mathbf{s}_i / r_{i,i} = 0 \quad (36)$$

Theorem 1: If the prognosed value λ_i^p (36) is equal to the preliminary value λ_0 used in the definition of the regularized vectors $\mathbf{z}_j = \mathbf{s}_j - \lambda_0 \mathbf{e}_j$ (14) constituting the non singular matrix $\mathbf{S}_\lambda^i(n-1)$ (34):

$$\lambda_i^p = \lambda_0 \quad (37)$$

Then the i -th eigenvalue λ_i of the matrix $\mathbf{S} = [\mathbf{s}_1, \dots, \mathbf{s}_n]^T$ (11) is equal to λ_i^p (36) and is linked to the below eigenvector $\mathbf{k}_i$ (5):

$$\mathbf{k}_i = \mathbf{r}_i^i(n-1) / \|\mathbf{r}_i^i(n-1)\| \quad (38)$$

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where is i -th column $\mathbf{r}_i^i(n-1)$ of the matrix $(S_\lambda^i(n-1))^{-1}$ (34).

The thesis of this theorem results directly from the construction described previously.

Remark 3: The consistency (*Definition 2*) of the hyperplane $H(\mathbf{z}_1, \dots, \mathbf{z}_{n-1})$ (30) with the hyperplane $H(\mathbf{r}_n(n-1), 0)$ (29) can be linked to the assumptions of the Theorem 1.

6 Fitting parameter λ

The problem of fitting the prognosed parameters λ_i^p (36) on the basis of the consistency condition (37) is now analysed. Let us consider at the beginning fitting the parameters λ_i^p (37) in an iterative procedure.

The proposed iterative procedure is based on the *Theorem 1*. The procedure starts with inversion of the matrix $\mathbf{S}_\lambda(n)$ composed of the n regularized vectors $\mathbf{z}_j = \mathbf{s}_j - \lambda_0 \mathbf{e}_j$ (14), where λ_0 is a large value of the parameter λ ($\lambda_0 \in \mathbb{R}^1$). A large value λ_0 of the parameter λ gives the nonsingular matrix $\mathbf{S}_\lambda(n)$ [7]:

$$\begin{aligned} \mathbf{S}_\lambda(n) &= [\mathbf{z}_1, \dots, \mathbf{z}_i, \dots, \mathbf{z}_n]^T = \\ &= [\mathbf{s}_1 - \lambda_0 \mathbf{e}_1, \dots, \mathbf{s}_i - \lambda_0 \mathbf{e}_i, \dots, \mathbf{s}_n - \lambda_0 \mathbf{e}_n]^T \end{aligned} \quad (39)$$

The inverse matrix $\mathbf{S}_\lambda(n)^{-1}$ is represented below as:

$$\mathbf{S}_\lambda(n)^{-1} = [\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_n] \quad (40)$$

The rows $\mathbf{z}_i$ (39) and the columns $\mathbf{r}_i$ of the matrix $\mathbf{S}_\lambda^{-1}(n)$ (40) fulfil the inverse equations (17).

As a result of the replacements of the n unit vectors $\mathbf{e}_j$ by the vectors regularized $\mathbf{z}_j = \mathbf{s}_j - \lambda_0 \mathbf{e}_j$ the complete inverse matrix $\mathbf{S}_\lambda(n)^{-1}$ (40) can be obtained. The temporary replacement of the vector $\mathbf{z}_i = \mathbf{s}_i - \lambda_0 \mathbf{e}_i$ by the unit vector $\mathbf{e}_i$ in the matrix $\mathbf{S}_\lambda(n)$ (39) allows to compute efficiently the i -th column $\mathbf{r}_i^i(n-1)$ of the matrix $(\mathbf{S}_\lambda^i(n-1))^{-1}$ (34) in accordance with the Gauss - Jordan transformation (17):

$$\mathbf{r}_i^i(n-1) = (1 / \mathbf{r}_i^T \mathbf{e}_i) \mathbf{r}_i = (1 / r_{ii}) \mathbf{r}_i \quad (41)$$

where $\mathbf{r}_i = [r_{i,1}, \dots, r_{i,n}]^T$ is the i -th column $\mathbf{r}_i$ of the matrix $\mathbf{S}_\lambda(n)^{-1}$ (40).

