
Support theorem for random evolution equations in hölderian norm

J.H. Andriatahina^{*1} ; R.N.B. Rakotoarisoa^{**2} and T.J. Rabeherimanana^{***3}

^{1,2,3} Mention: Mathematics and Informatica,
Sciences and Technology Domain,
Antananarivo University, B.P. 906, Ankatso 101,

**Original Research
Article**

Abstract

In this paper, we purpose to prove the support theorem of the random evolution equation

$$dX(t) = \sigma(X(t), Z(t))dW_t + b(X(t), Y(t))dt \quad (E)$$

where $X = \{X(t), t \in [0; 1]\}$ is the solution of (E) considered as which a random-variable to value in $C^{\alpha,0}([0; 1]; \mathbb{R}^d)$. The coefficients σ and b satisfied Lipschitz condition and linear growth and depend to $Z = \{Z(t), t \in [0; 1]\}$ and $Y = \{Y(t), t \in [0; 1]\}$ and W is a d-dimensional brownian motion.

For this, we use the method of approximations of the solution of random evolution equations and extend the results well-known of diffusion processes.

Keywords: Approximation, Random evolution equations, Support theorem, hölderian norm.

2010 Mathematics Subject Classification: PRIMARY 60H30 and SECONDARY 60G07

1 Introduction.

Stroock and Varadhan (1992) was began first to precise a support theorem for stochastic differential equations(SDE) in finite dimensional state spaces and with finite dimensional Wiener processes. Their research was next employed by different authors. Ben Arous; Gradinaru and Ledoux (1994)

* E-mail: joceandhaj@gmail.com

**E-mail: rbenja90@gmail.com

***E-mail: rabeherimenana.toussaint@yahoo.fr

was extended for these results in Hölder-norm for $\alpha < \frac{1}{4}$, and Ben Arous and Gradinaru (1993) for $\alpha \in [0, \frac{1}{2}]$. Aida (1990) used these methods for SDEs in separable Hilbert spaces driven by Wiener processes without infinitesimal generators.

Furthermore, in Millet and Sanz-Solé (1994), another methods as an approximation method for both inclusion was founded for obtained a support theorem for diffusion processes. This approach follow the next framework. Stochastic integral is changed into a deterministic one. In Bally and Millet (1995), Bally and Millet (1995), Cardon-Weber and Millet (2000), Kusuoka and Stroock (1993), Millet and Sanz-Solé (1994), approximations were follow a main step to characterize the support of the distribution of each stochastic evolution equations, treated in each of them. Nakayama (2004) established a support theorem for the one dimensional heat or Burger's equation for general mild solution in Hilbert spaces.

But at 2002, Ledoux; Qian and Zhang (2002) was proved a support theorem for diffusion processes via rough path. This approach consist to use continuity theorem of Lyons for rough path in p-variation topology to transfer support theorem for geometric rough path associated with brownian motion to support theorem for solution to SDE with respect to p-variation distance.

In this paper, we purpose to give in $C^{\alpha,0}([0, 1]; \mathbb{R}^d)$ a characterization of the topological support law of the process $X := \{X_t, t \in [0, 1]\}$ solution of the random evolution equation (see Mellouk (2000)):

$$\begin{cases} dX(t) = \sigma(X(t), Z(t))dW_t + b(X(t), Y(t))dt \\ X(0) = x \end{cases} \quad (1.1)$$

where $x \in \mathbb{R}^d$, W is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, Y is a progressively measurable random process which satisfies some integrability conditions, and Z is random process such that topological support of Z is a compact subset of $C^{\alpha,0}([0, 1]; \mathbb{R}^d)$. Furthermore, W is independent of (Y, Z) and: $\sigma : \mathbb{R}^d \times \mathbb{R}^l \rightarrow \mathbb{R}^d \otimes \mathbb{R}^k$ and $b : \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^d$ satisfy the following hypotheses: (H_0) : the function $b(x, y)$ is jointly measurable in (X, Y) and their exist a constant $C_b > 0$ such that

$$\begin{aligned} |b(x, y)| &\leq K(1 + |x|) \quad \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^m \\ |b(x, y) - b(x', y')| &\leq C_b(|x - x'| + |y - y'|) \quad \forall x, x' \in \mathbb{R}^d; y, y' \in \mathbb{R}^m. \end{aligned}$$

(H_1) : The function σ is C^2 and jointly measurable in (x, z) with linear growth and their exist a constant $C_\sigma > 0$ such that

$$\begin{aligned} |\sigma(x, z)| &\leq K \quad \forall (x, z) \in \mathbb{R}^d \times \mathbb{R}^l \\ |\sigma(x, z) - \sigma(x', z')| &\leq C_\sigma(|x - x'| + |z - z'|) \quad \forall x, x' \in \mathbb{R}^d; z, z' \in \mathbb{R}^l. \end{aligned}$$

Let $\alpha > 0$, and denote by $C^\alpha([0, 1]; \mathbb{R}^d)$ the set of α -Hölder continuous functions, i.e of continuous functions $f : [0, 1] \rightarrow \mathbb{R}^d$ such that

$$\|f\|_\alpha = \sup_{\{0 \leq |t-s| \leq 1\}} \frac{|f(t) - f(s)|}{|t - s|^\alpha} < \infty.$$

Define the Hölderian modulus of continuity of f by

$$\omega_\alpha(f, \delta) = \sup_{0 < |t-s| \leq \delta} \frac{|f(t) - f(s)|}{|t - s|^\alpha}.$$

The norm $\|\cdot\|_\alpha$ is called the α -Hölder norm. It is well known that the trajectories X are α -Hölder continuous for $\alpha \in [0, \frac{1}{2}]$. It is well known that $C^\alpha([0, 1]; \mathbb{R}^d)$ is not separable but its closed subspace, defined by

$$C^{\alpha,0}([0, 1]; \mathbb{R}^d) = \{f \in C^\alpha([0, 1]; \mathbb{R}^d); \lim_{\delta \rightarrow 0} \omega_\alpha(f, \delta) = 0\},$$

is separable.

Let $\mathcal{H} = \{h(t) = \int_0^t \dot{h}(s)ds, \dot{h} \in \mathbf{L}^2([0, 1])\}$, the Cameron-Martin space associated of the norm $\|h\|_{\mathcal{H}} = (\int |\dot{h}_t|^2 dt)^{\frac{1}{2}}$. For all $h \in \mathcal{H}$, $\psi \in \text{supp}Y$, and $\chi \in \text{supp}Z$, $S(h, \psi, \chi)$ is a solution of differential ordinary equation.

$$\begin{cases} dS(h, \psi, \chi)_t = (b(S(h, \psi, \chi)_t, \psi_t) - \frac{1}{2}(\nabla_x \sigma)(S(h, \psi, \chi)_t, \chi_t))dt + \sigma(S(h, \psi, \chi)_t, \chi_t)\dot{h}_t dt \\ S(h, \psi, \chi)_0 = x \end{cases} \quad (1.2)$$

where $\text{supp}Y$ is the support of distribution of Y , and $\text{supp}Z$ is the support of distribution of Z .

