

Analysis of a repairable k -out-of- n : G system with expert repairman's multiple vacations and shut off rule

**Original Research
Article**

Received: XX December 20XX

Accepted: XX December 20XX

Online Ready: XX December 20XX

Abstract

This paper investigates an k -out-of- n : G repairable system with expert repairman's multiple vacations and continuous operation rule where the operating times and the repair times of components follow exponential distributions. When there is no failed component in the system, the expert repairman leaves for a vacation, the duration of which is governed by a phase type distribution. Once an operating component fails during his/her vacation period, it is repaired by an ordinary repairman. The ordinary repairman becomes inactivated when there is no failed component or the expert repairman returns from the vacation. Employing the matrix analytical method, we obtain the transient system availability, the rate of occurrence of failures at time t , the system reliability, the stationary availability, the stationary rate of occurrence of failures, and the mean time to failure of the system. Further, we discuss the time-dependent behaviour of the system reliability measures under different initial system states. Finally, special cases of the system are given to show the correctness of our results.

Keywords: k -out-of- n : G system; multiple vacations; continuous operation; matrix analytical method; reliability measures

2010 Mathematics Subject Classification: 60K25; 60G10

1 Introduction

In the area of reliability theory, k -out-of- n : G systems are commonly applied in industrial systems. Its applications include, but are not limited to, the radar systems, the redundant inertial navigation system in an airborne integrated avionics system, the hydraulic control system with multiple pumps, and the communication systems. A k -out-of- n : G system is composed of n components which are all new and start to operate simultaneously at the beginning, and the system is down as soon as the number of operating components goes down to $k - 1$. For comprehensive analysis of k -out-of- n : G systems, the reader are directed to the monographs by Cao and Cheng (2006), and Kuo and Zuo (2003).

Due to the application background in **system design and industry**, k -out-of- n : G systems have received considerable attention in recent years. For example, Cekyay and Özekici (2015), Cheng et al. (2013), Krishnamoorthy and Ushakumari (2001), Smiddestombes et al. (2006, 2009), Ushakumari and Krishnamoorthy (2004). They studied such a system with different assumptions. Zhang and Wu (2009) considered a repairable $k/n(F)$ system (i.e., $(n - k + 1)$ -out-of- n : G system) with one piece of replaceable repair equipment where the operating times and the repair times of components, the fixing times and the replacement times of the repair equipment all follow exponential distributions. Moreover, the well-known phase type (PH) distribution which was first **analysed** in detail by Neuts (1981) has been introduced in k -out-of- n : G systems. The PH distribution is dense in the continuous non-negative class of distributions, and any such distribution can be approached by a PH distribution. This property enables us to make stochastic models more flexible and plausible. Chakravathy (2001) considered a k -out-of- n : G repairable system with (N, T) policy and one replaceable repair equipment, in which the repair times, the fixing times and the vacation times are governed by PH distributions. Various steady-state performance measures are obtained by **using** the matrix analytical method. Later, Ruiz-Castro and Li (2011) investigated a discrete k -out-of- n : G system with R repairmen where random variables follow discrete phase type distributions. We refer the reader to Chakravathy and Gómez-Corral (2009), Pérez-Ocón et al. (2006) and Wu and Tang (2015) for more details.

In reliability engineering, enterprises usually hire an expert repairman to repair failed components in order to improve the reliability and the availability of a system. However, vast majority of small-size and medium-size enterprises can not afford to hire a full-time expert repairman. Thus, the expert repairman ought to have two roles: one for looking after the components and one for other duties. Under this situation, the expert repairman will leave the system for other duties when there is no failed component. During this period, if an operating component fails, it is repaired by an ordinary repairman. Upon returning from his/her vacation, if there are still some failed components in the system, the expert repairman starts to repair them immediately. Otherwise, he/she leaves the system again for other duties. For example, Liu et al. (2015a,b). But, this vacation policy being considered in k -out-of- n : G system is little more than a beginning. With this knowledge, the case of expert repairman's multiple vacations is taken into account in this paper.

Another important factor in k -out-of- n : G systems is the shut off rule which describes the status of non-failed components when the system is down. Most commonly applied shut off rules are suspended animation (SA) and continuous operation (CO). The shut off rule of SA eliminates the possibility of additional failures of the non-failed components when the system is down, while the shut off rule of CO assumes that all the non-failed components are still subject to failure when the system is down. A great many publications are provided for the availability and reliability of k -out-of- n : G systems supposing the shut off rule is the SA. 2006, Li et al. (2006) studied a k -out-of- n : G system with CO rule. Some of $(k - 1)$ operating components are suspended and the others are still subject to failure whenever the system is down. Employing the Markov analysis method, formulas for various system reliability measures are obtained. Later, Moghaddass et al. (2011a,b) considered k -out-of- n : G system with CO rule where random variables all follow exponential distributions. However, the CO as the shut off rule for k -out-of- n : G system has so far been limited.

