

where eccentricity $e \in (0, 1)$ and the mean anomaly $M \in [0, \pi]$ are known parameters. We have selected this equation as a function test in the analysis of various semilocal convergence results for the Newton-Raphson method, which when is applied to (1.1) gives the sequence

$$E_{n+1} = E_n - \frac{f(E_n)}{f'(E_n)}, \quad n \geq 0. \quad (1.2)$$

The main idea of the paper is to construct the corresponding majoring function for the mentioned theories for the case $E_0 = \pi$ and then to give conditions on the eccentricity e that guarantee that these majoring functions have real roots. In this way, we establish conditions for the solution for Kepler's equation in terms of eccentricity e .

The authors consider the results given by Kantorovich [Kantorovich (1948)], Gutiérrez [Gutiérrez (1995)], α -theory of Smale [Smale (1986)] and α -theory of Wang-Zhao [Wang (1997)]. In addition, the authors obtain new results of existence and uniqueness of solution for the equation (1.1). In this they generalized the study in Dilone-Gutiérrez (2011) for the case $E_0 = M$.

We present now the semilocal convergence theorems that we use in this paper. Although they can be stated in a Banach space setting, we show here the version for real valued functions. The details of the demonstration of the previous theorems can be found in Diloné [Dilone (2013)].

Theorem 1.1 (Kantorovich's Theorem). *Let us consider $f : I \rightarrow \mathbb{R}$, where I is an open interval in $\mathbb{R}$, a differentiable function in I . Let us assume, without loss of generality, that $f(x_0) \neq 0$ and*

- i) $x_0 \in I$.
- ii) $f'(x_0) \neq 0$.
- iii) $\left| \frac{f(x_0)}{f'(x_0)} \right| \leq \beta$.
- iv) $\left| \frac{f'(x) - f'(y)}{f'(x_0)} \right| \leq \gamma |x - y|, \quad \forall x, y \in I$.
- v) $h = \gamma\beta \leq 1/2$.
- vi) $t^* = \frac{1 - \sqrt{1 - 2h}}{2\beta}$ it is such that $(x_0 - t^*, x_0 + t^*) \subset I$.

Then, the Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0, \quad (1.3)$$

is well defined and $\{x_n\}_{n=0}^\infty$ converges to $x^* \in I$, where $f(x^*) = 0$. In addition, this solution is located in the range

$$[x_0 - t^*, x_0 + t^*]. \quad (1.4)$$

and it is unique in

$$(x_0 - t^{**}, x_0 + t^{**}). \quad (1.5)$$

As it was pointed out by Argyros et al [Argyros (2012)] the Lipschitz condition in Kantorovich's theorem can be weakened by combining a center-Lipschitz with the usual Lipschitz condition.

Theorem 1.2 (Argyros-Hilout's Theorem). *Let us consider $f : I \rightarrow \mathbb{R}$, where I is an open interval in $\mathbb{R}$ and f a differentiable function defined in I . Let us suppose that there exist $x_0 \in I$, $L > L_0$ and $\eta > 0$ such that*

- i) $\left| \frac{f(x_0)}{f'(x_0)} \right| \leq \eta$,
- ii) $\left| \frac{f'(x) - f'(x_0)}{f'(x_0)} \right| \leq L_0 |x - x_0|$ for all $x \in I$,
- iii) $\left| \frac{f'(x) - f'(y)}{f'(x_0)} \right| \leq L |x - y|$ for all $x, y \in I$,

$$(iv) \quad h_A = l r \leq \frac{1}{2},$$

where

$$l = \frac{1}{8} (4l_0 + \sqrt{4l_0^2 + 8l_0^2} + \sqrt{4l_0^2}) r$$

✓ **2.2** $\{x \in \mathbb{R}, |x - x_0| \leq t^*\} \leq I$ where $t^* = \lim_{n \rightarrow \infty} t_n$ and t_n is the scalar sequence given by

$$t_0 = 0, \quad t_1 = \eta, \quad t_2 = \eta + \frac{L_0 \eta^2}{2(1 - L_0 \eta)}, \quad t_{n+2} = t_{n+1} + \frac{L(t_{n+1} - t_n)^2}{2(1 - L_0 t_{n+1})} \quad \forall n = 1, 2, \dots$$

Then, the Newton-Raphson method defined in (1.3) is well defined and $\{x_n\}_{n=0}^{\infty}$ converges to $x^* \in I$, where $f(x^*) = 0$.

Theorem 1.3 (Gutiérrez's theorem or Kantorovich's theorem under strong conditions). Let us consider $f : I \rightarrow \mathbb{R}$, where I is an open interval in $\mathbb{R}$, a differentiable function in I . Let us assume, without loss of generality, that $f(x_0) \neq 0$ and that

- i) $\left| \frac{f(x_0)}{f'(x_0)} \right| \leq a.$
- ii) $\left| \frac{f''(x_0)}{f'(x_0)} \right| \leq b.$
- iii) $\left| \frac{f''(x) - f''(x_0)}{f'(x_0)} \right| \leq K|x - x_0|.$

If the majorizing polynomial

$$\phi_G(t) = a - t + \frac{b}{2}t^2 + \frac{K}{6}t^3, \quad (1.6)$$

has two positive real roots t^* and t^{**} , then the Newton-Raphson method defined in (1.3) well defined and $\{x_n\}_{n=0}^{\infty}$ converges to $x^* \in I$, where $f(x^*) = 0$. This solution is located in (1.4) and it is unique in (1.5).

