

Generalized Hyers-Ulam stability of a cubic reciprocal functional equation

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Abstract

In this paper, we obtain the generalized Hyers-Ulam stability of a new cubic reciprocal functional equation of the form

$$f(2x + y) + f(2x - y) = \frac{4f(x)f(y) \left[4f(y) + 3f(x)^{\frac{2}{3}} f(y)^{\frac{1}{3}} \right]}{\left(4f(y)^{\frac{2}{3}} - f(x)^{\frac{2}{3}} \right)^3}$$

in non-Archimedean fields with $f(y) \neq \frac{1}{8}f(x)$. We also extend the results related to Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of powers of norms for the same equation.

Keywords: Reciprocal functional equation; Reciprocal-quadratic functional equation; Generalized Hyers-Ulam stability; non-Archimedean field

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1 Introduction

The classical theory of stability of functional equations is instigated by the question of S.M. Ulam [14] in the year 1940. In the subsequent year, D.H. Hyers [5] was the foremost mathematician to establish the result connected with the stability of functional equations. The result obtained by D.H. Hyers is called as Hyers-Ulam stability of functional equation. Later in the year 1950, T. Aoki [1] made further simplification to the theorem of D.H. Hyers. In the year 1978, Th.M. Rassias [13] took a broad view in the Hyers' result by taking the upper bound as sum of powers of norms. The theorem proved

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by Th.M. Rassias persuaded a lot of mathematicians to work on the stability of various functional equations and the result obtained by Th.M. Rassias is recognized as Hyers-Ulam-Rassias stability of functional equation. J.M. Rassias [6] provided further generalization of the result of D.H. Hyers by using weaker conditions controlled by a product of different powers of norms. The result proved by J.M. Rassias is termed as Ulam-Gavruta-Rassias stability of functional equation. Further, in the year 1994, P. Gavruta [4] provided a generalization of Th.M. Rassias' theorem by replacing a general control function as upper bound. The stability result ascertained by P. Gavruta is celebrated as generalized Hyers-Ulam stability of functional equation. In the year 2008, J.M. Rassias et al. [7] investigated the stability of quadratic functional equation

$$f(mx + y) + f(mx - y) = 2f(x + y) + 2f(x - y) + 2(m^2 - 2)f(x) - 2f(y)$$

for any arbitrary but fixed real constant m with $m \neq 0$; $m \neq \pm 1$; $m \neq \pm\sqrt{2}$ using mixed product-sum of powers of norms. This stability result acquired by J.M. Rassias is known as J.M. Rassias stability involving mixed product-sum of powers of norms.

In the year 2010, K. Ravi and B.V. Senthil Kumar [8] proved Ulam-Gavruta-Rassias stability for a new reciprocal type functional equation

$$f(x + y) = \frac{f(x)f(y)}{f(x) + f(y)} \quad (1.1)$$

where $f : X \rightarrow \mathbb{R}$ is a mapping with X as the space of non-zero real numbers. The reciprocal function $g(x) = \frac{c}{x}$ is a solution of the functional equation (1.1).

The other results pertaining to stability of different reciprocal type functional equations can be found in ([9], [10], [11]).

A. Bodaghi and S.O. Kim [2] introduced and studied the Ulam-Gavruta-Rassias stability for the quadratic reciprocal functional equation

$$f(2x + y) + f(2x - y) = \frac{2f(x)f(y)[4f(y) + f(x)]}{(4f(y) - f(x))^2}. \quad (1.2)$$

The quadratic reciprocal function $f(x) = \frac{c}{x^2}$ is a solution of the functional equation (1.2).

A. Bodaghi and Y. Ebrahimoost [3] generalized the equation (1.2) as

$$f((a + 1)x + ay) + f((a + 1)x - ay) = \frac{2f(x)f(y)[(a + 1)^2f(y) + a^2f(x)]}{((a + 1)^2f(y) - a^2f(x))^2} \quad (1.3)$$

where $a \in \mathbb{Z}$ with $a \neq 0$ and established the generalized Hyers-Ulam-Rassias stability for the functional equation (1.3) in non-Archimedean fields.