The i -th column $\mathbf{r}_i^i(n-1)$ (41) of the matrix $(\mathbf{S}_\lambda^i(n-1))^{-1}$ (34) allows to compute the prognosis λ_i^p (36) for each eigenvalue λ_i :

$$(\forall i \in \{1, \dots, n\}) \quad \lambda_i^p = \mathbf{r}_i^i(n-1)^T \mathbf{s}_i / \mathbf{r}_i^i(n-1)^T \mathbf{e}_i = \mathbf{r}_i^T \mathbf{s}_i / r_{i,i} \quad (42)$$

The proposed iterative procedure is based on the comparisons of the prognosed value λ_i^p (42) with the actual value λ_0 used in the definition of the regularized vectors $\mathbf{z}_j = \mathbf{s}_j - \lambda_0 \mathbf{e}_j$ (14). The below rules based on comparison of the difference $|\lambda_i^p - \lambda_0|$ with a small parameter ε ($\varepsilon \geq 0$) are proposed:

$$\text{if } |\lambda_i^p - \lambda_0| > \varepsilon, \text{ then } \lambda_0 = \lambda_i^p, (\forall i \in \{1, \dots, n\}) \mathbf{z}_i = \mathbf{s}_i - \lambda_0 \mathbf{e}_i, \text{ and} \quad (43)$$

the new matrix $S_{\lambda^i}(n-1)^{-1}$ (34) is designed from the vectors $\mathbf{z}_i$

$$\text{if } |\lambda_i^p - \lambda_0| \leq \varepsilon, \text{ then } \lambda_i = \lambda_i^p, \text{ and the procedure is stopped.} \quad (44)$$

In accordance with above rules the iterative procedure is stopped if the difference $|\lambda_i^p - \lambda_0|$ becomes small. In this case the last prognosed value λ_i^p (42) can play the role of the i -th eigenvalue λ_i .

The proposed iterative procedure could give a rather approximate than exact values of eigenvalues λ_i . But even approximate solution λ_i^p (44) can be useful in some practical issues. For example, the solution λ_i^p (44) can be used as a starting point for exact computations.

7 Fitting eigenvalues through inducing linear dependency

Alternative procedure of the eigenvalues λ_i computation is considered here. The proposed procedure is not iterative and is linked directly to the induced linear dependency (25) between the regularized vectors $\mathbf{s}_j - \lambda \mathbf{e}_j$ (14). The equation (25) describes the linear dependency of the vector $\mathbf{s}_n - \lambda \mathbf{e}_n$ on the remaining $n - 1$ vectors $\mathbf{s}_j - \lambda \mathbf{e}_j$ ($1 \leq j \leq n - 1$). Similarly to (25) equation can be formulated for each of the vectors $\mathbf{s}_j = \mathbf{s}_i - \lambda \mathbf{e}_i$ ($i = 1, \dots, n$):

$$(\forall i \in \{1, \dots, n\}) \quad \mathbf{s}_i - \lambda \mathbf{e}_i = \alpha_{i,1}(\mathbf{s}_1 - \lambda \mathbf{e}_1) + \dots + \alpha_{n,n}(\mathbf{s}_n - \lambda \mathbf{e}_n) \quad (45)$$

where $\mathbf{s}_i = [s_{i,1}, \dots, s_{i,n}]^T$ ($\mathbf{s}_i \in R^n$), ($\forall j \in \{1, \dots, n\}$) $\alpha_{i,j} \in R^1$, and $\alpha_{i,i} = 0$.

The equations (45) can be transformed in the below manner:

$$(\forall i \in \{1, \dots, n\}) \quad (46)$$

$$\mathbf{s}_i - \alpha_{i,1} \mathbf{s}_1 - \dots - \alpha_{i,n} \mathbf{s}_n = \lambda (\mathbf{e}_i - \alpha_{i,1} \mathbf{e}_1 - \dots - \alpha_{i,n} \mathbf{e}_n)$$

where $\alpha_{i,i} = 0$.

