Generalized Hyers-Ulam stability of a cubic reciprocal functional equation

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Research Article

Abstract

In this paper, we obtain the generalized Hyers-Ulam stability of a new cubic reciprocal functional equation of the form

$$f(2x+y)+f(2x-y)=\frac{4f(x)f(y)\left[4f(y)+3f(x)^{\frac{2}{3}}f(y)^{\frac{1}{3}}\right]}{\left(4f(y)^{\frac{2}{3}}-f(x)^{\frac{2}{3}}\right)^{3}}$$

in non-Archimedean fields with $f(y)\neq\frac{1}{8}f(x)$. We also extend the results related to Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of powers of norms for the same equation.

Keywords: Reciprocal functional equation; Reciprocal-quadratic functional equation; Generalized Hyers-Ulam stability; non-Archimedean field

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1 Introduction

The classical theory of stability of functional equations is instigated by the question of S.M. Ulam [14] in the year 1940. In the subsequent year, D.H. Hyers [5] was the foremost mathematician to establish the result connected with the stability of functional equations. The result obtained by D.H. Hyers is called as Hyers-Ulam stability of functional equation. Later in the year 1950, T. Aoki [1] made further simplification to the theorem of D.H. Hyers. In the year 1978, Th.M. Rassias [13] took a broad view in the Hyers' result by taking the upper bound as sum of powers of norms. The theorem proved

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by Th.M. Rassias persuaded a lot of mathematicians to work on the stability of various functional equations and the result obtained by Th.M. Rassias is recognized as Hyers-Ulam-Rassias stability of functional equation. J.M. Rassias [6] provided further generalization of the result of D.H. Hyers by using weaker conditions controlled by a product of different powers of norms. The result proved by J.M. Rassias is termed as Ulam-Gavruta-Rassias stability of functional equation. Further, in the year 1994, P. Gavruta [4] provided a generalization of Th.M. Rassias' theorem by replacing a general control function as upper bound. The stability result ascertained by P. Gavruta is celebrated as generalized Hyers-Ulam stability of functional equation. In the year 2008, J.M. Rassias et al. [7] investigated the stability of quadratic functional equation

$$f(mx+y)+f(mx-y)=2f(x+y)+2f(x-y)+2\left(m^2-2\right)f(x)-2f(y)$$

for any arbitrary but fixed real constant m with $m \neq 0$; $m \neq \pm 1$; $m \neq \pm\sqrt{2}$ using mixed product-sum of powers of norms. This stability result acquired by J.M. Rassias is known as J.M. Rassias stability involving mixed product-sum of powers of norms.

In the year 2010, K. Ravi and B.V. Senthil Kumar [8] proved Ulam-Gavruta-Rassias stability for a new reciprocal type functional equation

$$f(x+y)=\frac{f(x)f(y)}{f(x)+f(y)} \tag{1.1}$$

where $f : X \to \mathbb{R}$ is a mapping with X as the space of non-zero real numbers. The reciprocal function $g(x)=\frac{c}{x}$ is a solution of the functional equation (1.1).

The other results pertaining to stability of different reciprocal type functional equations can be found in ([9], [10], [11]).

A. Bodaghi and S.O. Kim [2] introduced and studied the Ulam-Gavruta-Rassias stability for the quadratic reciprocal functional equation

$$f(2x+y)+f(2x-y)=\frac{2f(x)f(y)[4f(y)+f(x)]}{(4f(y)-f(x))^2}. \tag{1.2}$$

The quadratic reciprocal function $f(x)=\frac{c}{x^2}$ is a solution of the functional equation (1.2).

A. Bodaghi and Y. Ebrahimdoost [3] generalized the equation (1.2) as

$$f((a+1)x+ay)+f((a+1)x-ay)=\frac{2f(x)f(y)\left[(a+1)^2f(y)+a^2f(x)\right]}{\left((a+1)^2f(y)-a^2f(x)\right)^2} \tag{1.3}$$

where $a \in \mathbb{Z}$ with $a \neq 0$ and established the generalized Hyers-Ulam-Rassias stability for the functional equation (1.3) in non-Archimedean fields.