In this paper, we **investigate** a k -out-of- n : G repairable system with **expert repairman's** multiple vacations and CO rule. Applying the matrix analytical method, the system availability and the rate of occurrence of failures of the system are deduced in transient regime and in stationary regime. Further, the system reliability and the mean time to failure of the system are discussed by using the theory of phase type distribution. We then numerically indicate the influence of various system parameters on the evolution of system reliability measures under different initial system states.

This paper is arranged as follows. Section 2 gives some definitions and assumptions of the model. The infinitesimal generator of the system is constructed in Section 3. System reliability measures are provided in Sections 4–6. Section 7 represents some numerical results. Finally, conclusions are drawn in Section 8.

2 Definitions and model description

2.1 Definitions

Definition 1. A distribution $H(x)$ on $[0, +\infty)$ is of phase type with representation (ϖ, S) , if it is the distribution of the time until absorption in a Markov process on the states $\{1, 2, \dots, r, r+1\}$ with infinitesimal generator

$$\begin{pmatrix} S_{r \times r} & S_{r \times 1}^0 \\ \mathbf{0}_{1 \times r} & 0_{1 \times 1} \end{pmatrix},$$

and initial probability vector $(\varpi_{1 \times r}, 0_{1 \times 1})$. Assuming that the states $\{1, 2, \dots, r\}$ are all transient, and the state $r+1$ is absorbing. Hence the matrix S can be interpreted as the rate transient matrix among the transient states, while S^0 represents the column vector of absorption rates. The matrix S is non-singular with negative diagonal entries and non-negative off-diagonal entries and satisfies $-S\mathbf{e}_{r \times 1} = S^0$. The probability distribution $H(x) = 1 - \varpi_{1 \times r} \exp(Sx)\mathbf{e}_{r \times 1}$, $x \geq 0$, and its mean $E(\chi) = -\varpi_{1 \times r} S^{-1} \mathbf{e}_{r \times 1}$.

2.2 Model description

The detailed assumptions of the system are given below.

Assumption 1. The k -out-of- n :G system is composed of n identical and independent components. The system is down until the number of components in the operating state goes down to $k-1$. When the system is down, the remaining $k-1$ operating components are still subject to failure.

Assumption 2. The operating time X of each component has an exponential distribution $F(t) = 1 - \exp(-\lambda t)$, $\lambda > 0, t \geq 0$. Failed components in the system form a single waiting line and receive repair provided by an ordinary repairman or an expert repairman in the order of their failures, i.e., FCFS discipline. The time Y_1 of the ordinary repairman repairing every failed component and the time Y of the expert repairman repairing every failed component follows an exponential distribution with parameter μ_1 and μ , respectively.

Assumption 3. The expert repairman leaves for a vacation whenever there is no failed component in the system. Once an operating component fails during his/her vacation period, it is repaired by an ordinary repairman. Upon returning from the vacation, the expert repairman starts to work finding there exist failed components. Otherwise, the expert repairman leaves the system for another vacation. The vacation time V is governed by a PH distribution with representation (θ, T) of order m . Its probability distribution and mean value are

$$V(t) = 1 - \theta \exp(Tt)\mathbf{e}_m, \quad t \geq 0, \quad E[V] = -\theta T^{-1} \mathbf{e}_m,$$

where

$$\theta = (\theta_1, \theta_2, \dots, \theta_m), \quad T = \begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,m} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m,1} & T_{m,2} & \cdots & T_{m,m} \end{pmatrix}, \quad T^0 = \begin{pmatrix} T_{1,1}^0 \\ T_{2,1}^0 \\ \vdots \\ T_{m,1}^0 \end{pmatrix}.$$

Assumption 4. The random variables X , Y_1 , Y and V are assumed to be independent of each other.

Further, we define the following notations for use in what follows. Let I_l be identity matrix of order l , and $\mathbf{e}_l$ be a column vector of order l of 1's. We denote by $\mathbf{0}_l$ a zero matrix of order l , by $\mathbf{0}_{l \times j}$ a zero matrix of dimension $l \times j$, and by $\mathbb{C}^{l \times q}$ the set of all $l \times q$ matrices over the field $\mathbb{C}$ of complex numbers.