Through multiplying both sides of the equality (46) for particular values of the index i by unit vectors $\mathbf{e}_j$ in sequence ($j = 1, \dots, n$) we obtain the below set of equations:

$$(\forall i \in \{1, \dots, n\}) \quad (47)$$

$$\begin{aligned} s_{i,1} - \alpha_{i,1}s_{1,1} - \dots - \alpha_{i,n}s_{n,1} &= -\lambda \alpha_{i,1} \\ s_{i,2} - \alpha_{i,1}s_{1,2} - \dots - \alpha_{i,n}s_{n,2} &= -\lambda \alpha_{i,2} \\ &\dots \\ &\dots \\ s_{i,i} - \alpha_{i,1}s_{1,i} - \dots - \alpha_{i,n}s_{n,i} &= \lambda \\ &\dots \\ &\dots \\ s_{i,n-1} - \alpha_{i,1}s_{1,n-1} - \dots - \alpha_{i,n}s_{n,n-1} &= -\lambda \alpha_{i,n-1} \\ s_{i,n} - \alpha_{i,1}s_{1,n} - \dots - \alpha_{i,n}s_{n,n} &= -\lambda \alpha_{i,n} \end{aligned}$$

For each value of the index i (for each regularized vector $\mathbf{z}_j = \mathbf{s}_j - \lambda \mathbf{e}_j$ (14)) the set of n equations (47) contains n unknowns: λ and $\alpha_{i,1}, \dots, \alpha_{i,n}$ (without the coefficient $\alpha_{i,i}$ which is equal to zero ($\alpha_{i,i} = 0$)). The $n - 1$ equations (47) with $j \neq i$ can be transformed in the below manner:

$$(\forall i \in \{1, \dots, n\}) \quad (48)$$

$$(\forall j \in \{1, \dots, n: j \neq i\})$$

$$\alpha_{i,j} = - (s_{i,j} - \alpha_{i,1} s_{1,j} - \dots - \alpha_{i,n} s_{n,j}) / \lambda$$

As a result, the i - th equation in the system (47) can be provided as:

$$\begin{aligned} \lambda = s_{i,i} + (s_{i,1} - \alpha_{i,1}s_{1,1} - \dots - \alpha_{i,n} s_{n,1}) s_{1,i} / \lambda + \dots \\ + (s_{i,n} - \alpha_{i,1}s_{1,n} - \dots - \alpha_{i,n} s_{n,n}) s_{n,i} / \lambda \end{aligned} \quad (49)$$

And we obtain the below quadratic equations for the unknown parameter λ_i :

$$(\forall i \in \{1, \dots, n\}) \quad \lambda^2 - s_{i,i} \lambda + C_i = 0 \quad (50)$$

where $\mathbf{S} = [\mathbf{s}_1, \dots, \mathbf{s}_n]^T$ (11), $(\forall i \in \{1, \dots, n\}) \mathbf{s}_i = [s_{i,1}, \dots, s_{i,n}]^T$, and

$$\begin{aligned} C_i = & \alpha_{i,1}(s_{1,1} s_{1,i} + \dots + s_{1,n} s_{n,i}) + \\ & \dots + \\ & \dots + \\ & + \alpha_{i,n}(s_{n,1} s_{1,i} + \dots + s_{n,n} s_{n,i}) - \\ & - s_{i,1} s_{1,i} - \dots - s_{i,n} s_{n,i} \end{aligned} \quad (51)$$

The solutions λ_i' of the i -th equation (50) can be used in designing of regularized vectors $\mathbf{z}_j(i)$ (14):

$$(\forall j \in \{1, \dots, n\}) \quad \mathbf{z}_j(i) = \mathbf{s}_j - \lambda_i' \mathbf{e}_j \quad (52)$$

The matrix $\mathbf{S}_{\lambda}^i(n-1)$ (34) can be composed from the $n-1$ regularized vectors $\mathbf{z}_j(i) = \mathbf{s}_j - \lambda_i' \mathbf{e}_j$ (52) and the unit vector $\mathbf{e}_i$ (52):

$$\mathbf{S}_{\lambda}^i(n-1) = [\mathbf{z}_1(i), \dots, \mathbf{z}_{i-1}(i), \mathbf{e}_i, \mathbf{z}_{i+1}(i), \dots, \mathbf{z}_n(i)]^T \quad (53)$$

If the matrix $\mathbf{S}_{\lambda}^i(n-1)$ (53) is not singular the inverse matrix $(\mathbf{S}_{\lambda}^i(n-1))^{-1}$ exists and can be represented as (34).