The aim of this paper is to prove the characterization of the support of $P \circ X^{-1}$ as the closure $\mathbb{S}$ of the set $\{S(h, \psi, \chi), h \in \mathcal{H}, \psi \in \text{supp}Y \text{ et } \chi \in \text{supp}Z\}$ in $C^{\alpha, 0}([0, 1]; \mathbb{R}^d)$. The proof used the approximation theorem for stochastic system using adapted linear interpolations ω_n of ω . Thus we check that $\|S(\omega^n, \psi, \chi) - X\|_{\alpha}$ and $\|X(\omega^n - \omega + h) - S(h, \psi, \chi)\|_{\alpha}$ converge to zero in $\mathbb{L}^2$. Since the law of the T_n defined by transformation T_n of ω defined by $T_n(\omega) = \omega^n - \omega + h$ is absolutely continuous with respect to P (by Girsanov's theorem). The second convergence yields that the support of $P \circ X^{-1}$ contains $\mathbb{S}$, while the first one provides the converse inclusion in the usual way.

This method has been used by Millet and Sanz-Solé (1994) for the diffusion process ($Z \equiv 0$ and $Y \equiv 0$). We adapted and extended this in our situation.

2 General results and approximations

In this section, we state criteria of convergence in Holder norms and a general theorem characterizing of the law of the Weiner functional. The following proposition is a consequence of the Garcia-Rodemich-Rumsey lemma (see e.g. Stroock and Varadhan (1979) p.60).

Proposition 2.1. (i) Let $(R^n(t))_{t \in [0, 1]}$ be a sequence of $\mathbb{R}^d$ -valued processes such that

(A1) For every $p \in [1, \infty)$ there exists C such that

$$\sup_n E(|R^n(t) - R^n(s)|^{2p}) \leq C|t - s|^p$$

for every $s, t \in [0, 1]$.

Then, for any $\lambda > 0$ and $\beta < \frac{p-1}{2p}$, there exists $C > 0$ such that

$$\sup_n P\left(\sup_{|s-t| \leq 1, s \neq t} \frac{|R^n(t) - R^n(s)|}{|t - s|^{\beta}} > \lambda\right) \leq C\lambda^{-2p}. \quad (2.1)$$

(ii) Let $(R^n(t))_{t \in [0, 1]}$ be a sequence of $\mathbb{R}^d$ -valued processes such that (A1) and the following assumption (A2) is satisfied:

(A2) For any $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} P\left(\sup_{0 \leq i \leq 2^n} |R^n(i2^{-n})| > \varepsilon\right) = 0.$$

Then, for any $\alpha \in [0, \frac{1}{2}]$, one has that

$$\lim_n P(\|R^n\|_{\alpha} > \varepsilon) = 0.$$

The following proposition state sufficient conditions for inclusions on the support of the law of the measurable map $F : \Omega \rightarrow E$, where $(E, \|\cdot\|)$ is a separable Banach space. The proof is straightforward.

Proposition 2.2. *Let $F : \Omega \rightarrow E$ be measurable.*

(i) *Let $\xi_1 : \mathcal{H} \rightarrow E$ be a measurable map, and let $H_n : \Omega \rightarrow \mathcal{H}$ be a sequence of variables such that for any $\varepsilon > 0$,*

$$\lim_n P\left(\|F(\omega) - \xi_1(H_n(\omega))\| > \varepsilon\right) = 0 \quad (2.2)$$

then

$$\text{supp}(P \circ F^{-1}) \subset \overline{\{\xi_1(h) : h \in \mathcal{H}\}}. \quad (2.3)$$

(ii) *Let $\xi_2 : \mathcal{H} \rightarrow E$ be a map and for fixed h let $T_n^h : \Omega \rightarrow \Omega$ be a sequence of measurable transformations such that $P \circ (T_n^h)^{-1} \ll P$, and for any $\varepsilon > 0$,*

$$\limsup_n P\left(\|F(T_n^h(\omega)) - \xi_2(h)\| < \varepsilon\right) = 0 \quad (2.4)$$

Then

$$\text{supp}(P \circ F^{-1}) \supset \overline{\{\xi_2(h) : h \in \mathcal{H}\}}. \quad (2.5)$$

Given a positive integer $n > 0$, let $D_n = \{i2^{-n} : 0 \leq i \leq 2^n\}$ denote the set of n -dyadic points. For $t \in [0, 1]$, $\frac{k}{2^n} \leq t < \frac{k+1}{2^n}$, set

$$\bar{t}_n = \frac{k}{2^n}, \quad \underline{t}_n = \frac{k-1}{2^n} \vee 0,$$

and let W^n be the adapted linear interpolation of ω defined by

$$W_t^n = W_{\bar{t}_n} + 2^n(t - \bar{t}_n)(W_{\bar{t}_n} - W_{\underline{t}_n}).$$

We consider the map $\xi_1 = \xi_2 = S(\cdot)$, $H_n(\omega) = \omega^n$, and $T_n^h(\omega) = \omega - \omega^n + h$. Girsanov's theorem implies that $P \circ (T_n^h)^{-1}$ is absolutely continuous with respect to P . Let $X^n(\omega) = X(\omega - \omega^n + h)$. Fix $\alpha < \frac{1}{2}$ and let $\beta \in]\alpha; \frac{1}{2}[$; $X - x$ and $X \circ T_n^h - x$, $S(\omega^n, \psi, \chi) - x$ a.s. belong to $\mathcal{C}^\beta([0, 1], \mathbb{R}^d)$ and have initial value 0, using Ciesielski (Ciesielski (1960)), on the isomorphisms spaces H_α , it is easy to see that they also belong to the separable Banach space H_0^α of $\mathcal{C}^\alpha([0, 1], \mathbb{R}^d)$ defined by:

$$H_0^\alpha = \{f \in \mathcal{C}^\alpha([0, 1], \mathbb{R}^d); f(0) = 0, |f(t) - f(s)| = o(|t - s|^\alpha) \text{ as } |t - s| \rightarrow 0\}.$$

Thus, by proposition 2.2, the equality $\text{supp } P \circ (X)^{-1} = \mathbb{S}$ will follow from the following convergence results for every $\delta > 0$.

$$\lim_n P(\|S(\omega^n, \psi, \chi) - X(\omega)\|_\alpha > \delta) = 0, \quad (2.6)$$

$$\lim_n P(\|X(\omega - \omega^n + h) - S(h, \psi, \chi)\|_\alpha > \delta) = 0. \quad (2.7)$$

Approximation of stochastic integrals of Riemann sums imply that $X^n(\omega)$ is solution of the SDE

$$X_t^n = x + \int_0^t \sigma(X_s^n, Z_s) dW_s - \int_0^t \sigma(X_s^n, Z_s) \dot{\omega}_s^n ds + \int_0^t \sigma(X_s^n, Z_s) \dot{h}_s ds + \int_0^t b(X_s^n, Y_s) ds. \quad (2.8)$$

Let X^n be a map which $\omega^n \mapsto X(\omega^n)$ a map which $(Y, Z) \mapsto X(\omega^n)(Y, Z)$ solution of (2.8). Put $X_t^n = X(\omega^n)(Y, Z) = X(\omega^n, Y_t, Z_t)$. $S(\omega^n, \psi, \chi)$ satisfies,

$$\begin{aligned} S(\omega^n, \psi, \chi)_t = & S(\omega^n, \psi, \chi)_0 + \int_0^t \sigma(S(\omega^n, \psi, \chi)_s, \chi_s) \dot{\omega}_s^n ds + \int_0^t \left(b(S(\omega^n, \psi, \chi)_s, \psi_s) \right. \\ & \left. - \frac{1}{2} (\nabla_x \sigma)(S(\omega^n, \psi, \chi)_s, \chi_s) \right) ds. \end{aligned} \quad (2.9)$$

