

the eq. (467) becomes

$$\begin{aligned} \frac{1}{r^2} \frac{d}{dr} \left[\frac{\kappa r^2}{\lambda \theta^n} \frac{n+1}{n} (\lambda \theta^n)^{1/n} \frac{d(\lambda \theta^n)}{dr} \right] &= -4\pi G \lambda \theta^n \\ \left[\frac{n+1}{4\pi G} \kappa \lambda^{\frac{1-n}{n}} \right] \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right) &= -\theta^n. \end{aligned} \quad (469)$$

We now make this equation dimensionless by introducing a radial variable ξ

$$\begin{aligned} \xi &\equiv \frac{r}{\alpha}, \\ \alpha &\equiv \sqrt{\frac{n+1}{4\pi G} \kappa \lambda^{\frac{1-n}{n}}}, \end{aligned} \quad (470)$$

to finally obtain the *Lane-Emden* equation for polytropes in hydrostatic equilibrium:

$$\begin{aligned} \alpha^2 \frac{1}{(\alpha \xi)^2} \frac{d}{d(\alpha \xi)} \left((\alpha \xi)^2 \frac{d\theta}{d(\alpha \xi)} \right) &= -\theta^n \\ \Rightarrow \frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) &= -\theta^n \end{aligned} \quad (471)$$

This is a second order ordinary differential equation, which means that it requires two boundary conditions in order to be well-defined:

1. Define the central density $\rho_c \equiv \lambda$. Then

$$\rho = \lambda \theta^n \quad \Rightarrow \quad \theta(0) = 1. \quad (472)$$

2. At $r = 0$, $\frac{dP}{dr} = -\rho g = -\rho_c g = 0$, because $g_c = 0$ (there is no mass inside zero radius). Therefore,

$$\frac{dP}{dr} = \kappa \gamma \rho^{\gamma-1} \frac{d\rho}{dr} \propto \frac{d\theta}{d\xi} \quad \Rightarrow \quad \left. \frac{d\theta}{d\xi} \right|_{\xi=0} = 0. \quad (473)$$

Analytic Solutions of the Lane-Emden Equation

The Lane-Emden equation can be analytically solved only for a few special, integer values of the index n : 0, 1 and 5. For all other values of n , we must resort to numerical solutions. However, it is beneficial from both pedagogical and intuitive standpoint to derive these analytical solutions, which is what we do next.

Analytic solution for $n=0$.

After substituting $n = 0$ into the Lane-Emden equation [eq. (471)], we obtain

$$\begin{aligned} \frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) &= -1 \quad \Rightarrow \quad \int \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) d\xi = - \int \xi^2 d\xi \\ \Rightarrow \xi^2 \frac{d\theta}{d\xi} &= -\frac{1}{3} \xi^3 + c_1 \quad \Rightarrow \quad \frac{d\theta}{d\xi} = -\frac{1}{3} \xi + \frac{c_1}{\xi^2}. \end{aligned} \quad (474)$$

But, using the boundary conditions, we obtain

$$\begin{aligned} \frac{d\theta}{d\xi}\bigg|_{\xi=0} = 0 &\implies c_1 = 0 \implies \frac{d\theta}{d\xi} = -\frac{1}{3}\xi \implies \theta = -\frac{1}{6}\xi^2 + c_2 \\ \implies \theta(0) = 1 &\implies c_2 = 1 \implies \theta_0 = 1 - \frac{1}{6}\xi^2. \end{aligned} \quad (475)$$

From the equation above, we see that this configuration has a boundary at $\xi = \sqrt{6}$, where $\theta_0 \rightarrow 0$.

Analytic solution for $n=1$.

After substituting $n = 1$ into the Lane-Emden equation [eq. (471)], we obtain

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta \implies \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\xi^2 \theta. \quad (476)$$

Introduce the variable χ

$$\chi(\xi) \equiv \xi \theta(\xi) \implies \theta \equiv \frac{\chi}{\xi}. \quad (477)$$

Then

$$\frac{d\theta}{d\xi} = \frac{d}{d\xi} \left(\frac{\chi}{\xi} \right) = \frac{\xi \chi' - \chi}{\xi^2}, \quad (478)$$

and the Lane-Emden equation in eq. (476) becomes

$$\begin{aligned} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) &= \frac{d}{d\xi} (\xi \chi') = \chi' + \xi \chi'' - \chi' = \xi \chi'' \\ \implies \frac{\xi \chi''}{\xi^2} &= -\frac{\chi}{\xi} \implies \chi'' = -\chi \implies \chi'' + \chi = 0. \end{aligned} \quad (479)$$

This is a harmonic oscillator with general solutions

$$\chi(\xi) = A \sin \xi + B \cos \xi, \quad (480)$$

or, in terms of $\theta \equiv \chi/\xi$

$$\theta(\xi) = A \frac{\sin \xi}{\xi} + B \frac{\cos \xi}{\xi}, \quad (481)$$

After imposing the first boundary condition, the general solution is obtained:

$$\begin{aligned} \theta(0) = 1 &\implies B = 0, \quad \text{because } \lim_{\xi \rightarrow 0} \frac{\cos \xi}{\xi} = \infty \\ A &= 1, \quad \text{because } \lim_{\xi \rightarrow 0} \frac{\sin \xi}{\xi} = 1. \\ \implies \theta_1(\xi) &= \frac{\sin \xi}{\xi}. \end{aligned} \quad (482)$$

The second boundary condition $\frac{d\theta}{d\xi}\bigg|_{\xi=0} = 0$ is explicitly satisfied, because, after applying L'Hospital's rule

$$\lim_{\xi \rightarrow 0} \frac{\xi \cos \xi - \sin \xi}{\xi^2} = \lim_{\xi \rightarrow 0} \frac{-\xi \sin \xi + \cos \xi - \cos \xi}{2\xi} = -\frac{1}{2} \lim_{\xi \rightarrow 0} \sin \xi = 0, \quad (483)$$

as required. From the eq. (482) above, we see that this configuration has a boundary at $\xi = \pi$, where $\theta_1 \rightarrow 0$.