K. Ravi et al. [12] investigated the generalized Hyers-Ulam-Rassias stability of a reciprocal-quadratic functional equation of the form

$$r(x + 2y) + r(2x + y) = \frac{r(x)r(y)[5r(x) + 5r(y) + 8\sqrt{r(x)r(y)}]}{[2r(x) + 2r(y) + 5\sqrt{r(x)r(y)}]^2} \quad (1.4)$$

in intuitionistic fuzzy normed spaces.

In this paper, we obtain the generalized Hyers-Ulam stability of a cubic reciprocal functional equation of the form

$$f(2x + y) + f(2x - y) = \frac{4f(x)f(y)[4f(y) + 3f(x)^{\frac{2}{3}}f(y)^{\frac{1}{3}}]}{(4f(y)^{\frac{2}{3}} - f(x)^{\frac{2}{3}})^3} \quad (1.5)$$

in non-Archimedean fields. We extend the results concerning Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of powers of norms for the equation (1.5). It is easy to see that the reciprocal cubic function $f(x) = \frac{1}{x^3}$ is a solution of the functional equation (1.5).

2 Preliminaries

In this section, we recall the basic concepts of non-Archimedean field, non-Archimedean norm and non-Archimedean space.

Definition 2.1. By a non-Archimedean field, we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$ and $|r + s| \leq \max\{|r|, |s|\}$ for all $r, s \in K$.

Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|rx\| = |r|\|x\|$ ($r \in \mathbb{K}, x \in X$);
- (iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean space. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space, we mean that every Cauchy sequence is convergent in the space.

An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial. Another example of a non-Archimedean valuation on a field $\mathbb{A}$ is the mapping

$$|r| = \begin{cases} 0 & \text{if } r = 0 \\ \frac{1}{r} & \text{if } r > 0 \\ -\frac{1}{r} & \text{if } r < 0 \end{cases}$$

for any $r \in \mathbb{A}$.

Let p be a prime number. For any non-zero rational number $x = p^r \frac{m}{n}$ in which m and n are coprime to the prime number p . Consider the p -adic absolute value $|x|_p = p^{-r}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|_p$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ which is denoted by $\mathbb{Q}_p$ is said to be the p -adic number field. Note that if $p > 2$, then $|2^n| = 1$ for all integers n .

3 Generalized Hyers-Ulam stability of equation (1.5)

In this section, we investigate the generalized Hyers-Ulam stability of equation (1.5) in non-Archimedean fields. We also establish the results pertaining to Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by product-sum of powers of norms.

Throughout this section, we consider that $\mathbb{A}$ and $\mathbb{B}$ is a non-Archimedean field and a complete non-Archimedean field, respectively. From now on, for a non-Archimedean field $\mathbb{A}$, we put $\mathbb{A}^* = \mathbb{A} - \{0\}$. For the purpose of simplification, let us define the difference operator $D_f : \mathbb{A}^* \times \mathbb{A}^* \rightarrow \mathbb{B}$ by

$$D_f(x, y) = f(2x + y) + f(2x - y) - \frac{4f(x)f(y) \left[4f(y) + 3f(x)^{\frac{2}{3}}f(y)^{\frac{1}{3}} \right]}{\left(4f(y)^{\frac{2}{3}} - f(x)^{\frac{2}{3}} \right)^3}$$

for all $x, y \in \mathbb{A}^*$.

In the following results, we assume $f(x) \neq 0$ and $x + y \neq 0$ for all $x, y \in \mathbb{R}$.

For proving our main results, we necessitate the following definition.

Definition 3.1. A mapping $f : \mathbb{A}^* \rightarrow \mathbb{B}$ is called as reciprocal cubic mapping if f satisfies equation (1.5).

Theorem 3.2. Let $\varphi : \mathbb{A}^* \times \mathbb{A}^* \rightarrow \mathbb{B}^*$ be a function such that

$$\lim_{n \rightarrow \infty} \left| \frac{1}{27} \right|^n \varphi \left(\frac{x}{3^{n+1}}, \frac{y}{3^{n+1}} \right) = 0 \quad (3.1)$$

for all $x, y \in \mathbb{A}^*$. Suppose that $f : \mathbb{A}^* \rightarrow \mathbb{B}$ is a mapping satisfying the inequality

$$|D_f(x, y)| \leq \varphi(x, y) \quad (3.2)$$

for all $x, y \in \mathbb{A}^*$. Then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \rightarrow \mathbb{B}$ such that

$$|f(x) - R_c(x)| \leq \max \left\{ \left| \frac{1}{27} \right|^j \varphi \left(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}} \right) : j \in \mathbb{N} \cup \{0\} \right\} \quad (3.3)$$

for all $x \in \mathbb{A}^*$.

