

SOME CONTRIBUTIONS TO THE SOLUTION OF CUBIC EQUATIONS

By

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ABSTRACT

In many applications, the solution of a cubic equation is required. This study improves on the existing methods of solving cubic equations. The general expression for the quadratic factor of a cubic equation is derived. The discriminant of the cubic equation, adopted in this study as D , comprised of two factors, D_1 and D_2 such that $D = D_1^2 - D_2^3$ and the value of D is shown to depend on the nature of the root of a given cubic equation. For a cubic equation to have three distinct real roots, $D < 0$, if $D > 0$, the equation has only one real root, and if $D = 0$, the equation has either two equal real roots or three equal real roots. A required and sufficient condition for a cubic equation to have three equal real roots is shown to be when $D_1 = 0$, $D_2 = 0$ and $D = 0$. It is also established that a cubic equation has two equal real roots if $D_1 \neq 0$, $D_2 \neq 0$, but $D_1^2 = D_2^3$ and $D = 0$. The solution of a cubic equation with three equal real roots is found using a method that depends on the corresponding reduced cubic equation. Finally numerical examples were used to demonstrate the established results.

Key Words: Cubic equation, discriminant, quadratic factor, root, cubic characteristic equations.

1. Introduction

Cubic equation is a special type of polynomial equations that is used in many fields of study. For example, Biney et al (1986) used a cubic equation to predict surface tension and spinodal limits. In time series analysis, characteristic equations emerging from moving average process of order three and autoregressive process of order three are all cubic equations (Box et al, 1994; Chatfield, 1995). Thus, any attempt to determine if a given moving average process of order three is invertible requires the solution of the cubic characteristic equation. Similarly, we need to solve the characteristic equation associated with an autoregressive model of order three in order to determine if it is stationary or not.

Let a , b , c and d be real constants such that $a \neq 0$. Then, a cubic equation in x is given by

$$ax^3 + bx^2 + cx + d = 0 \quad (1.1)$$

For details on solutions of equation (1.1), see Ostrowki (1966), Hahn(2005), Bessels(2006), Mukundan(2010), Weisstein (2014) and Adewumi(2014) among others.

A good number of methods of solving cubic equations have been discussed in the literature. Some of these methods are based on a reduced cubic equations formed by making the substitution $y = x + \frac{b}{3a}$ (Hahn, 2005; Bessels, 2006). This substitution leads to the following reduced cubic equation

$$y^3 + \alpha y + \beta = 0 \quad (1.2)$$

where $\alpha = \frac{2b^2 - 9ac}{9a^2}$ and $\beta = \frac{27a^2d - 2b^3 - 9abc}{27a^3}$. Munkundan (2010) provided the solution of cubic equations in form of (1.2). Notably, his method appears not to be suitable when $\alpha = 0$.

If we divide (1.1) by a , we obtain

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0 \quad (1.3)$$

Formulae for finding the roots of cubic equations usually depend on what value the discriminant takes. Following (Adewumi, 2014), the discriminant for determining the nature of the roots of (1.3) takes the form

$$D = D_1^2 - D_2^3 \quad (1.4)$$

where

$$D_1 = \frac{2\left(\frac{b}{a}\right)^3 - 9\left(\frac{b}{a}\right)\left(\frac{c}{a}\right) + 27\left(\frac{d}{a}\right)}{54} \quad (1.5)$$

and

$$D_2 = \frac{\left(\frac{b}{a}\right)^2 - 3\left(\frac{c}{a}\right)}{9} \quad (1.6)$$

where D can be positive, negative or zero as pointed out by Dickson (1914).

There is no question that there are already methods of solving cubic equations as contained in mathematical science literature, however, the overall aim of this paper is to improve on the existing methods of solution. For this purpose the subsequent part of this paper is organized as follows; we give the formulae for obtaining the roots of cubic equations based on the corresponding values of the discriminants as well as numerical examples in section 2 while the conclusion and references are contained in sections three and four, respectively.