Thus both processes (X^n) and $(S(\omega^n, \psi, \chi))$ are particular cases of the different processes (K^n) and $(\tilde{K}^n)$, solutions of the following stochastic differential equations:

$$\begin{aligned} K_t^n = & x + \int_0^t F(K_s^n, Z_s) dW_s + \int_0^t G(K_s^n, Z_s) \dot{\omega}_s^n ds + \int_0^t H(K_s^n, Z_s) \dot{h}_s ds \\ & + \int_0^t I(K_s^n, Z_s) ds + \int_0^t B(K_s^n, Y_s) ds, \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} \tilde{K}_t^n = & x + \int_0^t F(\tilde{K}_s^n, \chi_s) dW_s + \int_0^t G(\tilde{K}_s^n, \chi_s) \dot{\omega}_s^n ds + \int_0^t H(\tilde{K}_s^n, \chi_s) \dot{h}_s ds \\ & + \int_0^t I(\tilde{K}_s^n, \chi_s) ds + \int_0^t B(\tilde{K}_s^n, \psi_s) ds, \end{aligned} \quad (2.11)$$

where the coefficients F, G, H, I and B satisfy the following conditions:

(C₁) $F, G, H, I : \mathbb{R}^d \times \mathbb{R}^l \rightarrow \mathbb{R}^d \otimes \mathbb{R}^k$ are globally Lipschitz functions i.e. there exists $C_F, C_G, C_H, C_I > 0$, such that

$$\begin{aligned} |F(x, z) - F(x', z')| &\leq C_F(|x - x'| + |z - z'|), \\ |G(x, z) - G(x', z')| &\leq C_G(|x - x'| + |z - z'|), \\ |H(x, z) - H(x', z')| &\leq C_H(|x - x'| + |z - z'|), \\ |I(x, z) - I(x', z')| &\leq C_I(|x - x'| + |z - z'|), \end{aligned}$$

$\forall x, x' \in \mathbb{R}^d; z, z' \in \mathbb{R}^l$.

(C₂) G is of class $\mathcal{C}^2$ with bounded partial derivative, there exists C_{G_x}, C_{G_y} et $C_{G_{xy}}$ such that

$$\begin{aligned} \left| \frac{\partial G}{\partial x}(x, z) \right| &\leq C_{G_x} \\ \left| \frac{\partial G}{\partial y}(x, z) \right| &\leq C_{G_y} \end{aligned}$$

$$\left| \frac{\partial^2 G}{\partial x_i \partial x_j}(x, z) \right| \leq C_{G_{xz}} \quad i, j = 1, 2.$$

(C₃) $B : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, there exists C_B

$$|B(x, y) - B(x', y')| \leq C_B(|x - x'| + |y - y'|) \quad \forall x, x' \in \mathbb{R}^d; y, y' \in \mathbb{R}^m.$$

Given the coefficients F, G, H, I and B , let (L_t) be a solution of the SDE:

$$\begin{aligned} L_t = x & + \int_0^t [F(L_s, Z_s) + G(L_s, Z_s)] dW_s + \int_0^t H(L_s, Z_s) \dot{h}_s ds + \int_0^t [B(L_s, Y_s) + I(L_s, Z_s)] ds \\ & + \int_0^t \nabla G(L_s, Z_s) [F(L_s, Z_s) + \frac{1}{2} G(L_s, Z_s)] ds. \end{aligned} \quad (2.12)$$

Fix $\alpha \in [0, \frac{1}{2}[$ and for every $\varepsilon > 0$, we consider the following convergences:

$$\lim_n P(\|K^n - L\|_\alpha > \varepsilon) = 0 \quad (2.13)$$

and

$$\lim_n P(\|\tilde{K}^n - L\|_\alpha > \varepsilon) = 0. \quad (2.14)$$

Then (2.6) et (2.7) are the particular cases of (2.14).

Indeed, for $F = 0; G = \sigma; H = 0; I = -\frac{1}{2}(\nabla \sigma)\sigma$ and $B = b$, we obtain (2.6) and for $F = \sigma; G =$

$-\sigma; H = \sigma, I = 0$ et $B = b$, we obtain (2.7).

For $s, t \in [0, 1], p \in [1, \infty[$, there exists an constant C depend of $p; C_F, C_H, C_I, C_G, C_B, C_{G_x}, C_{G_z}, \|\dot{h}\|$ such that

$$E(|L_t - L_s|^{2p}) \leq C|t - s|^p. \quad (2.15)$$

Indeed, for $s, t \in [0, 1]$ and $p \in [1, +\infty)$, we have

$$\begin{aligned} L_t - L_s = & \int_s^t [F(L_u, Z_u) + G(L_u, Z_u)] dW_u + \int_s^t H(L_u, Z_u) \dot{h}_u du + \int_s^t [B(L_u, Y_u) + I(L_u, Z_u)] du \\ & + \int_s^t \nabla G(L_u, Z_u) [F(L_u, Z_u) + \frac{1}{2} G(L_u, Z_u)] du. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}|L_t - L_s|^{2p} \leq & 2^{2p-1} \left\{ \mathbb{E} \left(\left| \int_s^t F(L_u, Z_u) + G(L_u, Z_u) dW_u \right|^{2p} \right) + \mathbb{E} \left(\left| \int_s^t H(L_u, Z_u) \dot{h}_u du \right|^{2p} \right) \right. \\ & + \mathbb{E} \left(\left| \int_s^t B(L_u, Y_u) + I(L_u, Z_u) du \right|^{2p} \right) \\ & \left. + \mathbb{E} \left(\left| \int_s^t \nabla G(L_u, Z_u) [F(L_u, Z_u) + \frac{1}{2} G(L_u, Z_u)] du \right|^{2p} \right) \right\}. \end{aligned}$$

Burkholder's inequality implies

$$\begin{aligned} \mathbb{E} \left| \int_s^t F(L_u, Z_u) + G(L_u, Z_u) dW_u \right|^{2p} & \leq \mathbb{E} \left(\int_s^t |F(L_u, Z_u) + G(L_u, Z_u)|^2 du \right)^p \\ & \leq C(p, C_F, C_G) |t - s|^p. \end{aligned} \quad (2.16)$$

Using Schwartz's inequality, we have

$$\begin{aligned} \mathbb{E} \left| \int_s^t H(L_u, Z_u) \dot{h}_u du \right|^{2p} & \leq \left(\int_s^t \mathbb{E} |H(L_u, Z_u)|^2 du \right)^p \left(\int_s^t \mathbb{E} |\dot{h}_u|^2 du \right)^p \\ & \leq C(p, C_H, \|\dot{h}\|_{\mathcal{H}}^2) |t - s|^p. \end{aligned} \quad (2.17)$$

By Hölder's inequality

$$\mathbb{E} \left(\left| \int_s^t B(L_u, Y_u) + I(L_u, Z_u) du \right|^{2p} \right) \leq C(p, C_B, C_I) |t - s|^p \quad (2.18)$$

and since $(\nabla G)G$ and $(\nabla G)F$ are Lipschitz, we obtain

$$\mathbb{E} \left(\left| \int_s^t \nabla G(L_u, Z_u) [F(L_u, Z_u) + \frac{1}{2} G(L_u, Z_u)] du \right|^{2p} \right) \leq C(p, C_F, C_G, C_{G_x}, C_{G_z}) |t - s|^p. \quad (2.19)$$

The moments estimate (2.16)-(2.19) imply (2.15).