Proof. Plugging (x, y) by (x, x) in (3.2), we obtain

$$\left| f(3x) - \frac{1}{27} f(x) \right| \leq \varphi(x, x) \quad (3.4)$$

for all $x \in \mathbb{A}^*$. Now, substituting x by $\frac{x}{3}$ in (3.4), we find

$$\left| f(x) - \frac{1}{27} f\left(\frac{x}{3}\right) \right| \leq \varphi\left(\frac{x}{3}, \frac{x}{3}\right) \quad (3.5)$$

for all $x \in \mathbb{A}^*$. Replacing x by $\frac{x}{27^n}$ in (3.5) and multiplying by $\left| \frac{1}{27^n} \right|^n$, we have

$$\left| \frac{1}{27^n} f\left(\frac{x}{3^n}\right) - \frac{1}{27^{n+1}} f\left(\frac{x}{3^{n+1}}\right) \right| \leq \left| \frac{1}{27} \right|^n \varphi\left(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}\right) \quad (3.6)$$

for all $x \in \mathbb{A}^*$. Hence we find that the sequence $\left\{ \frac{1}{27^n} f\left(\frac{x}{3^n}\right) \right\}$ is a Cauchy sequence by (3.1) and (3.6). Since $\mathbb{B}$ is complete, we can define a mapping R_c given by

$$R_c(x) = \lim_{n \rightarrow \infty} \frac{1}{27^n} f\left(\frac{x}{3^n}\right). \quad (3.7)$$

For each $x \in \mathbb{A}^*$ and non-negative integers n , we have

$$\begin{aligned} \left| \frac{1}{27^n} f\left(\frac{x}{3^n}\right) - f(x) \right| &= \left| \sum_{i=0}^{n-1} \left\{ \frac{1}{27^{i+1}} f\left(\frac{x}{3^{i+1}}\right) - \frac{1}{27^i} f\left(\frac{x}{3^i}\right) \right\} \right| \\ &\leq \max \left\{ \left| \frac{1}{27^{i+1}} f\left(\frac{x}{3^{i+1}}\right) - \frac{1}{27^i} f\left(\frac{x}{3^i}\right) \right| : 0 \leq i < n \right\} \\ &\leq \max \left\{ \left| \frac{1}{27} \right|^i \varphi\left(\frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}\right) : 0 \leq i < n \right\}. \end{aligned} \quad (3.8)$$

The inequality (3.3) is true using (3.7) and letting $n \rightarrow \infty$ in the above inequality (3.8). Using (3.1), (3.2) and (3.7), we have for all $x, y \in \mathbb{A}^*$

$$\begin{aligned} |D_{R_c}(x, y)| &= \lim_{n \rightarrow \infty} \left| \frac{1}{27} \right|^n \left| D_f \left(\frac{x}{3^n}, \frac{y}{3^n} \right) \right| \\ &\leq \lim_{n \rightarrow \infty} \left| \frac{1}{27} \right|^n \varphi \left(\frac{x}{3^n}, \frac{y}{3^n} \right) = 0. \end{aligned}$$

Thus the mapping R_c satisfies (1.5) and hence it is cubic reciprocal mapping. In order to prove the uniqueness of R_c , let us consider $R'_c : \mathbb{A}^* \rightarrow \mathbb{B}$ be another cubic reciprocal mapping satisfying (3.3). Then we have

$$\begin{aligned} &|R_c(x) - R'_c(x)| \\ &= \lim_{m \rightarrow \infty} \left| \frac{1}{27} \right|^m \left| R_c \left(\frac{x}{3^m} \right) - R'_c \left(\frac{x}{3^m} \right) \right| \\ &\leq \lim_{m \rightarrow \infty} \left| \frac{1}{27} \right|^m \max \left\{ \left| R_c \left(\frac{x}{3^m} \right) - f \left(\frac{x}{3^m} \right) \right|, \left| f \left(\frac{x}{3^m} \right) - R'_c \left(\frac{x}{3^m} \right) \right| \right\} \\ &\leq \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \max \left\{ \max \left\{ \left| \frac{1}{27} \right|^{i+m} \varphi \left(\frac{x}{3^{i+m+1}}, \frac{x}{3^{i+m+1}} \right) : m \leq i \leq n+m \right\} \right\} \\ &= 0 \end{aligned}$$

for all $x \in \mathbb{A}^*$, which shows that R_c is unique. Hence the proof is complete. $\square$