3 Infinitesimal generator

The repairable k -out-of- n : G system can be studied as a block-structured continuous-time Markov chain (CTCM). To see this, denote by $L(t)$ the number of failed components (either waiting or being repaired) at time t , $L(t) = i (i = 0, 1, \dots, n)$. Define $J(t)$ to be the state of the expert repairman at time t , and

$$J(t) = \begin{cases} 0, & \text{the expert repairman is repairing failed component at time } t, \\ j, & \text{the expert repairman is on vacation at phase } j, j = 1, 2, \dots, m. \end{cases}$$

According to the model assumptions, one checks easily that the stochastic process $\mathbb{X} = \{L(t), J(t) : t \geq 0\}$ is a CTMC with state space given by

$$\Omega = \{(0, j), j = 1, 2, \dots, m\} \cup \{(i, j), i = 1, 2, \dots, n, j = 0, 1, \dots, m\},$$

where the state $(0, j), j = 1, 2, \dots, m$ means that there is no failed component and the expert repairman is on vacation at phase j , the state $(i, 0), i = 1, 2, \dots, n$ means that there are i failed components and the expert repairman is repairing failed components, the state $(i, j), i = 1, 2, \dots, n, j = 1, 2, \dots, m$ means that there are i failed components and the expert repairman is on vacation at phase j .

By partitioning the state space into levels with respect to the number of failed components and using lexicographical sequence for these states, we have that the infinitesimal generator matrix Q of $\mathbb{X}$ is of dimension $m + n(m + 1)$, exhibiting the following block-structured form

$$Q = \begin{pmatrix} A_0 & C_0 & & & & \\ B_1 & A_1 & C_1 & & & \\ & B_2 & A_2 & C_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_{n-1} & A_{n-1} & C_{n-1} \\ & & & & B_n & A_n \end{pmatrix}. \quad (3.1)$$

Each block of the matrix Q are given as follows

$$\begin{aligned} B_1 &= \begin{pmatrix} \mu \theta_{1 \times m} \\ \mu_1 I_m \end{pmatrix}, \quad B_r = \begin{pmatrix} \mu & \mathbf{0}_{1 \times m} \\ \mathbf{0}_{m \times 1} & \mu_1 I_m \end{pmatrix}, \quad r = 2, 3, \dots, n, \\ A_0 &= T - n\lambda I_m + T^0 \theta, \quad A_r = \begin{pmatrix} -\mu - (n-r)\lambda & \mathbf{0}_{1 \times m} \\ T^0 & T - ((n-r)\lambda + \mu_1) I_m \end{pmatrix}, \quad r = 1, 2, \dots, n-1, \\ A_n &= \begin{pmatrix} -\mu & \mathbf{0}_{1 \times m} \\ T^0 & T - \mu_1 I_m \end{pmatrix}, \\ C_0 &= (\mathbf{0}_{m \times 1}, n\lambda I_m), \quad C_r = \begin{pmatrix} (n-r)\lambda & \mathbf{0}_{1 \times m} \\ \mathbf{0}_{m \times 1} & (n-r)\lambda I_m \end{pmatrix}, \quad r = 1, 2, \dots, n-1. \end{aligned}$$

4 Transient analysis

In this section, we will deduce the transient behaviour of the availability and the rate of occurrence of failures. First, define the following probabilities

$$\begin{aligned} P_{i,j}(t) &= P\{L(t) = i, J(t) = j\}, \quad (i, j) \in \Omega, \\ P_0(t) &= (P_{0,1}(t), P_{0,2}(t), \dots, P_{0,m}(t)), \\ P_i(t) &= (P_{i,0}(t), P_{i,1}(t), \dots, P_{i,m}(t)), \quad i = 1, 2, \dots, n. \end{aligned}$$

The Kolmogorov forward equation for the Markov process $\mathbb{X}$ can be written in matrix format as

$$\begin{cases} \frac{d}{dt}P(t) = P(t)Q, \\ P(0) = \omega_{1 \times (m+n(m+1))}(r), \end{cases} \quad (4.1)$$

where $P(t) = (P_0(t), P_1(t), \dots, P_n(t))$, $\omega_{1 \times (m+n(m+1))}(r)$ is a row vector of dimension $m+n(m+1)$ with 1 in the r th position and 0 others.

The solution of the equation (4.1) may be written as

$$P(t) = P(0)e^{Qt}. \quad (4.2)$$

Let the distinct eigenvalues of the matrix $Q \in \mathbb{C}^{(m+n(m+1)) \times (m+n(m+1))}$ be $\zeta_1, \zeta_2, \dots, \zeta_l$, with respective algebraic multiplicities $\tau_1, \tau_2, \dots, \tau_l$ where $\tau_1 + \tau_2 + \dots + \tau_l = m + n(m+1)$. Using the existence assertion of the Jordan canonical form theorem (see (Roger, 2011, Theorem 3.1.11) for details), there is a nonsingular matrix $\Gamma \in \mathbb{C}^{(m+n(m+1)) \times (m+n(m+1))}$ such that

$$\Gamma^{-1}Q\Gamma = J = \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_l \end{pmatrix}, \quad (4.3)$$

where $J_v = \begin{pmatrix} \zeta_v & 1 & & \\ & \zeta_v & 1 & \\ & & \ddots & \ddots \\ & & & \zeta_v & 1 \\ & & & & \zeta_v \end{pmatrix}_{\tau_v \times \tau_v}$, $v = 1, 2, \dots, l$ is a Jordan matrix.