2. Finding the roots of cubic equations

Prior to the contribution of this paper, it is important to mention that Cardano(1545) established a formula called Cardano's formula for finding the roots of (1.1). The Cardano's formula for finding the roots of (1.1) as contained in Janson(2010) is given by

$$-\frac{b}{3} + \sqrt[3]{-\frac{q}{2} + \sqrt{-\frac{\Delta}{108}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{-\frac{\Delta}{108}}} \quad (2.1)$$

where

$$\left. \begin{aligned} p &= c - \frac{1}{3}b^3 \\ q &= d - \frac{1}{3}bc + \frac{2}{27}b^3 \text{ and} \\ \Delta &= b^2c^2 - 4c^3 - 4b^3d + 18bcd - 27d^2 = -4p^3 - 27q^2 \end{aligned} \right\} \quad (2.2)$$

However, it also important to mention that the Cardano's formula as noted by Janson (2010) does not handle correctly the exceptional case when $p = q = 0$, hence the need for a robust method of solution and this partly the essence of this paper.

As noted earlier, the nature of the roots of a cubic equation depends on whether the associated discriminant is positive, negative or zero. Three cases are discussed in this section.

Case I: Root of a cubic equation when $D < 0$

Let , x_1 , x_2 and x_3 denote the roots of a cubic equation. If $D < 0$, then Equation (1.1) has three distinct real roots such that (Adewumi, 2014)

$$x_1 = -2\sqrt{D_2} \cos\left(\frac{\theta}{3}\right) - \frac{b}{3}, \quad (2.3)$$

$$x_2 = -2\sqrt{D_2} \cos\left(\frac{\theta + 2\pi}{3}\right) - \frac{b}{3}, \quad (2.4)$$

and

$$x_3 = -2\sqrt{D_2} \cos\left(\frac{\theta - 2\pi}{3}\right) - \frac{b}{3}. \quad (2.5)$$

Note that θ is measured in radians and $\theta = \cos^{-1}\left(\frac{D_1}{\sqrt{D_2^3}}\right)$.

Lemma 1: Let r be a real root of the cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

Then,

Comment [Z1]: Added newly to bring Cardano (1545) who first established the roots of a cubic equation

$$\frac{ax^3 + bx^2 + cx + d}{x - r} = ax^2 + (ar + b)x + ar^2 + br + c \quad (2.6)$$

Proof:

Since r is a real root of $ax^3 + bx^2 + cx + d = 0$, r is therefore a factor of $ax^3 + bx^2 + cx + d$. By long division approach,

$$\frac{ax^3 + bx^2 + cx + d}{x - r} = ax^2 + (ar + b)x + ar^2 + br + c + (ar^3 + br^2 + cr + d)$$

Since r is a root, it is known that $ar^3 + br^2 + cr + d = 0$. Hence,

$$\frac{ax^3 + bx^2 + cx + d}{x - r} = ax^2 + (ar + b)x + ar^2 + br + c.$$

This completes the proof.

Theorem 1: If $r = x_1$ is a real root of the cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

the other two roots are:

$$x_2, x_3 = \frac{-(ar + b) \pm \sqrt{(ar + b)^2 - 4a(ar^2 + br + c)}}{2a} \quad (2.7)$$

Proof:

According to Lemma 1, the quadratic factor of $ax^3 + bx^2 + cx + d$, is $ax^2 + (ar + b)x + (ar^2 + br + c)$ provided $r = x_1$ is a real root of $ax^3 + bx^2 + cx + d = 0$, the other roots are therefore the values of x for which $ax^2 + (ar + b)x + (ar^2 + br + c) = 0$. Using the quadratic formula, we obtain the roots to be

$$x_2, x_3 = \frac{-(ar + b) \pm \sqrt{(ar + b)^2 - 4a(ar^2 + br + c)}}{2a}$$

Q.E.D.

Example 1.