Example 1: Let us consider the vectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ in the three dimensional feature space R^3 :

$$(\forall j \in \{1, 2, 3\}) \quad \mathbf{s}_j = [s_{j,1}, s_{j,2}, s_{j,3}]^T \in R^3 \quad (54)$$

The three regularized vectors $\mathbf{z}_j$ (14) are introduced by using the unit vectors $\mathbf{e}_j$:

$$(\forall j \in \{1, 2, 3\}) \quad \mathbf{z}_j = \mathbf{s}_j - \lambda \mathbf{e}_j. \quad (55)$$

where $\lambda \in R^1$.

The equation of the induced linear dependency (45) of the vector $\mathbf{z}_3$ on the vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ can be given in the below form:

$$\mathbf{s}_3 - \lambda \mathbf{e}_3 = \alpha_{3,1} (\mathbf{s}_1 - \lambda \mathbf{e}_1) + \alpha_{3,2} (\mathbf{s}_2 - \lambda \mathbf{e}_2) \quad (56)$$

or (46)

$$\mathbf{s}_3 - \alpha_{3,1} \mathbf{s}_1 - \alpha_{3,2} \mathbf{s}_2 = \lambda (\mathbf{e}_3 - \alpha_{3,1} \mathbf{e}_1 - \alpha_{3,2} \mathbf{e}_2) \quad (57)$$

where $\alpha_{j,i} \in \mathbb{R}^1$.

By multiplying both sides of the equation (57) by unit vectors $\mathbf{e}_j$ we obtain three scalar equations:

$$\begin{aligned} s_{3,1} - \alpha_{3,1} s_{1,1} - \alpha_{3,2} s_{2,1} &= -\lambda \alpha_{3,1} \\ s_{3,2} - \alpha_{3,1} s_{1,2} - \alpha_{3,2} s_{2,2} &= -\lambda \alpha_{3,2} \\ s_{3,3} - \alpha_{3,1} s_{1,3} - \alpha_{3,2} s_{2,3} &= \lambda \end{aligned} \quad (58)$$

with the unknown parameters $\alpha_{3,1}$, $\alpha_{3,2}$ and λ .

By eliminating the parameter λ we obtain two quadratic equations:

$$\begin{aligned} s_{3,1} - \alpha_{3,1} s_{1,1} - \alpha_{3,2} s_{2,1} &= - (s_{3,3} - \alpha_{3,1} s_{1,3} - \alpha_{3,2} s_{2,3}) \alpha_{3,1} \\ s_{3,2} - \alpha_{3,1} s_{1,2} - \alpha_{3,2} s_{2,2} &= - (s_{3,3} - \alpha_{3,1} s_{1,3} - \alpha_{3,2} s_{2,3}) \alpha_{3,2} \end{aligned} \quad (59)$$

The unknown parameter $\alpha_{3,1}$ can be extracted from the second equation:

$$\begin{aligned} \alpha_{3,1} &= [s_{3,2} - s_{2,2} \alpha_{3,2} - s_{3,3} \alpha_{3,2} + s_{2,3} \alpha_{3,2}] / (s_{1,2} + s_{1,3} / \alpha_{3,2}) = \\ &= [s_{3,2} - \alpha_{3,2} (s_{2,2} + s_{3,3} - s_{2,3})] / (s_{1,2} \alpha_{3,2} + s_{1,3}) / \alpha_{3,2} = \\ &= [s_{3,2} \alpha_{3,2} - (s_{2,2} + s_{3,3} - s_{2,3}) \alpha_{3,2}^2] / (s_{1,2} \alpha_{3,2} + s_{1,3}) \end{aligned} \quad (60)$$

By substituting $\alpha_{3,1}$ (60) to the first equation (59) we obtain the equation for the unknown parameter $\alpha_{3,2}$.

Theorem 2: If the matrix $\mathbf{S}_\lambda^{(n-1)}$ (53) is not singular, then the solution λ_i' of the i -th system of n equations (47) is equal to the i -th eigenvalue λ_i (4) of the matrix $\mathbf{S} = [\mathbf{s}_1, \dots, \mathbf{s}_n]^T$ (11):

$$\lambda_i = \lambda_i' \quad (61)$$

The i -th eigenvalue λ_i is linked to the below eigenvector $\mathbf{k}_i$ (38):

$$\mathbf{k}_i = \mathbf{r}_i^i(n-1) / \|\mathbf{r}_i^i(n-1)\| \quad (62)$$

where $\mathbf{r}_i^i(n-1)$ is the i -th column of the matrix $(S_{\lambda}^i(n-1))^{-1}$ (34) inverse to the matrix $S_{\lambda}^i(n-1)$ (53).