We consider for all $s, t \in [0, 1], p \in [1, \infty[$,

$$\sup_n E(\|K_t^n - K_s^n\|^{2p}) \leq C|t - s|^p; \quad (2.20)$$

$$\sup_n E(\|\tilde{K}_t^n - \tilde{K}_s^n\|^{2p}) \leq C|t - s|^p \quad (2.21)$$

and

$$\lim_n E \left(\sup_{0 \leq i \leq 2^n} |K_{i2^{-n}}^n - L_{i2^{-n}}|^2 \right) = 0; \quad (2.22)$$

$$\lim_n E \left(\sup_{0 \leq i \leq 2^n} |\tilde{K}_{i2^{-n}}^n - L_{i2^{-n}}|^2 \right) = 0. \quad (2.23)$$

3 Main result and discussion

The following theorem is the main result of this paper.

Theorem 3.1. *Let σ and b be functions such that conditions (H_0) and (H_1) are satisfied and let X be a solution of random evolution equation (1.1). Then, for any $\alpha \in [0, \frac{1}{2}]$, the support of the probability $P \circ X^{-1}$ in $C^\alpha([0, 1], \mathbb{R}^d)$ is the closure $\mathbb{S}$ of the set $\{S(h, \psi, \chi), h \in \mathcal{H}, \psi \in \text{supp}Y, \chi \in \text{supp}Z\}$ where $S(h, \psi, \chi)$ is given by (1.2).*

For the proof of this theorem, it suffice to checking the moments estimate (2.21) and (2.23) by proposition 2.2. Otherwise, using (2.13), (2.20) and (2.22) together with the following results, we obtain (2.23).

Proposition 3.1. *Let (R) be a random processes and let $\nu \in \text{Supp}R$. Then*

$$\lim_n E \left(\sup_{0 \leq i \leq 2^n} |\nu_{i2^{-n}} - R_{i2^{-n}}|^2 \right) = 0. \quad (3.1)$$

Proof. Since $P(R \in \text{supp}R) = 1$, there exists $\tilde{\nu} \in \text{supp}R$ such that $\tilde{\nu} = R(t)$ for any $t \in [0, 1]$. Then, we prove that $E \left(\sup_{0 \leq i \leq 2^n} |\nu_{i2^{-n}} - \tilde{\nu}_{i2^{-n}}|^2 \right) \rightarrow 0$ when integer n is large enough. And the following results in Mellouk (2000) complete our proof: $\sup_{0 \leq i \leq 2^n} |\nu_{i2^{-n}} - \tilde{\nu}_{i2^{-n}}| \rightarrow 0$ as n is large enough. $\square$

The following theorem is the one of version of the Arzelà-Ascoli Theorem(see Mellouk (2000) pp-979).

Theorem 3.2. *A set $A \subset C^{\alpha;0}([0, 1], \mathbb{R}^d)$ has compact closure in $C^\alpha([0, 1], \mathbb{R}^d)$ if and only if the following two conditions holds:*

$$\sup_{f \in A} \|f\|_\alpha < \infty$$

and

$$\lim_{\delta \downarrow 0} \sup_{f \in A} \omega_\alpha(f, \delta) = 0.$$

The following results are consequences of this theorem. Since $\text{supp}Z$ is a compact subset of $C^{\alpha;0}([0, 1], \mathbb{R}^d)$, the above theorem 3.2 implies

$$\lim_n \sup_{t \in [0;1]} |\chi_s - \chi s_n| = 0 \quad (3.2)$$

and by the triangular inequality together with the proposition 3.1 and (3.2), we obtain

$$\lim_n \sup_{t \in [0;1]} |Z_s - Z s_n| = 0. \quad (3.3)$$

We will consider two cases for the proof of the main result.

3.1 Bounded case

We check that the moments estimates (2.20); (2.21); (2.22) et (2.23) are true. Furthermore **we will prove the following propositions.**

Proposition 3.2. *Let F, G, H, I and B be the functions such that conditions $(C_1) - (C_3)$ are satisfied and let (K^n) and $(\tilde{K}^n)$ be the respective solutions of (2.10) and (2.11). Then given $p \in [1; \infty)$, there exists a constant C depend on $p, C_F, C_G, C_H, C_I, C_B, C_{G_x}, C_{G_z}$ and $\|h\|_{\mathcal{H}}$ such that for every $s, t \in [0; 1]$, we obtain (2.20) and (2.21).*

Proposition 3.3. *Assume F, G, H, I and B the functions such that conditions $(C_1) - (C_3)$ are satisfied and that the solution (K^n) of (2.10) satisfies (2.20). Then, we obtain (2.22).*

Proof of the proposition 3.2. the proof of (2.20) and (2.21) are similar. It is enough to show one of them.

Fix $p \in [1; \infty)$, $s, t \in \mathbb{R}$. Then, for every $n \geq 1$;

$$\mathbb{E}(|K_t^n - K_s^n|^{2p}) \leq C(T_1 + T_2 + T_3 + T_4 + T_5) \quad (3.4)$$

with

$$\begin{aligned} T_1 &= \mathbb{E} \left(\left| \int_s^t F(K_u^n, Z_u) dW_u \right|^{2p} \right), \\ T_2 &= \mathbb{E} \left(\left| \int_s^t G(K_u^n, Z_u) \dot{\omega}_u^n du \right|^{2p} \right), \\ T_3 &= \mathbb{E} \left(\left| \int_s^t H(K_u^n, Z_u) \dot{h}_u du \right|^{2p} \right), \\ T_4 &= \mathbb{E} \left(\left| \int_s^t I(K_u^n, Z_u) du \right|^{2p} \right), \\ T_5 &= \mathbb{E} \left(\left| \int_s^t B(K_u^n, Y_u) du \right|^{2p} \right). \end{aligned}$$

Burkholder's inequality on T_1 , from Schwartz and Hölder's inequalities, we have

$$T_1 + T_3 + T_4 + T_5 \leq C(n, p, C_F, C_H, C_I, C_B, \|h\|_{\mathcal{H}}^2) |t - s|^p.$$

Finally, $T_2 \leq T_{2,1}(n) + T_{2,2}(n)$, where

$$\begin{aligned} T_{2,1}(n) &= \mathbb{E} \left(\left| \int_s^t G(K_{\underline{u}_n}^n, Z_{\underline{u}_n}) \dot{\omega}_u^n du \right|^{2p} \right), \\ T_{2,2}(n) &= \mathbb{E} \left(\left| \int_s^t |G(K_u^n, Z_u) - G(K_{\underline{u}_n}^n, Z_{\underline{u}_n})| |\dot{\omega}_u^n| du \right|^{2p} \right). \end{aligned}$$

