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Trigonometric Solutions of the Lane-Emden Equation

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Abstract

With new approach, trigonometric solutions of the well-known Lane-Emden equation for $m = 0, 1$ are derived. To the best of author's knowledge these trigonometric solutions oscillating are not reported in the existing literature yet. New solutions are compared with the classical ones.

Keywords: Lane-Emden equation; special equations; trigonometric solutions; exact solutions
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1 Introduction

The well known Lane-Emden (LE) equation is

$$y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0 \tag{1.1}$$

subject to the initial conditions

$$y(0) = 1 \quad \text{and} \quad y'(0) = 0. \tag{1.2}$$

where m is real number. Since the LE equation is very important it has been the focus of several studies [1, 2, 3, 4, 5, 6] and more studies which are not reported here. In general, the solution of the Lane-Emden equation can not be given analytically but only numerically [6, 8, 9, 10]. It is solved analytically only for $m = 0, 1$ and 5 in [3]. For the other values of m, analytic solutions are still missing and thus the equation must be dealt with other techniques to get at least either approximate solutions or exact solutions. It is confirmed that the power series solutions obtained in [T Tanriverdi, Harran University, Turkey, Unpublished results] satisfying (1.1) and (1.2) guarantees that numerical results obtained in [6, 8, 9, 10] are correct. In these article, with new approach trigonometric solutions which are not known yet are derived for $m = 0, 1$.

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2 Main Results

One always rewrites (1.1) as

$$\left(x^2y'\right)' + x^2y^m = 0. \tag{2.1}$$

One of the main problems with the LE equation is the singularity occurring at zero which is a challenging task for many analysts to get its solution. Now set

$$x = \sin\theta. \tag{2.2}$$

Then one obtains

$$\frac{dy}{dx} = \frac{1}{\cos\theta}\frac{dy}{d\theta}. \tag{2.3}$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{1}{\cos\theta}\frac{dy}{d\theta}\right)}{d\theta}\frac{1}{\cos\theta}. \tag{2.4}$$

Plugging (2.2), (2.3) and (2.4) into (2.1) one obtains

$$\left(\frac{\sin^2\theta}{\cos\theta}\frac{dy}{d\theta}\right)' + \cos\theta\sin^2\theta y^m = 0. \tag{2.5}$$

So one formally has the following theorem.

Theorem 2.1. *Equations* (2.1) *and* (2.5) *are equivalent.*

Theorem 2.2. *Let* $m = 0$ *hold. Then* (2.5) *has solution*

$$y(\theta) + \frac{\sin^2\theta}{6} - \frac{c_1}{\sin\theta} + c_2 = 0.$$

Proof.

$$\left(\frac{\sin^2\theta}{\cos\theta}\frac{dy}{d\theta}\right)' + \cos\theta\sin^2\theta = 0. \tag{2.6}$$

Integrating (2.6) one obtains

$$\frac{\sin^2\theta}{\cos\theta}\frac{dy}{d\theta} + \frac{\sin^3\theta}{3} + c_1 = 0$$

where c_1 is a constant. One more integration yields

$$y(\theta) + \frac{\sin^2\theta}{6} - \frac{c_1}{\sin\theta} + c_2 = 0$$

where c_2 is a constant. □

Corollary 2.3. *Let* (1.2) *and* $m = 0$ *hold. Then* (2.5) *has solution*

$$y(\theta) = 1 - \frac{\sin^2\theta}{6}.$$

Remark 2.1. Let (1.2) hold. Then classical solution for $m = 0$ is a parabole $y(x) = 1 - \frac{x^2}{6}$. In the new case, we have an oscillating solution $y(\theta) = 1 - \frac{\sin^2(\theta)}{6}$.

Remark 2.2. Let (1.2) hold. The bound for classical solution is $|y(x)| \leq 1 + \frac{|x|^2}{6}$. The new solution is bounded and oscillating around line $y(x) = \frac{7}{6}$ for any x since $|y(x)| \leq \frac{7}{6}$. The classical solution has a similar bound when x goes to zero. But when x gets bigger then the bound for the old one grows infinitely.

Theorem 2.4. *Let $m = 1$ hold. Then (2.5) has solution*

$$y = c_1 \frac{\cos(\sin\theta)}{\sin\theta} + c_2 \frac{\sin(\sin\theta)}{\sin\theta}.$$

Proof. It is easy to see that with $m = 1$,

$$y = \frac{\sin(\sin\theta)}{\sin\theta}$$

satisfies (2.5). If one substitutes

$$y = \frac{\sin(\sin\theta)}{\sin\theta} u(\theta)$$

into (2.5) then one gets

$$u'' + (2\cos\theta\cot(\sin\theta) + \tan\theta)u' = 0. \tag{2.7}$$

Now integrate (2.7) one gets

$$u' = c\frac{\cos\theta}{\sin^2(\sin\theta)}.$$

One more integration yields

$$u = -c\cot(\sin\theta) + c_2.$$

So general solution of (2.5) with $-c = c_1$ is

$$y = c_1 \frac{\cos(\sin\theta)}{\sin\theta} + c_2 \frac{\sin(\sin\theta)}{\sin\theta}.$$

□

Corollary 2.5. *Let (1.2) and $m = 1$ hold. Then (2.5) has solution*

$$y(\theta) = \frac{\sin(\sin\theta)}{\sin\theta}.$$

Remark 2.3. Let (1.2) hold. Both the classical solution $y(x) = \frac{\sin(x)}{x}$ and the new solution $y(\theta) = \frac{\sin(\sin\theta)}{\sin\theta}$ are oscillating. It is easy to see that the classical one has infinitely many zeros when $x = k\pi$, k is an integer and differs from zero, and $-1 \leq y(x) \leq 1$. However, the new one does not have any zeros and oscillating around line $y(x) = 1$ for any x. If $x = k\pi$ where k is an integer, then the new one $y(x) \to 1$ by using L Hospital rule.

One can prove Theorem 2.4 differently from above.

Proof. In (2.5) set $\sin\theta y(\theta) = u(\theta)$ then one gets

$$u'' + \tan(\theta)u' + \cos^2(\theta)\sin^{1-m}(\theta)u^m = 0. \tag{2.8}$$

Now, for $m = 1$ one gets a solution in [7]

$$u(\theta) = c_1\cos(\sin\theta) + c_2\sin(\sin\theta).$$

Therefore, the desired solution is

$$y(\theta) = c_1 \frac{\cos(\sin\theta)}{\sin\theta} + c_2 \frac{\sin(\sin\theta)}{\sin\theta}.$$

□

Remark 2.4. It is important to note that the relation of (2.8) to the LE equation is not mentioned in [7] with $m = 1$.

3 Conclusion

New trigonometric solutions, which are completely oscillating that is not reported yet in the literature, are derived for $m = 0, 1$. Equation (2.8) which is not studied well and its connection to the LE is reported.

Competing Interests

Author has declared that no competing interests exist.

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