K. Ravi et al. [12] investigated the generalized Hyers-Ulam-Rassias stability of a reciprocal-quadratic functional equation of the form

$$r(x+2y)+r(2x+y)=\frac{r(x)r(y)\left[5r(x)+5r(y)+8\sqrt{r(x)r(y)}\right]}{\left[2r(x)+2r(y)+5\sqrt{r(x)r(y)}\right]^2} \tag{1.4}$$

in intuitionistic fuzzy normed spaces.

In this paper, we obtain the generalized Hyers-Ulam stability of a cubic reciprocal functional equation of the form

$$f(2x+y)+f(2x-y)=\frac{4f(x)f(y)\left[4f(y)+3f(x)^{\frac{2}{3}}f(y)^{\frac{1}{3}}\right]}{\left(4f(y)^{\frac{2}{3}}-f(x)^{\frac{2}{3}}\right)^3} \tag{1.5}$$

in non-Archimedean fields. We extend the results concerning Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of powers of norms for the equation (1.5). It is easy to see that the reciprocal cubic function $f(x)=\frac{1}{x^3}$ is a solution of the functional equation (1.5).

2 Preliminaries

In this section, we recall the basic concepts of non-Archimedean filed, non-Archimedean norm and non-Archimedean space.

Definition 2.1. By a non-Archimedean field, we mean a field K equipped with a function (valuation) $|\cdot|$ from K into $[0,\infty)$ such that $|r|=0$ if and only if $r=0$, $|rs|=|r||s|$ and $|r+s|\leq \mathrm{m}ax\{|r|,|s|\}$ for all $r,s\in K$.

Clearly $|1|=|-1|=1$ and $|n|\leq 1$ for all $n\in\mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|.|$. A function $||.||:X\to\mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

(i) $||x||=0$ if and only if $x=0$;

(ii) $||rx||=|r|||x||\qquad (r\in\mathbb{K},x\in X)$;

(iii) the strong triangle inequality (ultrametric); namely,

$$||x+y||\leq \max\{||x||,||y||\}\qquad (x,y\in X).$$

Then $(X,||.||)$ is called a non-Archimedean space. Due to the fact that

$$\|x_n-x_m\|\leq \max\left\{\|x_{j+1}-x_j\|:m\leq j\leq n-1\right\}\qquad (n>m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1}-x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space, we mean that every Cauchy sequence is convergent in the space.

An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking everything but 0 into 1 and $|0|=0$. This valuation is called trivial. Another example of a non-Archimedean valuation on a field $\mathbb{A}$ is the mapping

$$|r|=\begin{cases}0 & \text{if } r=0\\ \frac{1}{r} & \text{if } r>0\\ -\frac{1}{r} & \text{if } r<0\end{cases}$$

for any $r\in\mathbb{A}$.

Let p be a prime number. For any non-zero rational number $x=p^r\frac{m}{n}$ in which m and n are coprime to the prime number p. Consider the p-adic absolute value $|x|_p=p^{-r}$ on $\mathbb{Q}$. It is easy to check that $|\cdot|$ is a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to $|\cdot|$ which is denoted by $\mathbb{Q}_p$ is said to be the p-adic number field. Note that if $p>2$, then $|2^n|=1$ for all integers n.

3 Generalized Hyers-Ulam stability of equation (1.5)

In this section, we investigate the generalized Hyers-Ulam stability of equation (1.5) in non-Archimedean fields. We also establish the results pertaining to Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by product-sum of powers of norms.

Throughout this section, we consider that $\mathbb{A}$ and $\mathbb{B}$ is a non-Archimedean field and a complete non-Archimedean field, respectively. From now on, for a non-Archimdean field $\mathbb{A}$, we put $\mathbb{A}^*=A-\{0\}$. For the purpose of simplification, let us define the difference operator $D_f:\mathbb{A}^*\times\mathbb{A}^*\to\mathbb{B}$ by

$$D_f(x,y)=f(2x+y)+f(2x-y)-\frac{4f(x)f(y)\left[4f(y)+3f(x)^{\frac{2}{3}}f(y)^{\frac{1}{3}}\right]}{\left(4f(y)^{\frac{2}{3}}-f(x)^{\frac{2}{3}}\right)^3}$$

for all $x, y \in \mathbb{A}^*$.

In the following results, we assume $f(x) \neq 0$ and $x + y \neq 0$ for all $x, y \in \mathbb{R}$.

For proving our main results, we necessitate the following definition.