To determine the nonsingular matrix Γ , we write $\Gamma = (\Gamma_1, \Gamma_2, \dots, \Gamma_l)$ with $\Gamma_v \in \mathbb{C}^{(m+n(m+1)) \times \tau_v}$, $v = 1, 2, \dots, l$. It follows from Eq.(4.3) that

$$(Q\Gamma_1, Q\Gamma_2, \dots, Q\Gamma_l) = (\Gamma_1 J_1, \Gamma_2 J_2, \dots, \Gamma_l J_l).$$

We know that

$$Q\Gamma_v = \Gamma_v J_v, \quad v = 1, 2, \dots, l. \quad (4.4)$$

Denote $\Gamma_v = (\Gamma_1^{(v)}, \Gamma_2^{(v)}, \dots, \Gamma_{\tau_v}^{(v)})$ where $\Gamma_{\kappa}^{(v)} \in \mathbb{C}^{(m+n(m+1)) \times 1}$, $\kappa = 1, 2, \dots, \tau_v$. It appears from Eq.(4.4) that

$$Q \begin{pmatrix} \Gamma_1^{(v)} \\ \Gamma_2^{(v)} \\ \vdots \\ \Gamma_{\tau_v}^{(v)} \end{pmatrix} = \begin{pmatrix} \Gamma_1^{(v)} \\ \Gamma_2^{(v)} \\ \vdots \\ \Gamma_{\tau_v}^{(v)} \end{pmatrix} \begin{pmatrix} \zeta_v & 1 & & \\ & \zeta_v & 1 & \\ & & \ddots & \ddots \\ & & & \zeta_v & 1 \\ & & & & \zeta_v \end{pmatrix}.$$

Hence we conclude that

$$Q\Gamma_1^{(v)} = \zeta_v \Gamma_1^{(v)}, \quad v = 1, 2, \dots, l, \quad (4.5)$$

$$Q\Gamma_{\kappa}^{(v)} = \Gamma_{\kappa-1}^{(v)} + \zeta_v \Gamma_{\kappa}^{(v)}, \quad \kappa = 2, 3, \dots, \tau_v, \quad v = 1, 2, \dots, l. \quad (4.6)$$

It appears from Eq.(4.5) that $\Gamma_1^{(v)}$ is the eigenvector of Q corresponding to the eigenvalue ζ_v . Once the explicit expression of $\Gamma_1^{(v)}$ is given, the explicit expressions of $\Gamma_{\kappa}^{(v)}$ ($\kappa = 2, 3, \dots, \tau_v$) can be

obtained by a recursion procedure. Moreover, the nonsingular matrix Γ is determined. Furthermore, we have

$$P(t) = P(0)e^{(\Gamma J \Gamma^{-1})t} = P(0)\Gamma e^{Jt}\Gamma^{-1} = \Upsilon e^{Jt}\Gamma^{-1}. \quad (4.7)$$

From Eq.(4.7), we can obtain the system state probabilities $P_{i,j}(t)((i,j) \in \Omega)$ explicitly. Then some important reliability measures are easily obtained.

- Availability

$$A(t) = \sum_{j=1}^m P_{0,j}(t) + \sum_{i=1}^{n-k} \sum_{j=0}^m P_{i,j}(t).$$

- The rate of occurrence of failures at time t

$$m_f(t) = k\lambda \sum_{j=0}^m P_{n-k,j}(t).$$

5 Steady-state analysis

We first define the steady-state system state probabilities

$$\begin{aligned} P_{i,j} &= \lim_{t \rightarrow \infty} P_{i,j}(t) = \lim_{t \rightarrow \infty} P\{L(t) = i, J(t) = j\}, (i,j) \in \Omega, \\ P_0 &= (P_{0,1}, P_{0,2}, \dots, P_{0,m}), P_i = (P_{i,0}, P_{i,1}, \dots, P_{i,m}), i = 1, 2, \dots, n, \\ P &= (P_0, P_1, \dots, P_n). \end{aligned}$$