Consider the cubic equation $x^3 - 9x^2 + 26x - 24 = 0$. Here, $a = 1$, $b = -9$, $c = 26$ and $d = -24$.

$$D_1 = \frac{2\left(\frac{b}{a}\right)^3 - 9\left(\frac{b}{a}\right)\left(\frac{c}{a}\right) + 27\left(\frac{d}{a}\right)}{54} = \frac{2(-9)^3 - 9 \times -9 \times 26 + 27 \times -24}{54} = 0,$$

$$D_2 = \frac{\left(\frac{b}{a}\right)^2 - 3\left(\frac{c}{a}\right)}{9} = \frac{(-9)^2 - (3 \times 26)}{9} = \frac{1}{3},$$

and

$$D = D_1^2 - D_2^3 = 0^2 - \left(\frac{1}{3}\right)^3 = -\frac{1}{27}$$

It is therefore certain that the equation above has three distinct roots. These roots are given by

$$x_1 = -2\sqrt{\frac{1}{3}} \cos\left(\frac{\cos^{-1}(0)}{3}\right) - \left(\frac{-9}{3}\right) = 2,$$

$$x_2 = -2\sqrt{\frac{1}{3}} \cos\left(\frac{\cos^{-1}(0) + 2\pi}{3}\right) - \left(\frac{-9}{3}\right) = 4,$$

and

$$x_3 = -2\sqrt{\frac{1}{3}} \cos\left(\frac{\cos^{-1}(0) - 2\pi}{3}\right) - \left(\frac{-9}{3}\right) = 3$$

Alternatively, we may consider one of the roots, say, $r = x_1 = 2$ and use Theorem 1 to find the other two roots. Consequently,

$$\begin{aligned} x_2, x_3 &= \frac{-(ar + b) \pm \sqrt{(ar + b)^2 - 4a(ar^2 + br + c)}}{2a} \\ &= \frac{-(2 - 9) \pm \sqrt{(-7)^2 - 4(4 - 18 + 26)}}{2} = 3, 4 \end{aligned}$$

Therefore, $x_2 = 3$ and $x_3 = 4$.

Example 2

Furthermore, consider the cubic equation, $2x^3 - 11x^2 + 18x - 8 = 0$, whose three distinct real roots are : $x_1 = 2$, $x_2 = 0.719$ and $x_3 = 2.781$ (Stroud and Booth(2007, p. 214)). Here $a = 2$, $b = -11$, $c = 18$ and $d = -8$, therefore

$$D_1 = \frac{2\left(\frac{-11}{2}\right)^3 - 9\left(\frac{-11}{2}\right)\left(\frac{18}{2}\right) + 27\left(\frac{-8}{2}\right)}{54} = \frac{19}{216}$$

$$D_2 = \frac{\left(\frac{-11}{2}\right)^2 - 3\left(\frac{18}{2}\right)}{9} = \frac{13}{36}$$

and

$$D = D_1^2 - D_2^3 = \left(\frac{19}{216}\right)^2 - \left(\frac{13}{36}\right)^3 = -\frac{1836}{46656} < 0.$$

If we consider one of the roots, $r = x_1 = 2$ and use the results of theorem 1, the other two roots are obtained as follows;

$$x_2, x_3 = \frac{-(4-11) \pm \sqrt{(4-11)^2 - 32}}{4} = 0.7192, 2.7808$$

The procedure above can equally be used to find the roots of cubic equations that have two equal real roots as shown in Example 4.