Definition 3.1. A mapping $f : \mathbb{A}^* \to \mathbb{B}$ is called as reciprocal cubic mapping if f satisfies equation 1.5.

Theorem 3.2. *Let $\varphi : \mathbb{A}^* \times \mathbb{A}^* \to \mathbb{B}^*$ be a function such that*

$$\lim_{n\to\infty} \left|\frac{1}{27}\right|^n \varphi\left(\frac{x}{3^{n+1}}, \frac{y}{3^{n+1}}\right) = 0 \tag{3.1}$$

for all $x, y \in \mathbb{A}^*$. Suppose that $f : \mathbb{A}^* \to \mathbb{B}$ is a mapping satisfying the inequality

$$|D_f(x, y)| \leq \varphi(x, y) \tag{3.2}$$

for all $x, y \in \mathbb{A}^*$. Then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \to \mathbb{B}$ such that

$$|f(x) - R_c(x)| \leq \max\left\{\left|\frac{1}{27}\right|^j \varphi\left(\frac{x}{3^{j+1}}, \frac{x}{3^{j+1}}\right) : j \in \mathbb{N} \cup \{0\}\right\} \tag{3.3}$$

for all $x \in \mathbb{A}^*$.

Proof. Plugging (x, y) by (x, x) in (3.2), we obtain

$$\left|f(3x) - \frac{1}{27}f(x)\right| \leq \varphi(x, x) \tag{3.4}$$

for all $x \in \mathbb{A}^*$. Now, substituting x by $\frac{x}{3}$ in (3.4), we find

$$\left|f(x) - \frac{1}{27}f\left(\frac{x}{3}\right)\right| \leq \varphi\left(\frac{x}{3}, \frac{x}{3}\right) \tag{3.5}$$

for all $x \in \mathbb{A}^*$. Replacing x by $\frac{x}{2^n}$ in (3.5) and multiplying by $\left|\frac{1}{27^n}\right|^n$, we have

$$\left|\frac{1}{27^n}f\left(\frac{x}{3^n}\right) - \frac{1}{27^{n+1}}f\left(\frac{x}{3^{n+1}}\right)\right| \leq \left|\frac{1}{27}\right|^n \varphi\left(\frac{x}{3^{n+1}}, \frac{x}{3^{n+1}}\right) \tag{3.6}$$

for all $x \in \mathbb{A}^*$. Hence we find that the sequence $\left\{\frac{1}{27^n}f\left(\frac{x}{3^n}\right)\right\}$ is a Cauchy sequence by (3.1) and (3.6). Since $\mathbb{B}$ is complete, we can define a mapping R_c given by

$$R_c(x) = \lim_{n\to\infty} \frac{1}{27^n}f\left(\frac{x}{3^n}\right). \tag{3.7}$$

For each $x \in \mathbb{A}^*$ and non-negative integers n, we have

$$\begin{aligned}\left|\frac{1}{27^n}f\left(\frac{x}{3^n}\right) - f(x)\right| &= \left|\sum_{i=0}^{n-1}\left\{\frac{1}{27^{i+1}}f\left(\frac{x}{3^{i+1}}\right) - \frac{1}{27^i}f\left(\frac{x}{3^i}\right)\right\}\right| \\ &\leq \max\left\{\left|\frac{1}{27^{i+1}}f\left(\frac{x}{3^{i+1}}\right) - \frac{1}{27^i}f\left(\frac{x}{3^i}\right)\right| : 0 \leq i < n\right\} \\ &\leq \max\left\{\left|\frac{1}{27}\right|^i \varphi\left(\frac{x}{3^{i+1}}, \frac{x}{3^{i+1}}\right) : 0 \leq i < n\right\}. \end{aligned} \tag{3.8}$$

The inequality (3.3) is true using (3.7) and letting $n \to \infty$ in the above inequality (3.8). Using (3.1), (3.2) and (3.7), we have for all $x, y \in \mathbb{A}^*$

$$\begin{aligned} |D_{R_c}(x,y)| &= \lim_{n\to\infty} \left|\frac{1}{27}\right|^n \left|D_f\left(\frac{x}{3^n}, \frac{y}{3^n}\right)\right| \\ &\leq \lim_{n\to\infty} \left|\frac{1}{27}\right|^n \varphi\left(\frac{x}{3^n}, \frac{y}{3^n}\right) = 0. \end{aligned}$$