It follows from the matrix equation $PQ = \mathbf{0}_{1 \times (m+n(m+1))}$ that

$$P_0 A_0 + P_1 B_1 = \mathbf{0}_{1 \times m}, \quad (5.1)$$

$$P_0 C_0 + P_1 A_1 + P_2 B_2 = \mathbf{0}_{1 \times (m+1)}, \quad (5.2)$$

$$P_i C_i + P_{i+1} A_{i+1} + P_{i+2} B_{i+2} = \mathbf{0}_{1 \times (m+1)}, i = 1, 2, \dots, n-2, \quad (5.3)$$

$$P_{n-1} C_{n-1} + P_n A_n = \mathbf{0}_{1 \times (m+1)}. \quad (5.4)$$

Further, the following normalizing equation should be satisfied

$$P_0 \mathbf{e}_m + \sum_{i=1}^n P_i \mathbf{e}_{m+1} = 1. \quad (5.5)$$

After some calculations, we conclude that

$$P_i = -P_0 C_0 (A_1 + \xi_2 B_2)^{-1} \prod_{r=2}^i \xi_r, i = 2, 3, \dots, n, \quad (5.6)$$

$$P_1 = -P_0 C_0 (A_1 + \xi_2 B_2)^{-1}, \quad (5.7)$$

$$P_0 \{A_0 - C_0 (A_1 + \xi_2 B_2)^{-1} B_1\} = \mathbf{0}_{1 \times m}, \quad (5.8)$$

where $\xi_i = -C_{i-1} (A_i + \xi_{i+1} B_{i+1})^{-1}$, $i = 2, 3, \dots, n-1$, $\xi_n = -C_{n-1} A_n^{-1}$ are all square matrices of order $m+1$.

Once P_0 being obtained, the stationary probability vector $P = (P_0, P_1, \dots, P_n)$ are determined. It follows from Eqs. (5.5)-(5.7) that

$$P_0 \left\{ \mathbf{e}_m - C_0 (A_1 + \xi_2 B_2)^{-1} \left[I_{m+1} + \sum_{i=2}^n \prod_{r=2}^i \xi_r \right] \mathbf{e}_{m+1} \right\} = 1. \quad (5.9)$$

Solving Eqs (5.8) and (5.9) simultaneously would get the P_0 . Further, we obtain the steady-state system state probabilities

$$P_{0,j} = P_0 e_m(j), \quad j = 1, 2, \dots, m,$$

$$P_{i,j} = P_i e_{m+1}(j+1), \quad i = 1, 2, \dots, n, \quad j = 0, 1, \dots, m.$$

Having computed the steady-state probabilities, we provide some system reliability measures straightforwardly.

- The steady-state system availability

$$A = \sum_{j=1}^m P_{0,j} + \sum_{i=1}^{n-k} \sum_{j=0}^m P_{i,j}.$$

- The steady-state rate of occurrence of failure

$$m_f = k\lambda \sum_{j=0}^m P_{n-k,j}.$$

- The expected number of failed components in the system

$$E[L] = \sum_{j=0}^m \sum_{i=1}^n iP_{i,j}.$$

- The probability that the expert repairman is on vacation

$$P_v = \sum_{i=0}^n \sum_{j=1}^m P_{i,j}.$$

6 System reliability and the mean time to failure

This section will use the theory of PH distribution to discuss the expressions of the system reliability and the mean time to failure of the system. To achieve this, we lump all failure states together to make one absorbing state, say “ $\otimes$ ”. Consider an absorbing Markov process with state space

$$(0, 1), (0, 2), \dots, (0, m), (1, 0), (1, 1), \dots, (1, m), \dots, (n-k, 0), \dots, (n-k, m), \otimes,$$

where the state “ $\otimes$ ” is absorbing and all other states are transient.

The infinitesimal generator Θ of this absorbing Markov process is given by

$$\Theta = \begin{pmatrix} S_{(m+(n-k)(m+1)) \times (m+(n-k)(m+1))} & S_{(m+(n-k)(m+1)) \times 1}^0 \\ 0_{1 \times (m+(n-k)(m+1))} & 0_{1 \times 1} \end{pmatrix},$$

where

$$S = \begin{pmatrix} A_0 & C_0 & & & \\ B_1 & A_1 & C_1 & & \\ & \ddots & \ddots & \ddots & \\ & & B_{n-k-1} & A_{n-k-1} & C_{n-k-1} \\ & & & B_{n-k} & A_{n-k} \end{pmatrix},$$

and the initial probability vector is $(\omega_{1 \times (m+(n-k)(m+1))}(r), 0_{1 \times 1})$. Here symbol $\omega_{1 \times (m+(n-k)(m+1))}(r)$ ($r = 1, 2, \dots, m + (n-k)(m+1)$) is determined by initial state of the system.