Theorem 3.3. Let $\varphi : \mathbb{A}^* \times \mathbb{A}^* \rightarrow \mathbb{B}^*$ be a function such that

$$\lim_{n \rightarrow \infty} |27|^n \varphi(3^n x, 3^n x) = 0 \quad (3.9)$$

for all $x, y \in \mathbb{A}^*$. Suppose that $f : \mathbb{A}^* \rightarrow \mathbb{B}$ is a mapping satisfying the inequality (3.2) for all $x, y \in \mathbb{A}^*$. Then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \rightarrow \mathbb{B}$ such that

$$|f(x) - R_c(x)| \leq \max \left\{ |27|^{i+1} \varphi(3^i x, 3^i x) : i \in \mathbb{N} \cup \{0\} \right\} \quad (3.10)$$

for all $x \in \mathbb{A}^*$.

Proof. Switching y to x in (3.2) and multiplying by $|27|$, we find

$$|27f(3x) - f(x)| \leq |27|\varphi(x, x) \quad (3.11)$$

for all $x \in \mathbb{A}^*$. Now, replacing x by $3^n x$ in (3.11) and multiplying by $|27|^n$, we obtain

$$|27^n f(3^n x) - 27^{n+1} f(3^{n+1} x)| \leq |27|^{n+1} \varphi(3^n x, 3^n x) \quad (3.12)$$

for all $x \in \mathbb{A}^*$. It is easy to see that the sequence $\{3^n f(3^n x)\}$ is a Cauchy sequence by letting $n \rightarrow \infty$ in (3.12) and using (3.9). Since $\mathbb{B}$ is complete, this Cauchy sequence converges to a mapping $R_c : \mathbb{A}^* \rightarrow \mathbb{B}$ defined by

$$R_c(x) = \lim_{n \rightarrow \infty} 27^n f(3^n x). \quad (3.13)$$

For each $x \in \mathbb{A}^*$ and non-negative integers n , we have

$$\begin{aligned} |27^n f(3^n x) - f(x)| &= \left| \sum_{i=0}^{n-1} 27^{i+1} f(3^{i+1} x) - 27^i f(3^i x) \right| \\ &\leq \max \left\{ |27^{i+1} f(3^{i+1} x) - 27^i f(3^i x)| : 0 \leq i < n \right\} \\ &\leq \max \left\{ |27|^{i+1} \varphi(3^i x, 3^i x) : 0 \leq i < n \right\}. \end{aligned} \quad (3.14)$$

Applying (3.13) and letting $n \rightarrow \infty$, we find that the inequality (3.10) holds. From (3.9), (3.2) and (3.13), we have for all $x, y \in \mathbb{A}^*$

$$\begin{aligned} |D_{R_c}(x, y)| &= \lim_{n \rightarrow \infty} |27|^n |D_f(3^n x, 3^n y)| \\ &\leq \lim_{n \rightarrow \infty} |27|^n \varphi(3^n x, 3^n y) = 0. \end{aligned}$$

Hence the mapping R_c satisfies (1.5) and it is cubic reciprocal mapping. Now, let $R'_c : \mathbb{A}^* \rightarrow \mathbb{B}$ be another cubic reciprocal mapping satisfying (1.5). Then we have

$$\begin{aligned} |R_c(x) - R'_c(x)| &= \lim_{m \rightarrow \infty} |27|^m |R_c(3^m x) - R'_c(3^m x)| \\ &\leq \lim_{m \rightarrow \infty} |27|^m \max \{ |R_c(3^m x) - f(3^m x)|, |f(3^m x) - R'_c(3^m x)| \} \\ &\leq \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \max \left\{ \max \left\{ |27|^{i+m+1} \varphi(3^{i+m} x, 3^{i+m} x) : m \leq i \leq n+m \right\} \right\} \\ &= 0 \end{aligned}$$