We have,

$$\sup_n T_{2,1}(n) \leq C(n, p, C_G) |t - s|^p.$$

Let $a > 1$, $b > 1$, such that $\frac{1}{a} + \frac{1}{b} = 1$; using Hölder's inequality

$$\begin{aligned} T_{2,2}(n) &\leq |t - s|^{2p-1} \int_s^t \mathbb{E} \{ |G(K_u^n, Z_u) - G(K_{\underline{u}_n}^n, Z_{\underline{u}_n})|^{2pa} \}^{\frac{1}{a}} \{ \mathbb{E} |\dot{\omega}_u^n|^{2pb} \}^{\frac{1}{b}} du \\ &\leq C(p, C_G) |t - s|^{2p-1} 2^{np} \int_s^t \mathbb{E} \{ (|K_u^n - K_{\underline{u}_n}^n| + |Z_u - Z_{\underline{u}_n}|)^{2pa} \}^{\frac{1}{a}} du. \end{aligned}$$

Furthermore, the proof of (2.20) reduce to check that this estimate follow the particular case $s = \underline{u}_n$ et $t = u$. These arguments imply that

$$\begin{aligned} \sup_{s_n} \mathbb{E} \left(\left| \int_{\underline{u}_n}^s \{ F(K_u^n, Z_u) dW_u + H(K_u^n, Z_u) \dot{h}_u du + [I(K_u^n, Z_u) + B(K_u^n, Y_u)] du \} \right|^{2p} \right) \\ \leq C(p, C_F, C_H, C_I, C_B, \|h\|_{\mathcal{H}}) 2^{-np}. \end{aligned}$$

Therefore, we should checking that for each $p \in [1; \infty)$,

$$\sup_s \mathbb{E} \left(\left| \int_{\underline{s}_n}^s G(K_u^n, Z_u) \dot{\omega}_u^n du \right|^{2p} \right) \leq C(n, p, C_G) 2^{-np}.$$

Indeed,

$$\begin{aligned} \mathbb{E} \left(\left| \int_{\underline{s}_n}^s G(K_u^n, Z_u) \dot{\omega}_u^n du \right|^{2p} \right) &\leq C \mathbb{E} \left((2^n \int_{\underline{s}_n}^{\bar{s}_n} |G(K_u^n, Z_u)| du)^{2p} |W_{\underline{s}_n} - W_{\underline{s}_n - 2^{-n} \vee 0}|^{2p} \right) \\ &\quad + C \mathbb{E} \left((2^n \int_{\underline{s}_n}^s |G(K_u^n, Z_u)| du)^{2p} |W_{\bar{s}_n} - W_{\underline{s}_n}|^{2p} \right) \\ &\leq C \left[\mathbb{E} \left(|W_{\underline{s}_n} - W_{\underline{s}_n - 2^{-n} \vee 0}|^{2p} \right) + \mathbb{E} \left(|W_{\bar{s}_n} - W_{\underline{s}_n}|^{2p} \right) \right] \\ &\leq C(n, p, C_G) 2^{-2n}. \end{aligned}$$

This inequality implies $\sup_n T_{2,2}(n) \leq C(n, p, C_G) |t - s|^p$. The proof of (2.20) is complete. $\square$

We will need the following lemmas for the proof of proposition 3.3. We are omitted their demonstrations which are in Millet and Sanz-Solé (1994).

Lemma 3.3. Suppose that (K^n) is a sequence of processes such that (2.20) holds. Let be f a globally Lipschitz function; then

$$\lim_n \mathbb{E} \left(\sup_{1 \leq k \leq 2^n} \left| \int_0^{k2^{-n}} f(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) \{\dot{\omega}_s^n ds - dW_s\} \right|^2 \right) = 0. \quad (3.5)$$

Lemma 3.4. Let $(J_t^n; t \in [0; 1])$ be a sequence of measurable processes such that there exists $p \in [1; \infty)$, $C > 0$ and an sequence $\alpha(n)$ such that

$$\lim_n \alpha(n) = 0, \text{ et } \sup_t \mathbb{E}(|J_t^n|^{2p}) \leq \alpha(n) 2^{-np} \quad (3.6)$$

then

$$\lim_n \mathbb{E} \left(\sup_{1 \leq k \leq 2^n} \left| \int_0^{k2^{-n}} J_s^n \dot{\omega}_s^n |ds| \right|^2 \right) = 0. \quad (3.7)$$

Proof of proposition 3.3. For the proof of (2.22), let $n \geq 1$, $t = k.2^{-n}$

$$\begin{aligned} K_t^n - L_t &= \int_0^t \left[(F + G)(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) - (F + G)(L_{\underline{s}_n}, Z_{\underline{s}_n}) \right] dW_s \\ &\quad + \int_0^t \left[H(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) - H(L_{\underline{s}_n}, Z_{\underline{s}_n}) \right] \dot{h}(s) ds + \int_0^t \left[I(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) - I(L_{\underline{s}_n}, Z_{\underline{s}_n}) \right] ds \\ &\quad + \int_0^t \left\{ B(K_{\underline{s}_n}^n, Y_s) + [(\nabla G)F + \frac{1}{2}(\nabla G)G](K_{\underline{s}_n}^n, Z_{\underline{s}_n}) - B(L_{\underline{s}_n}, Y_s) \right. \\ &\quad \left. - [(\nabla G)F + \frac{1}{2}(\nabla G)G](L_{\underline{s}_n}, Z_{\underline{s}_n}) \right\} ds + \sum_{\beta=1}^6 A_{\beta}^n(t) \end{aligned}$$

where :

$$\begin{aligned}
 A_1^n(t) &= \int_0^t \left[F(K_s^n, Z_s) - F(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) - (F + G)(L_s, Z_s) + (F + G)(L_{\underline{s}_n}, Z_{\underline{s}_n}) \right] dW_s, \\
 A_2^n(t) &= \int_0^t \left[H(K_s^n, Z_s) - H(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) - H(L_s, Z_s) + H(L_{\underline{s}_n}, Z_{\underline{s}_n}) \right] \dot{h}(s) ds, \\
 A_3^n(t) &= \int_0^t \left[I(K_s^n, Z_s) - I(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) - I(L_s, Z_s) + I(L_{\underline{s}_n}, Z_{\underline{s}_n}) \right] ds, \\
 A_4^n(t) &= \int_0^t \left\{ B(K_s^n, Y_s) - B(K_{\underline{s}_n}^n, Y_s) - [B(L_s, Y_s) + [(\nabla G)F + \frac{1}{2}(\nabla G)]G(L_s, Z_s)] \right. \\
 &\quad \left. + B(L_{\underline{s}_n}, Y_s) + [(\nabla G)F + \frac{1}{2}(\nabla G)G](L_{\underline{s}_n}, Z_{\underline{s}_n}) \right\} ds, \\
 A_5^n(t) &= \int_0^t G(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) [\dot{\omega}_s^n ds - dW_s], \\
 A_6^n(t) &= \int_0^t [G(K_s^n, Z_s) - G(K_{\underline{s}_n}^n, Z_{\underline{s}_n})] \dot{\omega}_s^n ds - [(\nabla G)F + \frac{1}{2}(\nabla G)G](K_{\underline{s}_n}^n, Z_{\underline{s}_n}) ds.
 \end{aligned}$$

Gronwall's lemma apply to

$$\varphi(t) = \mathbb{E} \left(\sup_{i \cdot 2^{-n} \leq t} |K_{i \cdot 2^{-n}}^n - L_{i \cdot 2^{-n}}|^2 \right)$$

imply that

$$\mathbb{E} \left(\sup_{0 \leq i \leq 2^n} |K_{i \cdot 2^{-n}}^n - L_{i \cdot 2^{-n}}|^2 \right) \leq C(n, C_F, C_G, C_H, C_I, \|h\|_{\mathcal{H}}) \sum_{\beta=1}^6 \mathbb{E} \left(\sup_{0 \leq i \leq 2^n} |A_{\beta}^n(i \cdot 2^{-n})|^2 \right).$$

Burkholder's inequality, (3.3) and proposition 3.2 imply that $\mathbb{E}(\sup_t |A_1^n(t)|^2) \leq C(n, C_F, C_G)2^{-n}$ since (2.12) is a particular case of (2.10).