Thus the mapping R_c satisfies (1.5) and hence it is cubic reciprocal mapping. In order to prove the uniqueness of R_c, let us consider $R'_C : \mathbb{A}^* \to \mathbb{B}$ be another cubic reciprocal mapping satisfying (3.3). Then we have

$$\begin{aligned} &\left|R_c(x) - R'_C(x)\right| \\ &= \lim_{m\to\infty} \left|\frac{1}{27}\right|^m \left|R_c\left(\frac{x}{3^m}x\right) - R'_c\left(\frac{x}{3^m}\right)\right| \\ &\leq \lim_{m\to\infty} \left|\frac{1}{27}\right|^m \max\left\{\left|R_c\left(\frac{x}{3^m}\right) - f\left(\frac{x}{3^m}\right)\right|, \left|f\left(\frac{x}{3^m}\right) - R'_c\left(\frac{x}{3^m}\right)\right|\right\} \\ &\leq \lim_{m\to\infty} \lim_{n\to\infty} \max\left\{\max\left\{\left|\frac{1}{27}\right|^{i+m} \varphi\left(\frac{x}{3^{i+m+1}}, \frac{x}{3^{i+m+1}}\right) : m \leq i \leq n+m\right\}\right\} \\ &= 0 \end{aligned}$$

for all $x \in \mathbb{A}^*$, which shows that R_c is unique. Hence the proof is complete. □

Theorem 3.3. *Let $\varphi : \mathbb{A}^* \times \mathbb{A}^* \to \mathbb{B}^*$ be a function such that*

$$\lim_{n\to\infty} |27|^n \varphi\left(3^n x, 3^n y\right) = 0 \tag{3.9}$$

for all $x, y \in \mathbb{A}^*$. Suppose that $f : \mathbb{A}^* \to \mathbb{B}$ is a mapping satisfying the inequality (3.2) for all $x, y \in \mathbb{A}^*$. Then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \to \mathbb{B}$ such that

$$|f(x) - R_c(x)| \leq \max\left\{|27|^{i+1}\varphi\left(3^i x, 3^i x\right) : i \in \mathbb{N} \cup \{0\}\right\} \tag{3.10}$$

for all $x \in \mathbb{A}^*$.

Proof. Switching y to x in (3.2) and multiplying by $|27|$, we find

$$|27f(3x) - f(x)| \leq |27|\varphi(x,x) \tag{3.11}$$

for all $x \in \mathbb{A}^*$. Now, replacing x by $3^n x$ in (3.11) and multiplying by $|27|^n$, we obtain

$$\left|27^n f\left(3^n x\right) - 27^{n+1} f\left(3^{n+1} x\right)\right| \leq |27|^{n+1}\varphi\left(3^n x, 3^n x\right) \tag{3.12}$$

for all $x \in \mathbb{A}^*$. It is easy to see that the sequence $\{3^n f\left(3^n x\right)\}$ is a Cauchy sequence by letting $n \to \infty$ in (3.12) and using (3.9). Since $\mathbb{B}$ is complete, this Cauchy sequence converges to a mapping $R_c : \mathbb{A}^* \to \mathbb{B}$ defined by

$$R_c(x) = \lim_{n\to\infty} 27^n f\left(3^n x\right). \tag{3.13}$$

For each $x \in \mathbb{A}^*$ and non-negative integers n, we have

$$\begin{aligned} |27^n f\left(3^n x\right) - f(x)| &= \left|\sum_{i=0}^{n-1} 27^{i+1} f\left(3^{i+1}x\right) - 27^i f\left(3^i x\right)\right| \\ &\leq \max\left\{\left|27^{i+1} f\left(3^{i+1}x\right) - 27^i f\left(3^i x\right)\right| : 0 \leq i < n\right\} \\ &\leq \max\left\{|27|^{i+1}\varphi\left(3^i x, 3^i x\right) : 0 \leq i < n\right\}. \end{aligned} \tag{3.14}$$

Applying (3.13) and letting $n \to \infty$, we find that the inequality (3.10) holds. From (3.9), (3.2) and (3.13), we have for all $x, y \in \mathbb{A}^*$

$$\begin{aligned} |D_{R_c}(x,y)| &= \lim_{n\to\infty} |27|^n \left|D_f\left(3^n x, 3^n y\right)\right| \\ &\leq \lim_{n\to\infty} |27|^n \varphi\left(3^n x, 3^n y\right) = 0. \end{aligned}$$