Proof: If the parameter λ_i' is the solution of the i -th system of n equations (47), then the i -th vector $\mathbf{z}_i(i) = \mathbf{s}_i - \lambda_i' \mathbf{e}_i$ is the linear combination (45) of the remaining $n-1$ vectors $\mathbf{z}_j(i) = \mathbf{s}_j - \lambda_i' \mathbf{e}_j$ ($j \neq i$).

If the matrix $S_{\lambda}^i(n-1)$ (53) is not singular, then the inverse matrix $(S_{\lambda}^i(n-1))^{-1}$ can be computed by using the iterative procedure based on the Gauss - Jordan transformation (17), (18) [12]. The i -th column $\mathbf{r}_i^i(n-1)$ of the inverse matrix $(S_{\lambda}^i(n-1))^{-1}$ (34) fulfils the $n-1$ equations (17):

$$(\forall j \in \{1, \dots, n: j \neq i\}) \quad \mathbf{r}_i^i(n-1)^T \mathbf{z}_j(i) = \mathbf{r}_i^i(n-1)^T (\mathbf{s}_j - \lambda_i' \mathbf{e}_j) = 0 \quad (63)$$

The solution λ_i' of the i -th system (47) allows to fulfill the equation (63) also by the i -th vector $\mathbf{z}_i(i) = \mathbf{s}_i - \lambda_i' \mathbf{e}_i$:

$$\mathbf{r}_i^i(n-1)^T \mathbf{z}_i(i) = \mathbf{r}_i^i(n-1)^T (\mathbf{s}_i - \lambda_i' \mathbf{e}_i) = 0 \quad (64)$$

The above equality results from the linear dependency (45) of the i -th vector $\mathbf{z}_i(i) = \mathbf{s}_i - \lambda_i' \mathbf{e}_i$ (52) on the remaining $n-1$ vectors $\mathbf{z}_j(i) = \mathbf{s}_j - \lambda_i' \mathbf{e}_j$ ($j \neq i$).

Taking into account the equations (63) and (64) we realize that the parameter λ_i' obtained from the equation (50) is the eigenvalue λ_i of the matrix $\mathbf{S} = [\mathbf{s}_1, \dots, \mathbf{s}_n]^T$ (11) with the eigenvector $\mathbf{k}_i$ given by (62). $\square$

Let us remark that the indexes i of the eigenvalues λ_i and the eigenvectors $\mathbf{k}_i$ (62) used by us are the same as the indexes of the unit vectors $\mathbf{e}_i$ in the regularized vectors $\mathbf{z}_i = \mathbf{s}_i - \lambda \mathbf{e}_i$ (14). But of course, the commonly used indexing (7) can be also implemented.

If the solutions λ_i' of the n systems of equations (47) are different then they can be ordered as follows (7):

$$\lambda_{i(1)}' > \lambda_{i(2)}' > \dots > \lambda_{i(n)}' \quad (65)$$

The following remark about the assumptions of the *Theorem 2* can be formulated on this base:

Remark 4: If the solutions λ_i' of particular sets of equations (47) are different (65) then each regularized matrix $\mathbf{S}_{\lambda}^i(n-1)$ (53) designed from $n-1$ vectors $\mathbf{z}_j(i) = \mathbf{s}_j - \lambda_i' \mathbf{e}_j$ (52) and the i -th unit vector $\mathbf{e}_i$ is not singular.

Justification for this remark may be given on the basis of the consistency of the hyperplanes $H(\mathbf{z}_1, \dots, \mathbf{z}_{n-1})$ (30) and $H(\mathbf{r}_n(n-1), 0)$ (29) defined by the parameter λ_i' (*Definition 2*).

8 Repeated eigenvalues

Let us consider again the sequence of the nonsingular matrices $\mathbf{S}_{\lambda}^i(k)$ (15) created in successive steps k ($k = 1, \dots, n-1$) from the unit matrix $\mathbf{I} = [\mathbf{e}_1, \dots, \mathbf{e}_n]$ through changing the unit vectors $\mathbf{e}_k$ into the regularized vectors $\mathbf{z}_k(i)$ (52) [9]. It is assumed here that the regularized vectors $\mathbf{z}_k(i) = \mathbf{s}_k - \lambda_i' \mathbf{e}_k$ are defined by the solution λ_i' of the i -th system of the n equations (47).