By Schwartz's inequality, (2.15) and (3.3), we have $\mathbb{E}(\sup_t |A_2^n(t)|^2) \leq C(n, C_H) \|h\|_{\mathcal{H}}^2 2^{-n}$ and $\mathbb{E}(\sup_t |A_3^n(t)|^2) \leq C(n, C_I) 2^{-n}$.

Since $(\nabla G)F$ and $(\nabla G)G$ are Lipschitz; proposition 3.2, (2.15) and (3.3) imply $\mathbb{E}(\sup_t |A_4^n(t)|^2) < \infty$. Lemma 3.3 yields $\lim_n \mathbb{E}(\sup_t |A_5^n(t)|^2) = 0$. Therefore the proof of (2.22) reduce to check that,

$$\begin{aligned}
 \lim_n \mathbb{E} \left(\sup_{0 \leq j \leq 2^n} \left| \int_0^{j \cdot 2^{-n}} \left[G(K_s^n, Z_s) - G(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) \right] \dot{\omega}_s^n ds \right. \right. \\
 \left. \left. - \int_0^{j \cdot 2^{-n}} \left[(\nabla G)F + \frac{1}{2}(\nabla G)G \right] (K_{\underline{s}_n}^n, Z_{\underline{s}_n}) \right|^2 \right) = 0.
 \end{aligned}$$

Indeed, by Taylor's formula

$$\left| G(K_s^n, Z_s) - G(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) - \nabla G(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) (K_s^n - K_{\underline{s}_n}^n) \right| \leq C(n, C_G, C_{G_x}, C_{G_z}) \left(|K_s^n - K_{\underline{s}_n}^n| + |Z_s - Z_{\underline{s}_n}| \right)^2.$$

Let

$$\begin{aligned}
 \varphi_n(s) &= \int_{\underline{s}_n}^s \left\{ [F(K_u^n, Z_u) - F(K_{\underline{u}_n}^n, Z_{\underline{u}_n})] dW_u + [G(K_u^n, Z_u) - G(K_{\underline{u}_n}^n, Z_{\underline{u}_n})] \dot{\omega}_u^n du \right. \\
 &\quad \left. + H(K_u^n, Z_u) \dot{h}(u) du + [B(K_u^n, Y_u) + I(K_u^n, Z_u)] du \right\}.
 \end{aligned}$$

Then

$$\mathbb{E} \left(\sup_{1 \leq j \leq 2^n} |A_6^n(j \cdot 2^{-n})|^2 \right) \leq c(n, C_H, C_I, C_B) \sum_{\beta=1}^6 B_{\beta}^n$$

where

$$\begin{aligned}
 B_1^n &= \mathbb{E} \left(\sup_{1 \leq j \leq 2^n} \left| \int_0^{j \cdot 2^{-n}} \left(|K_s^n - K_{\underline{s}_n}^n| + |Z_s - Z_{\underline{s}_n}| \right)^2 |\dot{\omega}_s^n| ds \right|^2 \right), \\
 B_2^n &= \mathbb{E} \left(\sup_{1 \leq j \leq 2^n} \left| \int_0^{j \cdot 2^{-n}} |\nabla G(K_s^n, Z_s) \varphi_n(s)| |\dot{\omega}_s^n| ds \right|^2 \right), \\
 B_3^n &= \mathbb{E} \left(\sup_{1 \leq j \leq 2^n} \left| \int_0^{j \cdot 2^{-n}} (\nabla G) F(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) \left(\int_{\underline{s}_n}^{\bar{s}_n} dW_u \right) \dot{\omega}_s^n ds \right. \right. \\
 &\quad \left. \left. - \int_0^{j \cdot 2^{-n}} (\nabla G) F(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) ds \right|^2 \right), \\
 B_4^n &= \mathbb{E} \left(\sup_{1 \leq j \leq 2^n} \left| \int_0^{j \cdot 2^{-n}} (\nabla G) F(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) \left(\int_{\bar{s}_n}^s dW_u \right) \dot{\omega}_s^n ds \right|^2 \right), \\
 B_5^n &= \mathbb{E} \left(\sup_{1 \leq j \leq 2^n} \left| \int_0^{j \cdot 2^{-n}} (\nabla G) G(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) \left(\int_{\bar{s}_n}^s \dot{\omega}_u^n du \right) \dot{\omega}_s^n ds \right. \right. \\
 &\quad \left. \left. - \frac{1}{2} \int_0^{j \cdot 2^{-n}} (\nabla G) G(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) ds \right|^2 \right), \\
 B_6^n &= \mathbb{E} \left(\sup_{1 \leq j \leq 2^n} \left| \int_0^{j \cdot 2^{-n}} (\nabla G) G(K_{\underline{s}_n}^n, Z_{\underline{s}_n}) \left(\int_{\bar{s}_n}^s \dot{\omega}_u^n du \right) \dot{\omega}_s^n ds \right|^2 \right).
 \end{aligned}$$

Let $J_s^n = \left(|K_s^n - K_{\underline{s}_n}^n| + |Z_s - Z_{\underline{s}_n}| \right)^2$, $\mathbb{E}|J_s^n|^{2p} \leq \alpha(n) 2^{-np}$ where $\alpha(n) = C_p 2^{-np}$, proposition 3.2 and lemma 3.4 imply $\lim_n B_1^n = 0$. Let $J_s^n = \nabla G(K_s^n, Z_s) \varphi_n(s)$, then proposition 3.2 and Hölder's inequality imply that if $a > 1$, $b > 1$ are conjugate exponents such that $\frac{1}{a} + \frac{1}{b} = 1$, for any $p \in [0, \infty)$ and $s \in [0, 1]$,

$$\begin{aligned}
 \mathbb{E}(|J_s^n|^{2p}) &\leq \{\mathbb{E}(|\nabla G(K_s^n, Z_s)|^{2pa})\}^{\frac{1}{a}} \{\mathbb{E}(|\varphi_n(s)|^{2pb})\}^{\frac{1}{b}} \\
 &\leq C(n, C_{G_x}, C_{G_z}) \{\mathbb{E}(|\varphi_n(s)|^{2pb})\}^{\frac{1}{b}}.
 \end{aligned} \tag{3.8}$$