Hence the mapping R_c satisfies (1.5) and it is cubic reciprocal mapping. Now, let $R'_C : \mathbb{A}^* \to \mathbb{B}$ be another cubic reciprocal mapping satisfying (1.5). Then we have

$$\begin{aligned} &\left|R_c(x) - R'_c(x)\right| \\ &\quad = \lim_{m\to\infty} |27|^m \left|R_c\left(3^m x\right) - R'_c\left(3^m x\right)\right| \\ &\quad \leq \lim_{m\to\infty} |27|^m \max\left\{\left|R_c\left(3^m x\right) - f\left(3^m x\right)\right|, \left|f\left(3^m x\right) - R'_c\left(3^m x\right)\right|\right\} \\ &\quad \leq \lim_{m\to\infty} \lim_{n\to\infty} \max\left\{\max\left\{|27|^{i+m+1} \varphi\left(3^{i+m} x, 3^{i+m} x\right) : m \leq i \leq n+m\right\}\right\} \\ &\quad = 0 \end{aligned}$$

for all $x \in \mathbb{A}^*$, which proves that R_c is unique. □

Theorem 3.4. *Let $f : \mathbb{A}^* \to \mathbb{B}$ be a mapping for which there exists a constant $\epsilon > 0$ (independent of x and y) such that the functional inequality*

$$|D_f(x,y)| \leq \frac{26\epsilon}{27} \tag{3.15}$$

holds for all $x, y \in \mathbb{A}^*$. Then the limit

$$R_c(x) = \lim_{n\to\infty} \frac{1}{27^n} f\left(\frac{x}{3^n}\right) \tag{3.16}$$

exists for all $x \in \mathbb{A}^*$, $n \in \mathbb{N}$ and $R_c : \mathbb{A}^* \to \mathbb{B}$ is the unique mapping satisfying the cubic reciprocal functional equation (1.5) such that

$$|f(x) - R_c(x)| \leq \epsilon \tag{3.17}$$

for every $x \in \mathbb{A}^*$.

Proof. Putting $y = x$ in (3.15), we get

$$\left|f(3x) - \frac{1}{27} f(x)\right| \leq \frac{26\epsilon}{27} \tag{3.18}$$

for all $x \in \mathbb{A}^*$. Now, substituting x by $\frac{x}{3}$ in (3.18), we find

$$\left|f(x) - \frac{1}{27} f\left(\frac{x}{3}\right)\right| \leq \frac{26\epsilon}{27} \tag{3.19}$$

for all $x \in \mathbb{A}^*$. Replacing x by $\frac{x}{3}$ in (3.19) and then dividing both sides by 3^3, we have

$$\left|\frac{1}{3^3} f\left(\frac{x}{3}\right) - \frac{1}{3^6} f\left(\frac{x}{3^2}\right)\right| \leq \frac{26\epsilon}{27^2} \tag{3.20}$$

for all $x \in \mathbb{A}^*$. It follows from (3.19) and (3.19) that

$$\left|f(x) - \frac{1}{3^6} f\left(\frac{x}{3^2}\right)\right| \leq \frac{26\epsilon}{27}\left(1 + \frac{1}{3^3}\right) \tag{3.21}$$

for all $x \in \mathbb{A}^*$. The above process can be repeated to obtain

$$\left| f(x) - \frac{1}{3^{3n}} f\left(\frac{x}{3^n}\right) \right| \leq \frac{26\epsilon}{27} \left(1 + \frac{1}{3^3} + \frac{1}{3^6} + ... + \frac{1}{3^{3(n-1)}}\right) \tag{3.22}$$

for all $x \in \mathbb{A}^*$ and all $n \in \mathbb{N}$. In order to prove the convergence of the sequence $\left\{\frac{1}{3^{3n}} f\left(\frac{x}{3^n}\right)\right\}$, we have if $n > k > 0$, then by (3.23)