There is no question that the result in theorem (1) is applicable when one of the roots is given or known, however, it can be obtained using the following expression derived from the results of Adewumi(2014)

$$x_1 = r = -\frac{b}{3} - \frac{2\sqrt{b^2 - 3ac}}{3a} \cos \left[\frac{1}{3} \cos^{-1} \left\{ \frac{2b^3 - 9abc + 27a^2d}{2\sqrt{(b^2 - 3ac)^3}} \right\} \right] \quad (2.8)$$

or by simply using the formula by Adewumi(2014) given by

$$x_1 = r = -\frac{b}{3} - 2\sqrt{D_2} \cos \left\{ \frac{1}{3} \cos^{-1} \left(\frac{D_1}{D_2\sqrt{D_2}} \right) \right\} \quad (2.9)$$

Comment [Z2]: Added newly

Case II: Root of a cubic equation when $D > 0$

For $D > 0$, the cubic equation has only one real root given by (Adewumi, 2014)

$$x_1 = S + T - \frac{b}{3} \quad (2.9)$$

where

$$S = \sqrt[3]{-D_1 + \sqrt{D}} \quad (2.10)$$

and

$$T = \sqrt[3]{-D_1 - \sqrt{D}} \quad (2.11)$$

Our Theorem 1 may help us to find the accompanying complex roots of the cubic equations when $D > 0$ as illustrated in Example 3.

Example 3.

If $x^3 + 2x^2 + 3x + 4 = 0$, then $a = 1$, $b = 2$, $c = 3$ and $d = 4$. Thus,

$$D_1 = \frac{2(2)^3 - 9 \times 2 \times 3 + 27 \times 4}{54} = \frac{35}{27},$$

$$D_2 = \frac{(2)^2 - 3 \times 3}{9} = -\frac{5}{9},$$

and

$$D = D_1^2 - D_2^3 = \left(\frac{35}{27}\right)^2 - \left(-\frac{5}{9}\right)^3 = \frac{1350}{729} > 0$$

It is then obvious that the cubic equation has only one real root. For this root, we have

$$x_1 = \sqrt[3]{-\frac{35}{27} + \sqrt{\frac{1350}{729}}} + \sqrt[3]{-\frac{35}{27} - \sqrt{\frac{1350}{729}}} - \left(\frac{2}{3}\right) = -1.6507$$

Consequently,

$$\begin{aligned} x_2, x_3 &= \frac{-(a+b) \pm \sqrt{(a+b)^2 - 4a(ar^2 + br + c)}}{2a} \\ &= \frac{-((-1.6507) + 2) \pm \sqrt{(0.3493)^2 - 4((-1.6507)^2 - 2 \times 1.6507 + 3)}}{2} \\ \Rightarrow x_2 &= -0.1747 - 1.5469i, x_3 = -0.1747 + 1.5469i \end{aligned}$$

Case III: Obtaining the roots of a cubic equation when $D = 0$.

In this section, we discuss the two cases for which $D = 0$. This is when the equation has two equal real roots or when the equation has three equal real roots.

Theorem 2: If a cubic equation has two equal real roots, with the condition that $D_1^2 = D_2^3$ when $D_1 \neq 0$, $D_2 \neq 0$, then $D = 0$. Let the roots of the equation be α , α and β . Then the corresponding equation is

$$x^3 + \left(\frac{b}{a}\right)x^2 + \left(\frac{c}{a}\right)x + \frac{d}{a} = (x - \alpha)^2(x - \beta)$$

$$= x^3 - (2\alpha + \beta)x^2 + (\alpha^2 + 2\alpha\beta)x - \alpha^2\beta = 0$$

Accordingly,

$$D_1 = \frac{2(-(2\alpha + \beta))^2 - 9(-(2\alpha + \beta))(\alpha^2 + 2\alpha\beta) - 27\alpha^2\beta}{54} = \frac{\alpha^3 - 3\alpha^2\beta + 3\alpha\beta^2 - \beta^3}{27}$$

and

$$D_2 = \frac{(2\alpha + \beta)^2 - 3(\alpha^2 + 2\alpha\beta)}{9} = \frac{\alpha^2 - 2\alpha\beta + \beta^2}{9}$$

It is easily seen that $D_1^2 = D_2^3$ and $D = 0$. Q. E. D

Example 4.