for all $x \in \mathbb{A}^*$, which proves that R_c is unique. $\square$

Theorem 3.4. Let $f : \mathbb{A}^* \rightarrow \mathbb{B}$ be a mapping for which there exists a constant $\epsilon > 0$ (independent of x and y) such that the functional inequality

$$|D_f(x, y)| \leq \frac{26\epsilon}{27} \quad (3.15)$$

holds for all $x, y \in \mathbb{A}^*$. Then the limit

$$R_c(x) = \lim_{n \rightarrow \infty} \frac{1}{27^n} f\left(\frac{x}{3^n}\right) \quad (3.16)$$

exists for all $x \in \mathbb{A}^*$, $n \in \mathbb{N}$ and $R_c : \mathbb{A}^* \rightarrow \mathbb{B}$ is the unique mapping satisfying the cubic reciprocal functional equation (1.5) such that

$$|f(x) - R_c(x)| \leq \epsilon \quad (3.17)$$

for all $x \in \mathbb{A}^*$.

Proof. Putting $y = x$ in (3.15), we get

$$\left| f(3x) - \frac{1}{27} f(x) \right| \leq \frac{26\epsilon}{27} \quad (3.18)$$

for all $x \in \mathbb{A}^*$. Now, substituting x by $\frac{x}{3}$ in (3.18), we find

$$\left| f(x) - \frac{1}{27} f\left(\frac{x}{3}\right) \right| \leq \frac{26\epsilon}{27} \quad (3.19)$$

for all $x \in \mathbb{A}^*$. Replacing x by $\frac{x}{3}$ in (3.19) and then dividing both sides by 3^3 , we have

$$\left| \frac{1}{3^3} f\left(\frac{x}{3}\right) - \frac{1}{3^6} f\left(\frac{x}{3^2}\right) \right| \leq \frac{26\epsilon}{27^2} \quad (3.20)$$

for all $x \in \mathbb{A}^*$. It follows from (3.19) and (3.19) that

$$\left| f(x) - \frac{1}{3^6} f\left(\frac{x}{3^2}\right) \right| \leq \frac{26\epsilon}{27} \left(1 + \frac{1}{3^3} \right) \quad (3.21)$$

for all $x \in \mathbb{A}^*$. The above process can be repeated to obtain

$$\left| f(x) - \frac{1}{3^{3n}} f\left(\frac{x}{3^n}\right) \right| \leq \frac{26\epsilon}{27} \left(1 + \frac{1}{3^3} + \frac{1}{3^6} + \dots + \frac{1}{3^{3(n-1)}} \right) \quad (3.22)$$

for all $x \in \mathbb{A}^*$ and all $n \in \mathbb{N}$. In order to prove the convergence of the sequence $\left\{ \frac{1}{3^{3n}} f\left(\frac{x}{3^n}\right) \right\}$, we have if $n > k > 0$, then by (3.23)

$$\begin{aligned} \left| \frac{1}{3^{3n}} f\left(\frac{x}{3^n}\right) - \frac{1}{3^{3k}} f\left(\frac{x}{3^k}\right) \right| &= \frac{1}{3^{3k}} \left| \frac{1}{3^{3(n-k)}} f\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^k}\right) \right| \\ &= \frac{1}{3^{3k}} \left| \frac{1}{3^{3(n-k)}} f\left(\frac{y}{3^{n-k}}\right) - f(y) \right| \end{aligned} \quad (3.23)$$

$$\leq \frac{1}{3^{3k}} \frac{26\epsilon}{27} \left(1 + \frac{1}{3^3} + \frac{1}{3^6} + \dots + \frac{1}{3^{3(n-k-1)}} \right) \quad (3.24)$$

$$\leq 3^{-3k} \epsilon \quad (3.25)$$

for all $x \in \mathbb{A}^*$ in which $y = \frac{x}{3^k}$. The above result shows that the mentioned sequence is a Cauchy sequence and thus limit (3.16) exists for all $x \in \mathbb{A}^*$. Taking that n tends to infinity in (3.21), we can see that inequality (3.17) holds for all $x \in \mathbb{A}^*$. Replacing (x, y) by $(\frac{x}{3^n}, \frac{y}{3^n})$, respectively in (3.15) and dividing both sides by 3^{3n} , we deduce that