$$\begin{aligned}
\left| \frac{1}{3^{3n}} f\left(\frac{x}{3^n}\right) - \frac{1}{3^{3k}} f\left(\frac{x}{3^k}\right) \right| &= \frac{1}{3^{3k}} \left| \frac{1}{3^{3(n-k)}} f\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^k}\right) \right| \\
&= \frac{1}{3^{3k}} \left| \frac{1}{3^{3(n-k)}} f\left(\frac{y}{3^{n-k}}\right) - f(y) \right| && (3.23) \\
&\leq \frac{1}{3^{3k}} \frac{26\epsilon}{27} \left(1 + \frac{1}{3^3} + \frac{1}{3^6} + ... + \frac{1}{3^{3(n-k-1)}}\right) && (3.24) \\
&\leq 3^{-3k}\epsilon && (3.25)
\end{aligned}$$

for all $x \in \mathbb{A}^*$ in which $y = \frac{x}{3^k}$. The above result shows that the mentioned sequence is a Cauchy sequence and thus limit (3.16) exists for all $x \in \mathbb{A}^*$. Taking that n tends to infinity in (3.21), we can see that inequality (3.17) holds for all $x \in \mathbb{A}^*$. Replacing (x, y) by $\left(\frac{x}{3^n}, \frac{y}{3^n}\right)$, respectively in (3.15) and dividing both sides by 3^{3n}, we deduce that

$$\frac{1}{3^{3n}} \left| D_f\left(3^n x, 3^n y\right) \right| \leq \frac{26\epsilon}{3^{3(n+1)}} \tag{3.26}$$

holds for all $x, y \in \mathbb{A}^*$. Taking that n tends to infinity in (3.26), we see that $R_c(x)$ satisfies (3.15) for all $x, y \in \mathbb{A}^*$. To prove that $R'_c(x)$ is a unique quadratic reciprocal functional satisfying (3.15) subject to (3.18), let us consider a $R'_c(x) : \mathbb{A}^* \to \mathbb{B}$ to be another quadratic reciprocal function which satisfies (3.15) and inequality (3.18). Clearly $R_c(x)$ and $R'_c(x)$ satisfy (3.18) and using (3.17), we get

$$\begin{aligned}
\left| R_c(x) - R'_c(x) \right| &= \lim_{n\to\infty} \frac{1}{3^{3n}} \left| R_c\left(\frac{x}{3^n}\right) - R'_c\left(\frac{x}{3^n}\right) \right| \\
&\leq \lim_{n\to\infty} \frac{1}{3^{3n}} \left\{ \left| R_c\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right) \right| + \left| f\left(\frac{x}{3^n}\right) - R'_c\left(\frac{x}{3^n}\right) \right| \right\} \\
&= 0
\end{aligned}$$

for all $x \in \mathbb{A}^*$, which proves that R_c is unique. □

Corollary 3.5. *Let $c_1 \geq 0$ and $p \neq -3$, be fixed constants. If $f : \mathbb{A}^* \to \mathbb{B}$ satisfies*

$$|D_f(x, y)| \leq c_1 \left(|x|^p + |y|^p\right)$$

for all $x, y \in \mathbb{A}^*$, then there exists a unique cubic reciprocal mapping $R_c : \mathbb{A}^* \to \mathbb{B}$ satisfying (1.5) and

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{2c_1}{|3|^p |3|^{p+3}} |x|^p, & \text{for } p > -3 \\ 2c_1 |3|^3 |3|^{p+3} |x|^p, & \text{for } p < -3 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. Considering $\varphi(x, y) = c_1 \left(|x|^p + |y|^p\right)$, for all $x, y \in \mathbb{A}^*$ in Theorem 3.2 with $p > -3$ and in Theorem 3.3 with $p < -3$ and proceeding by similar arguments as in Theorems 3.2 and 3.3. □

Corollary 3.6. *Let $f : \mathbb{A}^* \to \mathbb{B}$ be a mapping and let there exist real numbers $\alpha, \beta : \rho = \alpha + \beta \neq -3$. Let there exists $c_2 \geq 0$ such that*

$$|D_f(x, y)| \leq c_2 |x|^\alpha |y|^\beta$$

for all $x, y \in \mathbb{A}^*$. *Then there exists a unique cubic reciprocal mapping* $R_c : \mathbb{A}^* \to \mathbb{B}$ *satisfying (1.5) and*

