

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0$$

From Poisson's equation

Equivalently, one can start with Poisson's equation,

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho$$

One can replace the gradient of the potential using the hydrostatic equilibrium, via

$$\frac{d\Phi}{dr} = -\frac{1}{\rho} \frac{dP}{dr}$$

which again yields the dimensional form of the Lane–Emden equation.

Solutions

For a given value of the polytropic index n , denote the solution to the Lane–Emden equation as $\theta_n(\xi)$. In general, the Lane–Emden equation must be solved numerically to find θ_n . There are exact, analytic solutions for certain values of n , in particular: $n = 0, 1, 5$. For n between 0 and 5, the solutions are continuous and finite in extent, with the radius of the star given by $R = \alpha \xi_1$, where $\theta_n(\xi_1) = 0$.

For a given solution θ_n , the density profile is given by

$$\rho = \rho_c \theta_n^n.$$

The total mass M of the model star can be found by integrating the density over radius, from 0 to ξ_1 .

The pressure can be found using the polytropic equation of state, $P = K \rho^{1+\frac{1}{n}}$, i.e.

$$P = K \rho_c^{1+\frac{1}{n}} \theta_n^{n+1}$$

Finally, if the gas is ideal, the equation of state is $P = k_B \rho T / \mu$, where k_B is the Boltzmann constant and m the mean molecular weight. The temperature profile is then given by

$$T = \frac{K \mu}{k_B} \rho_c^{1/n} \theta_n$$

Exact solutions

There are only three values of the polytropic index n that lead to exact solutions.

For $n = 0$

If $n = 0$, the equation becomes

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + 1 = 0$$

Re-arranging and integrating once gives

$$\xi^2 \frac{d\theta}{d\xi} = C_1 - \frac{1}{3}\xi^3$$

Dividing both sides by ξ^2 and integrating again gives

$$\theta(\xi) = C_0 - \frac{C_1}{\xi} - \frac{1}{6}\xi^2$$

The boundary conditions $\theta(0) = 1$ and $\theta'(0) = 0$ imply that the constants of integration are $C_0 = 1$ and $C_1 = 0$.

For $n = 1$

When $n = 1$, the equation can be expanded in the form

$$\frac{d^2\theta}{d\xi^2} + \frac{2}{\xi} \frac{d\theta}{d\xi} + \theta = 0$$

One assumes a power series solution:

$$\theta(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$$

This leads to a recursive relationship for the expansion coefficients:

$$a_{n+2} = -\frac{a_n}{(n+3)(n+2)}$$

This relation can be solved leading to the general solution:

$$\theta(\xi) = a_0 \frac{\sin \xi}{\xi} + a_1 \frac{\cos \xi}{\xi}$$

The boundary condition for a physical polytrope demands that $\theta(\xi) \rightarrow 1$ as $\xi \rightarrow 0$. This requires that $a_0 = 1$, $a_1 = 0$, thus leading to the solution:

$$\theta(\xi) = \frac{\sin \xi}{\xi}$$

For $n = 5$

We start from with the Lane–Emden equation:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) + \theta^5 = 0$$

Rewriting for $\frac{d\theta_5}{d\xi}$ produces:

$$\frac{d\theta_5}{d\xi} = \frac{1}{2} \left(1 + \frac{\xi^2}{3} \right)^{\frac{3}{2}} \frac{2\xi}{3} = \frac{\xi^3}{3 \left[1 + \frac{\xi^2}{3} \right]^{\frac{3}{2}}}$$

Differentiating with respect to ξ leads to:

$$\theta_5^5 = \frac{\xi^2}{\left[1 + \frac{\xi^2}{3} \right]^{\frac{3}{2}}} + \frac{3\xi^2}{9 \left[1 + \frac{\xi^2}{3} \right]^{\frac{5}{2}}} = \frac{9}{9 \left[1 + \frac{\xi^2}{3} \right]^{\frac{5}{2}}}$$

Reduced, we come by:

$$\theta_5^5 = \frac{1}{\left[1 + \frac{\xi^2}{3} \right]^{\frac{5}{2}}}$$

Therefore, the Lane–Emden equation has the solution

$$\theta(\xi) = \frac{1}{\sqrt{1 + \xi^2/3}}$$

when $n = 5$. This solution is finite in mass but infinite in radial extent, and therefore the complete polytrope does not represent a physical solution.

Numerical solutions

In general, solutions are found by numerical integration. Many standard methods require that the problem is formulated as a system of first-order ordinary differential equations. For example,

$$\begin{aligned} \frac{d\theta}{d\xi} &= -\frac{\phi}{\xi^2} \\ \frac{d\phi}{d\xi} &= \theta^n \xi^2 \end{aligned}$$

Here, $\phi(\xi)$ is interpreted as the dimensionless mass, defined by $m(r) = 4\pi\alpha^3\rho_c\phi(\xi)$. The relevant initial conditions are $\phi(0) = 0$ and $\theta(0) = 1$. The first equation represents hydrostatic equilibrium and the second represents mass conservation.

Homologous variables

Homology-invariant equation