According to the definition of the PH distribution, we know that the reliability of the system is

$$R(t) = \omega_{1 \times (m+(n-k)(m+1))}(r) \exp(S_{(m+(n-k)(m+1)) \times (m+(n-k)(m+1))} t) e_{(m+(n-k)(m+1)) \times 1}.$$

The mean time to failure of the system (MTTF) is given by

$$\text{MTTF} = -\omega_{1 \times (m+(n-k)(m+1))}(r) S_{(m+(n-k)(m+1)) \times (m+(n-k)(m+1))}^{-1} e_{(m+(n-k)(m+1)) \times 1}.$$

7 Numerical examples

Example 1. This example studies the time-dependent behaviour of the system availability $A(t)$, the rate of occurrence of failures $m_f(t)$, and the system reliability $R(t)$. We choose $k = 7$, $n = 15$, $\lambda = 0.55$, $\mu_1 = 1.5$, $\mu = 4.5$. The vacation time follows a PH distribution with representation (θ, T) of order 4 and mean value 5.3333, where

$$\theta = (1, 0, 0, 0), \quad T = \begin{pmatrix} -\frac{3}{4} & \frac{3}{4} & 0 & 0 \\ 0 & -\frac{3}{4} & \frac{3}{4} & 0 \\ 0 & 0 & -\frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 & -\frac{3}{4} \end{pmatrix}, \quad T^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{3}{4} \end{pmatrix}.$$

The numerical results for the state probabilities at time t under initial state $(0, 1)$ are displayed in Table 1. We observe from Fig. 1 that these curves display violent fluctuations in the early stage. The higher the probability that the expert repairman returns from his/her vacation, the bigger the system availability and the system reliability are, and the smaller the rate of occurrence of failures is. But after some time units this fluctuations of them tend to disappear. Further, one can easily see that the convergence of these reliability measures are independent of initial conditions, which is as we have anticipated. We observe from Table 2 that the mean time to failure is a decreasing function of the number of failed components at the beginning. Table 2 indicates MTTF is also sensitive to system initial states.

Table 1. System state probabilities at time t under initial state $(0, 1)$.

(i, j)	$P_{i,j}(0.5)$	$P_{i,j}(1.0)$	$P_{i,j}(3.0)$	$P_{i,j}(5.0)$	$P_{i,j}(10)$	$P_{i,j}(15)$
(0, 1)	0.034819	0.003629	0.000416	0.000959	0.003171	0.003837
(0, 2)	0.013043	0.002635	0.000056	0.000106	0.000372	0.000461
(0, 3)	0.002445	0.000985	0.000018	0.000014	0.000052	0.000066
(0, 4)	0.000306	0.000246	0.000010	0.000025	0.000008	0.000011
(1, 0)	0.000066	0.000223	0.000735	0.001708	0.005403	0.006434
(1, 1)	0.087506	0.011455	0.000396	0.000881	0.003031	0.003723
(1, 2)	0.032808	0.008529	0.000124	0.000186	0.000676	0.000850
(1, 3)	0.006151	0.003194	0.000065	0.000035	0.000129	0.000168
(1, 4)	0.000769	0.000798	0.000041	0.000008	0.000025	0.000034
(2, 0)	0.000115	0.000537	0.002057	0.004776	0.014758	0.017441
(2, 1)	0.139205	0.027020	0.000413	0.000804	0.002883	0.003602
(2, 2)	0.052199	0.020225	0.000271	0.000259	0.000971	0.001246
(2, 3)	0.009788	0.007580	0.000208	0.000067	0.000243	0.000323
(2, 4)	0.001223	0.001895	0.000143	0.000022	0.000056	0.000079

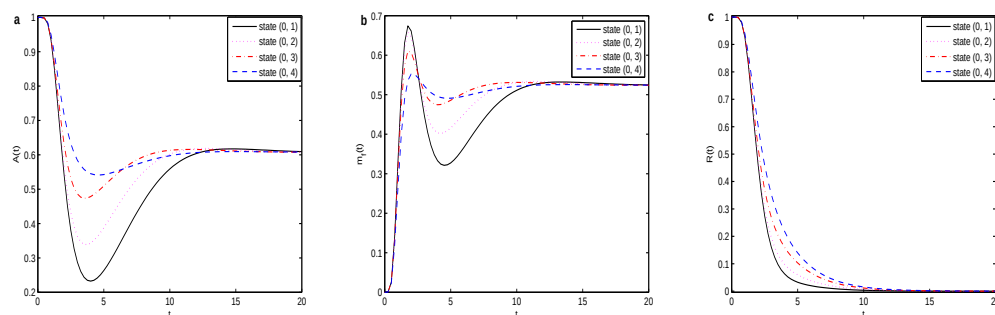


Fig. 1. Illustration of system reliability measures under different initial states.