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{c_2}{|3|^\rho |3|^{\rho+3}} |x|^\rho, & \text{for } \rho > -3 \\ c_2 |3|^3 |3|^{+\rho+3} |x|^\rho, & \text{for } \rho < -3 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. The required results are obtained by choosing $\varphi(x, y) = c_2 |x|^\alpha |y|^\beta$, for all $x, y \in \mathbb{A}^*$ in Theorem 3.2 with $\rho > -3$ and in Theorem 3.3 with $\rho < -3$, the proof of the corollary is complete. □

Corollary 3.7. *Let* $c_3 \geq 0$ *and* $r \neq -3$ *be real numbers, and* $f : \mathbb{A}^* \to \mathbb{B}$ *be a mapping satisfying the functional inequality*

$$|D_f(x, y)| \leq c_3 \left(|x|^{\frac{r}{2}} |y|^{\frac{r}{2}} + (|x|^r + |y|^r) \right)$$

for all $x, y \in \mathbb{A}^*$. *Then there exists a unique cubic reciprocal mapping* $R_c : \mathbb{A}^* \to \mathbb{B}$ *satisfying (1.5) and*

$$|f(x) - R_c(x)| \leq \begin{cases} \frac{3c_3}{|3|^r |3|^{r+3}} |x|^r, & \text{for } r > -3 \\ 3c_3 |3|^3 |3|^{r+3} |x|^r, & \text{for } r < -3 \end{cases}$$

for every $x \in \mathbb{A}^*$.

Proof. The proof follows immediately by taking $\varphi(x, y) = \left(|x|^{\frac{r}{2}} |y|^{\frac{r}{2}} + (|x|^r + |y|^r) \right)$ in Theorem 3.2 with $r > -3$ and in Theorem 3.3 with $r < -3$. □

References

[1] Aoki, T. (1950). On the stability of the linear transformation in Banach spaces, J. Math.Soc. Japan, 2, 64–66.

[2] Bodaghi, A. and Kim, S.O. (2014). Approximation on the quadratic reciprocal functional equation, J. Function Spaces and Appl., 2014, Article ID532463, 5 pages.

[3] Bodaghi, A. and Ebrahimdoost, Y. (2016). On the stability of quadratic reciprocal functional equation in non-Archimedean fields, Asian-European J. Math., 9(1), 1650002, 9 pages.

[4] Gavruta, P. (1994). A generalization of the Hyers-Ulam-Rassias stability of approximately additive mapppings, J. Math. Anal. Appl. 184, 431–436.

[5] Hyers, D.H. (1941). On the stability of the linear functional equation, Proc. Nat. Acad. Sci. U.S.A. 27, 222–224.

[6] Rassias, J.M. (1982). On approximation of approximately linear mappings by linear mappings, J. Funct. Anal. 46, 126–130.

[7] Ravi, K., Arunkumar, M. and Rassias, J.M. (2008). Ulam stability for the orthogonally general Euler-Lagrange type functional equation, Int. J. Math. Stat. 3(A08), 36–46.

[8] Ravi, K. and Senthil Kumar, B.V. (2010). Ulam-Gavruta-Rassias stability of Rassias reciprocal functional equation, Global J. of Appl. Math. and Math. Sci. 3(1-2), 57–79.

[9] Ravi, K., Rassias, J.M. and Senthil Kumar, B.V. (2010). Ulam stability of generalized reciprocal functional equation in several variables, Int. J. App. Math. Stat. 19(D10), 1–19.

[10] Ravi, K., Rassias, J.M. and Senthil Kumar, B.V. (2011). Ulam stability of reciprocal difference and adjoint functional equations, Aust. J. Math. Anal. Appl., 8(1), Art. 13, 1–18.

[11] Ravi, K., Thandapani, E. and Senthil Kumar, B.V. (2011). Stability of reciprocal type functional equations, PanAmerican Math. Journal, 21(1), 59–70.

[12] Ravi, K., Rassias, J.M., Senthil Kumar, B.V. and Bodaghi, A. (2014). Intuitionistic fuzzy stability of a reciprocal-quadratic functional equation, Int. J. Appl. Sci. Math., 1(1), 9–14.

[13] Rassias, T.M. (1978). On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72, 297–300.

[14] Ulam, S.M. (1940). Problems in Modern Mathematics, Chapter VI, Wiley-Interscience, New York.