$$\frac{1}{3^{3n}} |D_f(3^n x, 3^n y)| \leq \frac{26\epsilon}{3^{3(n+1)}} \quad (3.26)$$

holds for all $x, y \in \mathbb{A}^*$. Taking that n tends to infinity in (3.26), we see that $R_c(x)$ satisfies (3.15) for all $x, y \in \mathbb{A}^*$. To prove that $R'_c(x)$ is a unique quadratic reciprocal functional satisfying (3.15) subject to (3.18), let us consider a $R'_c(x) : \mathbb{A}^* \rightarrow \mathbb{B}$ to be another quadratic reciprocal function which satisfies (3.15) and inequality (3.18). Clearly $R_c(x)$ and $R'_c(x)$ satisfy (3.18) and using (3.17), we get

$$\begin{aligned} |R_c(x) - R'_c(x)| &= \lim_{n \rightarrow \infty} \frac{1}{3^{3n}} \left| R_c\left(\frac{x}{3^n}\right) - R'_c\left(\frac{x}{3^n}\right) \right| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{3^{3n}} \left\{ \left| R_c\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right) \right| + \left| f\left(\frac{x}{3^n}\right) - R'_c\left(\frac{x}{3^n}\right) \right| \right\} \\ &= 0 \end{aligned}$$

for all $x \in \mathbb{A}^*$, which proves that R_c is unique. $\square$

Corollary 3.5. Let $c_1 \geq 0$ and $p \neq -3$, be fixed constants. If $f : \mathbb{A}^* \rightarrow \mathbb{B}$ satisfies

$$|D_f(x, y)| \leq c_1 (|x|^p + |y|^p)$$

for all $x, y \in \mathbb{A}^*$, then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \rightarrow \mathbb{B}$ satisfying (1.5) and

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{2c_1}{|3|^p |3|^{p+3}} |x|^p, & \text{for } p > -3 \\ 2c_1 |3|^3 |3|^{p+3} |x|^p, & \text{for } p < -3 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. Considering $\varphi(x, y) = c_1 (|x|^p + |y|^p)$, for all $x, y \in \mathbb{A}^*$ in Theorem 3.2 with $p > -3$ and in Theorem 3.3 with $p < -3$ and proceeding by similar arguments as in Theorems 3.2 and 3.3. $\square$

Corollary 3.6. Let $f : \mathbb{A}^* \rightarrow \mathbb{B}$ be a mapping and let there exist real numbers $\alpha, \beta : \rho = \alpha + \beta \neq -3$. Let there exists $c_2 \geq 0$ such that

$$|D_f(x, y)| \leq c_2 |x|^\alpha |y|^\beta$$

for all $x, y \in \mathbb{A}^*$. Then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \rightarrow \mathbb{B}$ satisfying (1.5) and

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{c_2}{|3|^\rho |3|^{\rho+3}} |x|^\rho, & \text{for } \rho > -3 \\ c_2 |3|^3 |3|^{\rho+3} |x|^\rho, & \text{for } \rho < -3 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. The required results are obtained by choosing $\varphi(x, y) = c_2 |x|^\alpha |y|^\beta$, for all $x, y \in \mathbb{A}^*$ in Theorem 3.2 with $\rho > -3$ and in Theorem 3.3 with $\rho < -3$, the proof of the corollary is complete. $\square$

Corollary 3.7. Let $c_3 \geq 0$ and $r \neq -3$ be real numbers, and $f : \mathbb{A}^* \rightarrow \mathbb{B}$ be a mapping satisfying the functional inequality

$$|D_f(x, y)| \leq c_3 \left(|x|^{\frac{r}{2}} |y|^{\frac{r}{2}} + (|x|^r + |y|^r) \right)$$

for all $x, y \in \mathbb{A}^*$. Then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \rightarrow \mathbb{B}$ satisfying (1.5) and

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{3c_3}{|3|^r |3|^{r+3}} |x|^r, & \text{for } r > -3 \\ 3c_3 |3|^3 |3|^{r+3} |x|^r, & \text{for } r < -3 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. The proof follows immediately by taking $\varphi(x, y) = \left(|x|^{\frac{r}{2}} |y|^{\frac{r}{2}} + (|x|^r + |y|^r) \right)$ in Theorem 3.2 with $r > -3$ and in Theorem 3.3 with $r < -3$. $\square$

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