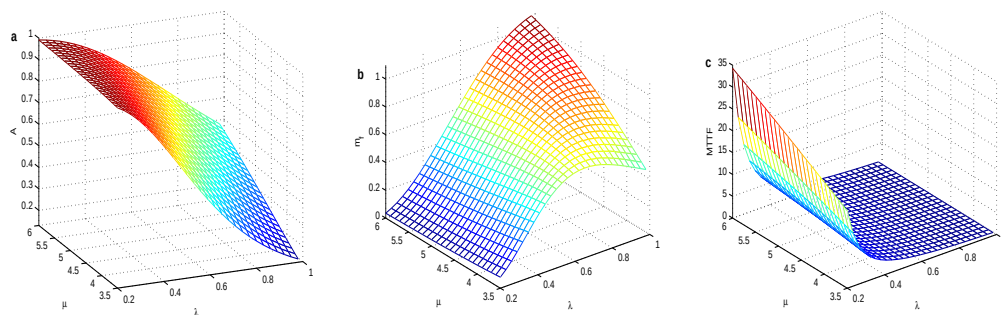
Table 2. The mean time to failure under different initial states.

(i, j)	MTTF	(i, j)	MTTF	(i, j)	MTTF	(i, j)	MTTF
(0, 1)	2.19597682	(2, 2)	2.02493162	(4, 3)	1.81866906	(6, 4)	1.58714981
(0, 2)	2.36295354	(2, 3)	2.31792519	(4, 4)	2.32865739	(7, 0)	1.98296504
(0, 3)	2.67140275	(2, 4)	2.81968345	(5, 0)	2.92553319	(7, 1)	0.72066197
(0, 4)	3.00471737	(3, 0)	3.29384181	(5, 1)	1.26130781	(7, 2)	0.75009790
(1, 0)	2.93837501	(3, 1)	1.69795543	(5, 2)	1.33372576	(7, 3)	0.83983839
(1, 1)	2.05958500	(3, 2)	1.81473601	(5, 3)	1.52460589	(7, 4)	1.11927825
(1, 2)	2.21370058	(3, 3)	2.08306299	(5, 4)	1.98922064	(8, 0)	1.18842427
(1, 3)	2.51988929	(3, 4)	2.60667178	(6, 0)	2.53762712	(8, 1)	0.39066849
(1, 4)	2.95702712	(4, 0)	3.17741838	(6, 1)	1.00725819	(8, 2)	0.40277980
(2, 0)	3.24237394	(4, 1)	1.48994740	(6, 2)	1.05736074	(8, 3)	0.44241327
(2, 1)	1.88813378	(4, 2)	1.58495743	(6, 3)	1.19923211	(8, 4)	0.58528452

Example 2. This example considers a repairable 6-out-of-9-G system with $\mu_1 = 1.2$, λ from 0.2 to 1.0, and μ from 3.5 to 6.0. We specify the expert repairman's vacation time with representation (θ, T) of order 4.

$$\theta = (1, 0, 0, 0), \quad T = \begin{pmatrix} -10 & 7 & 1 & 1 \\ 2 & -15 & 9 & 1 \\ 1 & 4 & -20 & 12 \\ 3 & 2 & 5 & -30 \end{pmatrix}, \quad T^0 = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 20 \end{pmatrix}, \quad E[V] = 0.2744.$$

The numerical results are shown in Fig. 2(a)-2(c). We observe from Fig. 2 that the system availability and the mean time to failure are monotonically decreasing function of λ , and the rate of occurrence of failures goes up with the increasing values of λ . This is because the larger value of λ , the operating components are more easily fail. At the same time, for fixed value of λ , the changes of μ have no noticeable effect on the system reliability measures.

**Fig. 2.** Three kinds of system reliability measures versus (λ, μ) .

Example 3. Setting $n = 2$, $k = 1$, $\mu_1 = 0$, $E[V] \rightarrow 0$, our model reduces to the 2-component parallel system. By using the Markov analysis method, (Cao and Cheng, 2006, pp. 216-219) obtained the

transient reliability measures as follows.

$$A^a(t) = \frac{2\lambda\mu + \mu^2}{2\lambda^2 + 2\lambda\mu + \mu^2} - \frac{2\lambda^2(s_2e^{s_1t} - s_1e^{s_2t})}{s_1s_2(s_1 - s_2)},$$

$$m_f^a(t) = \frac{2\lambda^2\mu}{2\lambda^2 + 2\lambda\mu + \mu^2} + \frac{2\lambda^2(s_1 + \mu)}{s_1(s_1 - s_2)}e^{s_1t} + \frac{2\lambda^2(s_2 + \mu)}{s_2(s_2 - s_1)}e^{s_2t},$$

$$R^a(t) = \frac{c_1e^{c_2t} - c_2e^{c_1t}}{c_1 - c_2},$$

where $s_1, s_2 = \frac{1}{2} \left[-(3\lambda + 2\mu) \pm \sqrt{\lambda^2 + 4\lambda\mu} \right]$, $c_1, c_2 = \frac{1}{2} \left[-(3\lambda + \mu) \pm \sqrt{\lambda^2 + 6\lambda\mu + \mu^2} \right]$.