Therefore, apply lemma 3.4, it suffices to check that $\sup_t \mathbb{E}(|\varphi_n(s)|^{2p}) \leq \alpha(n) 2^{-2p}$ with $\lim \alpha(n) = 0$. Burkholder's inequality, Hölder inequality, (3.3) and proposition 3.3 yield

$$\begin{aligned}
 \mathbb{E}(|\varphi_n(s)|^{2p}) &\leq C \mathbb{E} \left[\left| \int_{\underline{s}_n}^s \left(|K_u^n - K_{\underline{u}_n}^n| + |Z_u - Z_{\underline{u}_n}| \right)^2 du \right|^p \right. \\
 &\quad \left. + 2^n \left(\int_{\bar{s}_n}^{\bar{s}_n} \left(|K_u^n - K_{\underline{u}_n}^n| + |Z_u - Z_{\underline{u}_n}| \right)^{2p} du \right) |W_{\bar{s}_n} - W_{(\bar{s}_n - 2^{-n}) \vee 0}|^{2p} \right) \\
 &\quad \left. + 2^n \left(\int_{\bar{s}_n}^s \left(|K_u^n - K_{\underline{u}_n}^n| + |Z_u - Z_{\underline{u}_n}| \right)^{2p} du \right) |W_{\bar{s}_n} - W_{\underline{s}_n}|^{2p} \right) \\
 &\quad \left. + \left(\sup \left\{ \left(\int_I |\dot{h}(u)|^2 du \right)^p : \lambda(I) \leq 2^{1-n} \right\} 2^{-(n-1)(p-1)} + 2^{-n(2p-1)} \right) \right. \\
 &\quad \left. \times \int_{\bar{s}_n}^s (1 + |K_u^n|^{2p}) du \right] \\
 &\leq C(p, C_B, C_F, C_G, C_H, C_I) 2^{-np} \alpha(n),
 \end{aligned}$$

where $\alpha(n) = 2^{-np} + \sup \{ (\int_I |\dot{h}(u)|^2 du)^p : \lambda(I) \leq 2^{1-n} \} \rightarrow 0$ as $n \rightarrow \infty$. Thus, lemma 3.4 implies that $\lim_n B_2^n = 0$.

Since $K_{(i-1)2^{-n}}^n$ and $(W_{(i+1)2^{-n}} - W_{i.2^{-n}})$ are independent,

$$\begin{aligned} B_3^n &= \mathbb{E} \left(\sup_{1 \leq k \leq 2^n} \left| \sum_{i=0}^{(k-2) \vee 0} (\nabla G) F(K_{(i-1)2^{-n} \vee 0}^n, Z_{(i-1)2^{-n} \vee 0}) \right. \right. \\ &\quad \left. \left. \times [(W_{(i+1)2^{-n}} - W_{i.2^{-n}})^2 - 2^{-n}] \right|^2 \right) \\ &\leq \sum_{i=0}^{2^n-2} \mathbb{E} ((\nabla G) F(K_{(i-1)2^{-n} \vee 0}^n, Z_{((i-1)2^{-n} \vee 0)}))^2 \\ &\quad \times \mathbb{E} [(W_{(i+1)2^{-n}} - W_{i.2^{-n}})^2 - 2^{-n}]^2 \\ &\leq C(C_F, C_{G_x}, C_{G_z}) 2^n 2^{-2n} \rightarrow 0 \text{ when } n \rightarrow +\infty. \end{aligned}$$

An similar computation yields

$$B_5^n \leq C(C_F, C_{G_x}, C_{G_z}) 2^n 2^{-2n} \rightarrow 0 \text{ when } n \rightarrow +\infty.$$

Using Doob's inequality and proposition 3.2; we have

$$\begin{aligned} B_4^n &\leq \mathbb{E} \left(\left| \int_0^1 2^n (\bar{s}_n + 2^{-n} - s) (\nabla G) F(K_{\bar{s}_n}^n, Z_{\bar{s}_n}) [W_{\bar{s}_n} - W_{\bar{s}_n}] dW_s \right|^2 \right) \\ &\leq C \int_0^1 \mathbb{E} (|(\nabla G) F(K_{\bar{s}_n}^n, Z_{\bar{s}_n})|^2) \mathbb{E} (|W_{\bar{s}_n} - W_{\bar{s}_n}|^2) ds \\ &\leq C(n, C_F, C_{G_x}, C_{G_z}) 2^{-n}; \end{aligned}$$

and the conjugate exponents $a > 1$ and $b > 1$,

$$\begin{aligned} B_6^n &\leq \mathbb{E} \left(\left| \int_0^1 (\nabla G) G(K_{\bar{s}_n}^n, Z_{\bar{s}_n}) \left(\int_{\bar{s}_n}^s \dot{\omega}_u^n du \right) dW_s \right|^2 \right) \\ &\leq C \int_0^1 \mathbb{E} ([(\nabla G) G(K_{\bar{s}_n}^n, Z_{\bar{s}_n})]^{2a})^{\frac{1}{a}} \mathbb{E} (|W_{\bar{s}_n} - W_{(\bar{s}_n - 2^{-n}) \vee 0}|^{2b})^{\frac{1}{b}} ds \\ &\leq C(n, C_{G_x}, C_{G_z}) 2^{-n}. \end{aligned}$$

This complete the proof of $\lim_n \mathbb{E} (\sup_{1 \leq k \leq 2^n} |A_6^n(k, 2^{-2n})|^2) = 0$, and hence that of the proposition. $\square$

Proposition 3.4. Assume F, G, H, I and B the functions such that conditions $(C_1) - (C_3)$ are satisfied and that the solution $(\tilde{K}_s^n)$ of (2.11) satisfies and (2.21). Then, we obtain (2.23).

Proof. For $t = i.2^{-n}$ with $i = 0$ to 2^n

$$\begin{aligned} \tilde{K}_t^n - L_t &= \int_0^t [(F + G)(\tilde{K}_{\bar{s}_n}^n, \chi_{\bar{s}_n}) - (F + G)(L_{\bar{s}_n}, Z_{\bar{s}_n})] dW_s \\ &\quad + \int_0^t [H(\tilde{K}_{\bar{s}_n}^n, \chi_{\bar{s}_n}) - H(L_{\bar{s}_n}, Z_{\bar{s}_n})] \dot{h}(s) ds + \int_0^t [I(\tilde{K}_{\bar{s}_n}^n, \chi_{\bar{s}_n}) - I(L_{\bar{s}_n}, Z_{\bar{s}_n})] ds \\ &\quad + \int_0^t \left\{ B(\tilde{K}_{\bar{s}_n}^n, \psi_s) + [(\nabla G) F + \frac{1}{2} (\nabla G) G](\tilde{K}_{\bar{s}_n}^n, \chi_{\bar{s}_n}) - B(L_{\bar{s}_n}, Y_s) \right. \\ &\quad \left. - [(\nabla G) F + \frac{1}{2} (\nabla G) G](L_{\bar{s}_n}, Z_{\bar{s}_n}) \right\} ds + \sum_{\beta=1}^6 Q_\beta^n(t) \end{aligned}$$

where :