Theorem 1.4 (Smale α -theory). Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ a continuous and differentiable function in an interval D where the following conditions hold:

- i) $\left| \frac{f(x_0)}{f'(x_0)} \right| \leq \beta, \quad f'(x_0) \neq 0.$
- ii) $\frac{1}{k!} \left| \frac{f^{(k)}(x_0)}{f'(x_0)} \right| \leq \gamma^{k-1}, \quad \text{for } k \geq 2.$
- iii) $\alpha = \beta\gamma \leq 3 - 2\sqrt{2}.$

If the majorizing function

$$\phi_S(t) = \beta - t + \sum_{k \geq 2} \gamma^{k-1} t^k = \beta - t + \frac{\gamma t^2}{1 - \gamma t}, \quad \text{for } 0 \leq t < \frac{1}{\gamma}, \quad (1.7)$$

has two positive real roots t^* and t^{**} then the Newton-Raphson method (1.3) is well defined and $\{x_n\}_{n=0}^{\infty}$ converges to $x^* \in I$, where $f(x^*) = 0$. In addition, this solution is located in the range defined in (1.4) and it is unique in (1.5).

Theorem 1.5 (Wang-Zhao α -theory). Let us consider $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ a continuous and differentiable function in an interval D where the following conditions hold:

- i) $f'(x_0) \neq 0.$
- ii) $\beta(x_0, f) = \left| \frac{f(x_0)}{f'(x_0)} \right|.$
- iii) $\frac{1}{k!} \left| \frac{f^{(k)}(x_0)}{f'(x_0)} \right| \leq \gamma_k, \quad k \geq 2.$

If the majorizing function

$$\phi_W(t) = \beta - t + \sum_{k \geq 2} \gamma_k t^k. \quad (1.8)$$

has two positive real roots t^* and t^{**} , then method defined in (1.3) is well defined and $\{x_n\}_{n=0}^{\infty}$ converges to $x^* \in I$, where $f(x^*) = 0$. This solution is located in the range defined in (1.4) and it is unique in (1.5).

In this case, we see that

$$\begin{aligned} f(E) &= E - e \sin E - M \Rightarrow f(E_0) = \pi - M \Rightarrow |f(E_0)| \leq \pi. \\ f'(E) &= 1 - e \cos E \Rightarrow f'(E_0) = 1 + e \\ f'(y) &= 1 - e \cos y. \\ \left| \frac{f(E_0)}{f'(E_0)} \right| &\leq \frac{\pi}{1+e} = \beta = \pi\delta. \\ \left| \frac{f'(E) - f'(y)}{f'(E_0)} \right| &= \frac{e}{1+e} |\cos E - \cos y| \leq \frac{e}{1+e} |E - y| \Rightarrow \gamma = \frac{e}{1+e} = e\delta \end{aligned}$$

so the majorizing polynomial is given by (2.3). Then

$$h = \beta\gamma = \pi e\delta^2 \leq \frac{1}{2} \Rightarrow \frac{\pi e}{(1+e)^2} \leq \frac{1}{2} \Rightarrow e \leq 0.247808.$$

The rest of the proof follows immediately from the theorem of Kantorovich. $\square$

REMARK. In some cases, it is possible to weaken the Lipschitz condition in Kantorovich's theorem, as it has been indicated in Theorem 1.1. In our case, for the Kepler's equation, $f(x) = x - e \sin x - M$, we have $f'(x) = 1 - e \cos x$.

So, for $x_0 = \pi$, we have

$$\begin{aligned} \frac{f'(x) - f'(y)}{f'(x_0)} &= \frac{e}{1+e} (\cos y - \cos x) \\ &= \frac{e}{1+e} \sum_{n \geq 1} \frac{(-1)^n}{(2n)!} (x^{2n} - y^{2n}) \\ &= \frac{e}{1+e} \left[\sup_{x,y \in I} \left(\frac{x+y}{2!} - \frac{x^3 + x^2y + xy^2 + y^3}{4!} + \frac{x^5 + x^4y + \dots + y^5}{6!} - \dots \right) \right] (x-y), \end{aligned}$$

and

$$L = \frac{e}{1+e} \sup_{x,y \in I} \left| \frac{x+y}{2!} - \frac{x^3 + x^2y + xy^2 + y^3}{4!} + \frac{x^5 + x^4y + \dots + y^5}{6!} - \dots \right|.$$

For the Lipschitz-center condition:

$$L_0 = \frac{e}{1+e} \sup_{x \in I} \left| \frac{x+\pi}{2!} - \frac{x^3 + x^2\pi + x\pi^2 + \pi^3}{4!} + \frac{x^5 + x^4\pi + \dots + \pi^5}{6!} - \dots \right|.$$

So, if $L \leq \gamma = \frac{e}{1+e}$ or $L_0 < L$, then the Theorem 1.2 could be applied, obtaining better bounds for the eccentricity e .

Theorem 2.2 (Gutiérrez's conditions). *If $e \leq 0.129927$ then Kepler's equation has a solution. This solution is located in (2.1) and it is unique in (2.2), where t^* and t^{**} are the positive roots of the majorizing polynomial*

$$\phi_G(t) = \pi\delta - t + \frac{1}{6}e\delta t^3, \quad (2.4)$$

with $\delta = 1/(1+e)$. In addition, the Newton-Raphson method, defined by (2.3), starting in $E_0 = \pi$, converges to this solution.

Proof. We take $E_0 = \pi$ in Gutiérrez's theorem to obtain the following conditions:

- i) $\left| \frac{f(E_0)}{f'(E_0)} \right| \leq a.$
- ii) $\left| \frac{f''(E_0)}{f'(E_0)} \right| \leq b.$