To validate the correctness of the analytical results of our model, we fix $k = 1$, $n = 2$, $\lambda = 0.55$, $\mu_1 = 0$, $\mu = 2.0$, $\theta = 1$, $T = -10^3$. Numerical results provided in Table 3 show that the formulae of reliability measures derived in this paper exactly match the one reported by Cao and Cheng (2006).

Table 3. Illustration of transient system reliability measures with time t .

	$A(t)$	$A^a(t)$	$m_f(t)$	$m_f^a(t)$	$R(t)$	$R^a(t)$
$t=0.5$	0.96805555	0.96810950	0.14133328	0.17591611	0.95562589	0.95570133
$t=1.0$	0.93768170	0.93777377	0.16801629	0.18236568	0.88273535	0.88291392
$t=1.5$	0.92256048	0.92266459	0.17458949	0.18047560	0.81026720	0.81053327
$t=2.0$	0.91588987	0.91599637	0.17669063	0.17903387	0.74285269	0.74318992
$t=3.0$	0.91185321	0.91195894	0.17779481	0.17803353	0.62406805	0.62450772
$t=5.0$	0.91101661	0.91112138	0.17801226	0.17781733	0.44038741	0.44091829
$t=10.0$	0.91099009	0.91109479	0.17801912	0.17781043	0.18422118	0.18467418
$t=15.0$	0.91099009	0.91109478	0.17801912	0.17781043	0.07706269	0.07734892

Example 4. Let $k = 1$, $\mu_1 = 0$, $E[V] \rightarrow 0$, then our model reduces to the n components parallel repairable system. Cao and Cheng (2006) derived the steady-state system reliability measures which provided below.

$$A^b = \frac{\sum_{i=1}^n \frac{1}{i!} \left(\frac{\mu}{\lambda} \right)^i}{\sum_{i=0}^n \frac{1}{i!} \left(\frac{\mu}{\lambda} \right)^i}, \quad m_f^b = \frac{\mu}{\sum_{i=0}^n \frac{1}{i!} \left(\frac{\mu}{\lambda} \right)^i}, \quad \text{MTTF}^b = \frac{1}{\mu} \sum_{i=1}^n \sum_{j=0}^{i-1} \frac{j!}{i!} \left(\frac{\mu}{\lambda} \right)^{i-j}.$$

To verify the correctness of the expressions obtained in the paper, we select $k = 1$, $\lambda = 0.7$, $\mu_1 = 0$, $\mu = 3.0$, $\theta = 1$, $T = -10^3$. Table 4 shows that the formulae obtained in this paper exactly agree with that given in Cao and Cheng (2006).

Table 4. System reliability measures for different values of n .

n	A	A^b	m_f	m_f^b	MTTF	MTTF ^b
2	0.92953905	0.93088858	0.21119955	0.20733427	5.09485762	5.20408163
3	0.96322515	0.96375357	0.11032172	0.10873930	10.89679549	11.07385814
4	0.97574040	0.97598784	0.07277873	0.07203648	17.51922759	17.72004720
5	0.98124884	0.98137599	0.05625348	0.05587204	23.52171837	23.70249499
6	0.98388141	0.98394869	0.04835577	0.04815392	28.07744612	28.21376722
7	0.98516542	0.98520037	0.04450374	0.04439888	31.09022357	31.17985226
8	0.98577659	0.98579383	0.04267020	0.04261849	32.89398648	32.94739782
9	0.98605212	0.98606002	0.04184363	0.04181994	33.91758593	33.94781634
10	0.98616774	0.98617107	0.04149678	0.04148679	34.50215159	34.51942427
11	0.98621257	0.98621386	0.04136229	0.04135842	34.86134481	34.87199879

8 Conclusions

This paper discussed a repairable k -out-of- n : G system with CO rule and expert repairman's multiple vacations, the duration of which follows a PH distribution. Various system reliability measures are discussed in transient regime and in stationary regime. In addition, numerical examples and special cases are given to illustrate how the various parameters influence the behavior of the system.

Using the theory of PH distribution, we deduced well-structured matrix expressions for the system reliability and the mean time to failure of the system. Compared with previous theoretical results, our expressions are more concise and numerically tractable. In the future work, an interesting extension is to consider such a system with T -policy or N -policy.

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