Given that $x^3 - 5x^2 + 7x - 3 = 0$, then $a = 1$, $b = -5$, $c = 7$ and $d = -3$. It can be shown that

$$D_1 = \frac{2\left(\frac{b}{a}\right)^3 - 9\left(\frac{b}{a}\right)\left(\frac{c}{a}\right) + 27\left(\frac{d}{a}\right)}{54} = \frac{2(-5)^3 - 9 \times -5 \times 7 + 27 \times -3}{54} = -\frac{8}{27},$$

$$D_2 = \frac{\left(\frac{b}{a}\right)^2 - 3\left(\frac{c}{a}\right)}{9} = \frac{(-5)^2 - 3 \times 7}{9} = \frac{4}{9},$$

and

$$D = D_1^2 - D_2^3 = \left(-\frac{8}{27}\right)^2 - \left(\frac{4}{9}\right)^3 = 0.$$

These roots are as follows

$$x_1 = -2\sqrt{\frac{4}{9}} \cos\left(\frac{\cos^{-1}(-1)}{3}\right) - \left(\frac{-5}{3}\right)$$

$$= 1$$

$$x_2 = -2\sqrt{\frac{4}{9}} \cos\left(\frac{\cos^{-1}(-1) + 2\pi}{3}\right) - \left(\frac{-5}{3}\right)$$

$$= 3$$

$$x_3 = -2\sqrt{\frac{4}{9}} \cos\left(\frac{\cos^{-1}(-1) - 2\pi}{3}\right) - \left(\frac{-5}{3}\right) = 1$$

With $x_1 = 1$, Theorem 1 can also be used to obtain $x_2 = 3$ and $x_3 = 1$.

From the foregoing, it is obvious that not much emphasis has been laid on finding the solution of a cubic equation which has three equal roots. First, we show that $D = 0$ and $D_1 = D_2 = 0$ if and only if the cubic equation has three equal real roots.

Lemma 2: The cubic equation $ax^3 + bx^2 + cx + d = 0$ has three real equal roots if $D_1 = 0$, $D_2 = 0$ and $D = 0$.

Proof:

Let each of the roots of the equation be r . Then

$$x^3 + \left(\frac{b}{a}\right)x^2 + \left(\frac{c}{a}\right)x + \frac{d}{a} = (x - r)^3 = x^3 - 3rx^2 + 3r^2x - r^3 = 0 \quad (2.12)$$

From (2.12), we have

$$D_1 = \frac{-54r^3 + 81r^3 - 27r^3}{54} = 0$$

and

$$D_2 = \frac{9r^2 - 9r^2}{9} = 0.$$

It then follows that $D = 0$. This completes the proof.

The reduced cubic equation corresponding to a cubic equation with three equal roots is given in the following theorem.

Theorem 3: If the cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

has three real equal roots, say r , and the substitution $y = x + \frac{b}{3a}$ (Hahn, 2005; Bessels, 2006) is made, then the corresponding reduced cubic equation is $y^3 = (x - r)^3 = 0$. Hence, $r = \frac{-b}{3a}$.

Proof:

Consider

$$x^3 + \left(\frac{b}{a}\right)x^2 + \left(\frac{c}{a}\right)x + \left(\frac{d}{a}\right) = (x - r)^3 = x^3 - 3rx^2 + 3r^2x - r^3 = 0 \quad (2.13)$$

Here, $y = x - r \Rightarrow x = y + r$ and substituting $x = y + r$ into (2.13) leads to

$$\begin{aligned} (y + r)^3 - 3r(y + r)^2 + 3r^2(y + r) - r^3 &= 0 \\ \Rightarrow y^3 = (x - r)^3 = 0 &\Rightarrow y = x - r = 0 \Rightarrow x = r \end{aligned}$$

We have shown that

$$y = x + \frac{b}{3a} = 0 \Rightarrow x = r = -\frac{b}{3a}. \quad \text{Q.E.D.}$$

Example 5.