$$\begin{aligned}
 Q_1^n(t) &= \int_0^t [F(\tilde{K}_s^n, \chi_s) - F(\tilde{K}_{s_n}^n, \chi_{s_n}) - (F+G)(L_s, Z_s) + (F+G)(L_{s_n}, Z_{s_n})] dW_s, \\
 Q_2^n(t) &= \int_0^t [H(\tilde{K}_s^n, \chi_s) - H(\tilde{K}_{s_n}^n, \chi_{s_n}) - H(L_s, Z_s) + H(L_{s_n}, Z_{s_n})] \dot{h}(s) ds, \\
 Q_3^n(t) &= \int_0^t [I(\tilde{K}_s^n, \chi_s) - I(\tilde{K}_{s_n}^n, \chi_{s_n}) - I(L_s, Z_s) + I(L_{s_n}, Z_{s_n})] ds, \\
 Q_4^n(t) &= \int_0^t \left\{ B(\tilde{K}_s^n, \psi_s) - B(\tilde{K}_{s_n}^n, \psi_{s_n}) - [B(L_s, Y_s) + [(\nabla G)F + \frac{1}{2}(\nabla G)G](L_s, Z_s)] \right. \\
 &\quad \left. + [B(L_{s_n}, Y_{s_n}) + [(\nabla G)F + \frac{1}{2}(\nabla G)G](L_{s_n}, Z_{s_n})] \right\} ds, \\
 Q_5^n(t) &= \int_0^t G(\tilde{K}_{s_n}^n, \chi_{s_n}) [\dot{\omega}_s^n ds - dW_s], \\
 Q_6^n(t) &= \int_0^t [G(\tilde{K}_s^n, \chi_s) - G(\tilde{K}_{s_n}^n, \chi_{s_n})] \dot{\omega}_s^n ds - [(\nabla G)F + \frac{1}{2}(\nabla G)G](\tilde{K}_{s_n}^n, \chi_{s_n}) ds.
 \end{aligned}$$

We have

$$\begin{aligned}
 |\tilde{K}_t^n - L_t|^2 &\leq c(n, C_F, C_G, C_H, C_I, C_B, C_{G_x}, C_{G_z}, \|h\|_{\mathcal{H}}^2) \\
 &\quad \times \left[\int_0^t \left\{ |\tilde{K}_{s_n}^n - L_{s_n}|^2 + |\chi_{s_n} - Z_{s_n}|^2 + |\psi_s - Y_s|^2 \right\} ds + \sum_{\zeta=1}^6 Q_{\zeta}^n(t) \right].
 \end{aligned}$$

Using Grownwal's lemma applied to

$$\tilde{\varphi}(t) = \mathbb{E} \left(\sup_{i, 2^{-n} \leq t} |\tilde{K}_{i, 2^{-n}}^n - L_{i, 2^{-n}}|^2 \right),$$

we obtain

$$\begin{aligned}
 \mathbb{E}(\sup_{i, 2^{-n} \leq t} |\tilde{K}_t^n - L_t|^2) &\leq C(n, C_F, C_G, C_H, C_I, C_B, C_{G_x}, C_{G_z}, \|h\|_{\mathcal{H}}^2) \left(\mathbb{E}(\sup_{i, 2^{-n} \leq t} \int_0^t |\chi_{s_n} - Z_{s_n}|^2 dt) \right. \\
 &\quad \left. + \mathbb{E}(\sup_{i, 2^{-n} \leq t} \int_0^t |\psi_s - Y_s|^2 dt) + \sum_{\zeta=1}^6 \mathbb{E}(\sup_{i, 2^{-n} \leq t} (Q_{\zeta}^n(t))) \right).
 \end{aligned}$$

By proposition 3.1, we obtain

$$\mathbb{E} \left(\sup_{i, 2^{-n} \leq t} |\tilde{K}_t^n - L_t|^2 \right) \leq C(n, C_F, C_G, C_H, C_I, C_B, C_{G_x}, C_{G_z}, \|h\|_{\mathcal{H}}) \left(\sum_{\zeta=1}^6 \mathbb{E} \left(\sup_{i, 2^{-n} \leq t} (Q_{\zeta}^n(t)) \right) \right).$$

Furthermore, the rest to prove that $\mathbb{E}(\sup_{i, 2^{-n} \leq t} (Q_j^n(t))) \rightarrow 0$ as n tend to ∞ for $j = 1$ to 6.

Thus, by Burkholder's inequality, (3.2), (3.3) and (2.15); we have

$$\mathbb{E}(\sup_t |Q_1^n(t)|^2) \leq C(n, C_F, C_G) 2^{-n}$$

and using the argument above and the idea of the proof of the moment estimate $A_i^n(t)$ for $i = 2$ to 6, we obtain

$$\sum_{\zeta=2}^6 \mathbb{E} \left(\sup_{i, 2^{-n} \leq t} (Q_{\zeta}^n(t)) \right) \leq C(n, C_F, C_G, C_H, C_B, C_{G_x}, C_{G_z}, \|h\|_{\mathcal{H}}) 2^{-n}.$$

□

Proposition 3.2, 3.3 and 3.4 complete the proof of the main result.

3.2 Unbounded case

In this section, we will check theorem 3.1 for b and σ unbounded. Define for $R > 0$,

$$m_R := \sup\{b(t, x, y), \sigma(t, x, z), t \in [0, 1], |x| < R, |y| < R, |z| < R\},$$

$b_i^R := -(m_R - 1) \vee b_i \wedge (m_R + 1)$ and $\sigma_{i,j}^R := -(m_R - 1) \vee \sigma_{i,j} \wedge (m_R + 1)$, $0 \leq i, j \leq 1$. Let $b_R := (b_i^R)_{1 \leq i \leq 2d}$ and $\sigma_R := (\sigma_{i,j}^R)_{1 \leq i, j \leq 2d}$, then $b(t, x, y) = b_R(t, x, y)$ and $\sigma(t, x, z) = \sigma_R(t, x, z)$ for $|x| < R$, $|y| < R$, and $|z| < R$. Furthermore b_R and σ_R verify Lipschitz condition. Consequently, for $R > 0$ large enough, $b(t, x, y) = b^{(R)}(t, x, y)$ and $\sigma(t, x, z) = \sigma^{(R)}(t, x, z)$ for $x, y, z \in \mathbb{R}^d$ with $|x| < R$, $|y| < R$, et $|z| < R$ such that $\sigma^{(R)}$ is globally Lipschitz and bounded and $b^{(R)}$ is of class C^1 and bounded.

Let X^R be the solution of differential equation

$$dX^R(t) = \sigma^{(R)}(X(t), Z(t))dW_t + b^{(R)}(X(t), Y(t))dt.$$

We need the following proposition for the proof of the main result.

Proposition 3.5. For all $t \in [0, 1]$,

$$\lim_{R \rightarrow \infty} \mathbb{E} \left(\sup_t |X(t) - X^R(t)| > \delta \right) = 0 \quad \delta > 0. \quad (3.9)$$

Proof. Consider for $t \in [0, 1]$,

$$X(t) - X^R(t) = [\sigma(X(t), Z(t)) - \sigma^{(R)}(X(t), Z(t))]dW_t + [b(X(t), Y(t)) - b^{(R)}(X(t), Y(t))]dt.$$

The construction of $b^{(R)}$ and of $\sigma^{(R)}$ which verify Lipschitz condition and the local uniqueness of the solution of truncated SDE for R large enough imply (3.9). $\square$

Finally, we may follow the argument in the case of bounded coefficients and we complete the proof of the main result.

4 Conclusion

In the first section, we state historical support theorem for different equations study by many authors before to give the aim of our work which prove the support theorem of random evolution equations in Hölder norm. In second section, the basic results and tools necessary are given for proving our main result. At third section, we propose two cases: b and σ are bounded or none for the proof of our main result.

Acknowledgment

The authors would like to thank the anonymous referees for their valuable comments and suggestions for improving the quality of this work.

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