In accordance with the *Remark 4*, all $n-1$ unit vectors $\mathbf{e}_k$ different from $\mathbf{e}_i$ ($\mathbf{e}_k \neq \mathbf{e}_i$) can be exchanged into regularized vectors $\mathbf{z}_k(i)$ (52) if the solutions λ_i' of particular sets of equations (47) are different (65). In this case, the resulting matrix $\mathbf{S}_{\lambda}^i(n-1)$ (53) is not singular. In accordance with the *Theorem 2*, the i -th eigenvalue λ_i (4) of the matrix $\mathbf{S} = [\mathbf{s}_1, \dots, \mathbf{s}_n]^T$ (11) is equal in this case to the solution λ_i' (61) and the eigenvector $\mathbf{k}_i$ is determined (62) by the i -th column of the inverse matrix $(\mathbf{S}_{\lambda}^i(n-1))^{-1}$ (34).

Let us assume now that the below nonsingular matrix $\mathbf{S}_{\lambda}^i(k)$ contains k regularized basic vectors $\mathbf{z}_j(i) = \mathbf{s}_j - \lambda_i' \mathbf{e}_j$ (52) ($j \in J_k$), and $n-k$ unit vectors $\mathbf{e}_j$. For example:

$$\mathbf{S}_{\lambda}^i(k) = [\mathbf{z}_1(i), \dots, \mathbf{z}_k(i), \mathbf{e}_{k+1}, \dots, \mathbf{e}_n]^T \quad (66)$$

The k' -th unit vector $\mathbf{e}_{k'}$ ($k' \neq i$) in the matrix $\mathbf{S}_{\lambda}^i(k)$ (66) can not be exchanged into the vector $\mathbf{z}_{k'}(i) = \mathbf{s}_{k'} - \lambda_{i'} \mathbf{e}_{k'}$ (52) if the below condition similar to (24) is met:

$$\mathbf{r}_{k'}^i(k)^T (\mathbf{s}_{k'} - \lambda_{i'} \mathbf{e}_{k'}) = 0 \quad (67)$$

where $\mathbf{r}_{k'}^i(k)$ is the k' -th column of the matrix $(\mathbf{S}_{\lambda}^i(k))^{-1}$ (34) inverse to $\mathbf{S}_{\lambda}^i(k)$:

$$(\mathbf{S}_{\lambda}^i(k))^{-1} = [\mathbf{r}_1^i(k), \dots, \mathbf{r}_{k'}^i(k), \dots, \mathbf{r}_n^i(k)] \quad (68)$$

Lemma 1: The regularized vector $\mathbf{z}_{k'}(i) = \mathbf{s}_{k'} - \lambda_{i'} \mathbf{e}_{k'}$ ($k' \neq i$) can not be inserted into the basis $\mathbf{S}_{\lambda}^i(k)$ (66) with preserving this matrix nonsingularity, if and only if this vector $\mathbf{z}_{k'}(i)$ is a linear combination (45) of the k basic vectors $\mathbf{z}_j(k) = \mathbf{s}_j - \lambda_{k'} \mathbf{e}_j$ ($j \in J_k$).

Proof: In accordance with the Gauss - Jordan transformation (18) the regularized vector $\mathbf{z}_{k'}(i) = \mathbf{s}_{k'} - \lambda_{i'} \mathbf{e}_{k'}$ (52) can not be inserted into the matrix $\mathbf{S}_{\lambda}^i(k)$ (66) if the condition (67) occurs. The transformation (18) cannot be used with the condition (67) because the division by zero appears.