For $x^3 - 3x^2 + 3x - 1 = 0$, we have $a = 1$, $b = -3$, $c = 3$ and $d = -1$.

$$D_1 = \frac{2(-3)^3 - 9 \times -3 \times 3 + 27 \times -1}{54} = 0,$$

$$D_2 = \frac{(-3)^2 - 3 \times 3}{9} = 0,$$

and

$$D = D_1^2 - D_2^3 = 0.$$

Since each of D_1, D_2 and D are equal to zero, the cubic equation has three real equal roots. Using Theorem 3, we find that

$$(x - 1)^3 = 0 \Rightarrow r = 1$$

The overall summary of the results of this study is given in Table 1 for the various cases considered.

3. Conclusion

The fundamental contribution of this paper is the establishment of an improved method of finding the solution of a cubic equation using the concept of quadratic factor. The general expression for the quadratic factor of a cubic equation was established. The discriminant of a cubic equation, D , which comprised D_1 and D_2 , such that $D = D_1^2 - D_2^3$ was shown to depend on the nature of the roots of a given equation. Whereas, in mathematical science literature, a cubic equation has: (i) three real roots if $D < 0$ (ii) one real root if $D > 0$ and (iii) either two equal real roots or three equal real roots if $D = 0$, therefore in this paper, the following required and sufficient conditions were established : (i) a cubic equation has three equal real roots if $D_1 = 0$ and $D_2 = 0$ which implies $D = 0$ while (ii) it has two equal roots if $D_1 \neq 0, D_2 \neq 0$ but

$D_1^2 = D_2^3 = 0$ which implies $D = 0$. More so, the general solution for a cubic equation, with three equal real roots were established using the method that depends on the corresponding reduced form. Finally numerical examples were used to backup the established results.

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Comment [Z3]: Added Newly

Comment [Z4]: Added newly

Table 1: Summary of the root of a cubic equation, $ax^3+bx^2+cx+d=0$, for the various cases

Case	Number of roots	Roots
Case I: $D = D_1^2 - D_2^3 < 0$	Three distinct real roots	Here assuming that the first root, $x_1 = r$ $x_2, x_3 = \frac{-(ar+b) \pm \sqrt{(ar+b)^2 - 4a(ar^2+br+c)}}{2a}$
Case II: $D = D_1^2 - D_2^3 > 0$	One real root and two complex roots	$x_1 = \sqrt[3]{-D_1 + \sqrt{D}} + \sqrt[3]{-D_1 - \sqrt{D}} - \frac{b}{3}$ and the two complex roots, x_2 and x_3 can be found using $x_2, x_3 = \frac{-(ar+b) \pm \sqrt{(ar+b)^2 - 4a(ar^2+br+c)}}{2a}$
Case III: (i) $D = D_1^2 - D_2^3 = 0$ where $D_1 \neq 0$ and $D_2 \neq 0$	Two equal real roots	$x_1 = -2\sqrt{D_2} \cos\left(\frac{\theta}{3}\right) - \frac{b}{3}$, $x_2 = -2\sqrt{D_2} \cos\left(\frac{\theta+2\pi}{3}\right) - \frac{b}{3}$ and $x_3 = -2\sqrt{D_2} \cos\left(\frac{\theta-2\pi}{3}\right) - \frac{b}{3}$, where either $x_1 = x_2$ or $x_1 = x_3$.
Case III: (ii) $D = D_1 = D_2 = 0$ where $D_1 \neq 0$ and $D_2 \neq 0$	Three equal real roots	$x_i = -\frac{b}{3a}$, $i=1,2,3$
$D_1 = \frac{2\left(\frac{b}{a}\right)^3 - 9\left(\frac{b}{a}\right)\left(\frac{c}{a}\right) + 27\left(\frac{d}{a}\right)}{54} \quad \text{and} \quad D_2 = \frac{\left(\frac{b}{a}\right)^2 - 3\left(\frac{c}{a}\right)}{9}$		