The columns $\mathbf{r}_{j'}^i(k)$ ($j' \notin J_k$) of the inversed matrix $(\mathbf{S}_{\lambda}^i(k))^{-1}$ (68) and the basic vectors $\mathbf{z}_j(k) = \mathbf{s}_j - \lambda_{k'} \mathbf{e}_j$ ($j \in J_k$) in the matrix $\mathbf{S}_{\lambda}^i(k)$ (66) fulfill the equations (17):

$$(\forall j' \notin J_k) (\forall j \in J_k) \mathbf{r}_{j'}^i(k)^T (\mathbf{s}_j - \lambda_{k'} \mathbf{e}_j) = 0 \quad (69)$$

So, if the vector $\mathbf{z}_{k'}(i) = \mathbf{s}_{k'} - \lambda_{i'} \mathbf{e}_{k'}$ ($k' \notin J_k$) is a linear combination (45) of the basic vectors $\mathbf{z}_j(k) = \mathbf{s}_j - \lambda_{k'} \mathbf{e}_j$ ($j \in J_k$) then the condition (67) appears. We can also infer that the linear dependency (45) is necessary for the condition (67) appearing [12]. $\square$

Theorem 3: If the k' -th regularized vector $\mathbf{z}_{k'}(i) = \mathbf{s}_{k'} - \lambda_{i'} \mathbf{e}_{k'}$ ($k' \notin J_k$) fulfills the condition (67) then then the k' -th eigenvalue $\lambda_{k'}$ of the matrix $\mathbf{S} = [\mathbf{s}_1, \dots, \mathbf{s}_n]^T$ (11) is equal to $\lambda_{i'}$:

$$\lambda_{k'} = \lambda_{j'} \quad (70)$$

and the k' -th eigenvector $\mathbf{k}_{k'}$ of the matrix $\mathbf{S}$ can be determined as

$$\mathbf{k}_{k'} = \mathbf{r}_{k'}^i(k) / \|\mathbf{r}_{k'}^i(k)\| \quad (71)$$

where $\mathbf{r}_{k'}^i(k)$ is the k' -th column of the inverse matrix $(\mathbf{S}_{\lambda}^k(k))^{-1}$ (67) which is linked to the k' -th unit vector $\mathbf{e}_{k'}$ ($k' \notin J_k$) in the basis $\mathbf{S}_{\lambda}^i(k)$ (66).

Remark 5: Each regularized vector $\mathbf{z}_{k'}(i) = \mathbf{s}_{k'} - \lambda_{i'} \mathbf{e}_{k'}$ ($k' \notin J_k$) which is a linear combination (45) of the k basic vectors $\mathbf{z}_j(k) = \mathbf{s}_j - \lambda_{k'} \mathbf{e}_j$ ($j \in J_k$) allows to determine the k' -th eigenvalue $\lambda_{k'}$ (70) and the eigenvector $\mathbf{k}_{k'}$ (71) of the matrix $\mathbf{S}$.

If none of $n - k$ regularized vector $\mathbf{z}_k(i) = \mathbf{s}_k - \lambda_{i'} \mathbf{e}_k$ ($k \notin J_k$) can be inserted into the basis $\mathbf{S}_{\lambda}^i(k)$ (66), then the repeated eigenvalues $\lambda_{k'} = \lambda_{j'}$ (70) are linked to $n - k$ eigenvectors $\mathbf{k}_{k'}$ (71) of the matrix $\mathbf{S}$.

9 Concluding remarks

The eigenvalue problem (4) has been decomposed in the proposed method into subproblems related to single regularized vectors $\mathbf{s}_i - \lambda \mathbf{e}_i$ ($i = 1, \dots, n$) (14). The equation (45) describes the induced linear dependency of the i -th regularized vector $\mathbf{s}_i - \lambda \mathbf{e}_i$ ($i = 1, \dots, n$) on the remaining $n - 1$ vectors $\mathbf{s}_j - \lambda \mathbf{e}_j$ ($j \neq i$). Each equation (45) of linear dependency of the i -th regularized vector $\mathbf{s}_i - \lambda \mathbf{e}_i$ (14) allows to find the i -th eigenvalue λ_i (54) through solving the quadratic equation (50). In accordance with *Theorem 3* the eigenvalue λ_i (54) allows to determine the eigenvector $\mathbf{k}_i$ (55) on the basis of the i -th column $\mathbf{r}_i^i(n - 1)$ of the inverse matrix $(\mathbf{S}_{\lambda}^i(n - 1))^{-1}$ (34).

Induced linear dependency (45) of the regularized vectors $\mathbf{s}_i - \lambda \mathbf{e}_i$ (14) plays a crucial role in the proposed solution of the eigenvalue problem. The considered approach to the eigenvalue problem can be linked to the regularization techniques of squared matrices by single unit vectors [7]. The presented approach should be useful, among others, in collinear biclustering based on the flat patterns extraction [11].

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